BOOK TWO

TEACHER'S EDITION

General Mathematics

A PROBLEM SOLVING APPROACH

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Guide for Teachers

General Mathematics, A Problem Solving Approach is designed to take into account the variety of interests, abilities, and purposes of the students who will be using this text. Three major objectives are given priority in this text:

1. to reveal how mathematics is used in daily affairs, including many areas of pupil interest and future need
2. to encourage the student to explore the nature of mathematics
3. to provide a continuing program of study of computation and problem solving that stresses understanding as well as skill

Through this approach, the student is given an opportunity to explore the structure of mathematics and applications. The text material encourages the student to develop his ability and to increase his interest in mathematics.

Specifically, this text is designed to help the student

1. study mathematics that is suited to his level of maturity
2. improve his computational skill and problem-solving ability
3. understand the nature, social role, and fascination of mathematics
4. recognize that mathematics is needed by all citizens
5. develop his understanding of mathematics and his skill in using it
6. develop a basis for further study of mathematics and related fields

To achieve these purposes, the text incorporates several continuing programs with periodic presentations and checkups, as appear requisite.

Provision for individual differences

The over-all structure of the text features frequent changes of pace. As a change from pure recitation, topics that provide opportunity for laboratory activities are included. These deal with geometric constructions, scale drawings, graphs, and models. For special interests, the text presents a variety of features:

1. *Research Projects:* outside reading, personal inquiry, and committee projects are suggested for the class.
2. *Exploration and Discovery:* These topics are especially useful for group discussion.
3. *Recreational Mathematics:* Mathematics offers an intellectual challenge of a quality offered by few other fields. The puzzle problems present an intellectual venture that has its own reward. Mathematical concepts and procedures provide opportunities for expression and appreciation that can be most gratifying to those who become aware of them. *Cryptanalysis* may provide a source of enjoyment beyond the text. Many publications, including newspapers, provide activity of this type.

The individualization of remedial work in skills and problem solving and the emphasis placed on vocabulary development provide for the needs of slower students.

Inventory Tests furnish periodic individualized checkups on the basic operations involving real numbers. These tests enable each student to determine for himself his strengths and weaknesses. If weak in any area, he can turn to related *Practice Exercises* and work to correct the deficiency. If he has no difficulty in the test, he may work on the *Experts' Corner* which follows the test.

The *Examples* and review suggestions provide a model for practicing correct procedures, and also limit the amount of individual guidance required of the teacher.

The problems of this text are designed to relate mathematics to student and adult activities. Many problems in this text will develop an idea or tell a story that calls for use of mathematics in the process. The students should be encouraged to identify similar situations from their own experiences. By raising questions in class about a problem situation, you can motivate the class to solve problems. Also, you can discuss or develop parallel situations using the local environment.

Emphasis on mathematical types of thinking

Mathematics is an area of language that makes inductive and deductive thinking possible without distraction by irrelevant elements. The student is required to draw valid conclusions from the available data. He is encouraged to recognize general mathematical properties among the many mathematical relationships presented.

Development of problem-solving ability is an important purpose for studying mathematics. The procedures are presented as a series of related steps. These steps are displayed at appropriate intervals to remind the student of their importance in his work. In using the problem-solving steps, we should emphasize that dead ends should not be considered mistakes. Class discussion of problems should lead the student to see that

1. few problems are really as difficult as they appear
2. everyone has to explore data at the outset
3. clues will emerge as the data are studied
4. tables or sketches are useful in organizing the data
5. action, not worry, speeds success

The instructional strategy enables the students to interact with classmates and the teacher to reinforce a positive attitude toward improving students' thinking when confronting mathematical situations.

Interpreting mathematical ideas

The ability to use and understand mathematical ideas as expressed by graphs, tables, formulas, percentage, and mathematical sentences and symbols is part of the literacy expected of all adults. Many employment tests emphasize the ability to interpret graphical presentations of data. The use of graphs to present data is a well accepted way of communication in news media. An organized treatment of these various methods of expressing ideas is included to develop "consumer" use and "producer" skill in interpreting and presenting ideas in the most effective way.

Maintenance of computational skills

Computational skills are essential in problem solving. The student who needs special practice must be identified; his special weaknesses must be called to his attention; and remedial practice should help him develop correct procedures. The *Inventory Tests* are to be used for this purpose. *Practice Exercises* provide additional opportunities to work on a given difficulty. Also, the daily work of the student should be periodically examined to detect weaknesses in computation.

Vocabulary development

Special attention is given to the development of the vocabulary necessary to deal with essential ideas—a "Words to Watch For" list appears at the beginning of each chapter. Vocabulary tests are provided at strategic intervals. A *Glossary* in the back of the book provides a convenient reference for students who need assistance with vocabulary in solving problems or in reviewing what they have already covered.

Class discussion

Special provision is made for topics suitable for class discussion. The importance of carefully directed class discussion in teaching mathematics cannot be emphasized too strongly. Mathematics is a part of our language. Group discussion is effective not only in the initial development of concepts and processes, but in the detection of incorrect procedures and concepts before they become well established.

You will be able to make more efficient use of time and effort if you develop certain techniques with the group rather than with individuals; for example, estimating answers, deciding the computations to make, and practicing problems with special difficulties. Working in a group stimulates students' originality and ingenuity in attacking and analyzing a problem. Together, they gain experience in exploring the problem situation until they find a clue leading to a solution. They learn that trial and error is the typical approach to solving a genuine problem. They learn that they can discover new facts and that careful analysis can yield interesting and valuable results. Without group activity, each student is apt to assume that he is the only one who cannot proceed directly to a logical analysis of the problem without preliminary exploration and searches for clues. Moreover, the benefits of group learning will carry over to individualized assignments.

Having the class look back at a problem that has been solved can be a very profitable experience. Rather than listening to the description of how some student solved it, it is more instructive to explore such questions as: Were there other ways of solving it? Which is the most efficient solution? Which is easiest? Why was this a division (or multiplication, etc.) situation? How can the relationships among the data be expressed in a simple mathematical sentence or formula?

From such analysis, the students should learn that while there are unlimited numbers of problem situations, the typical mathematical relationships fall into a few basic patterns. It will take many discussions to reveal the common elements among situations that lead to relationships such as:

$$c = np \qquad d = rt \qquad A = lw \qquad C = 2\pi r$$

Further examination shows that these are variations of a single relationship. Also, these relationships reveal an equivalent set of formulas which show division situations:

$$p = \frac{c}{n} \qquad r = \frac{d}{t} \qquad l = \frac{A}{w} \qquad r = \frac{C}{2\pi}$$

Emphasize that the analysis leading to such generalizations is quite different from learning to apply a rule of thumb leading to a formula. Encourage the class to explicate the mathematical relationships and to identify those that are essential to the solution.

By experience and discussion, the class can learn to look for similarities among situations that on the surface appear to be quite different. Once students learn that there are essential relationships to be identified, they can attack a complex problem with new confidence.

Teaching schedule

As a rule, a typical or "average" class will require more time on those chapters that emphasize computations and problem solving than will a rapid class. The latter can devote more time to chapters that provide optional topics and projects.

The allotment of time proposed here takes this into account, and is to be taken as a guide, rather than as rigid prescription. If you find a chapter arousing special interest in the class, or one that provides opportunity for you to develop a continuing project, it may well deserve extra time in your schedule.

This is a suggested schedule for a 36-week school year:

Chapter	*Average*	*Rapid*	*Chapter*	*Average*	*Rapid*
1	3 wk.	3 wk.	7	3 wk.	2 wk.
2	4 wk.	3 wk.	8	2 wk.	2 wk.
3	3 wk.	3 wk.	9	3 wk.	3 wk.
4	3 wk.	4 wk.	10	4 wk.	4 wk.
5	3 wk.	4 wk.	11	3 wk.	2 wk.
6	3 wk.	4 wk.	12	2 wk.	2 wk.

Teaching Commentary

The *Teaching Commentary* provides additional assistance in presenting and interpreting material. You will notice that an overview of each chapter is followed by specific suggestions for a page or topic. You will also notice that our treatment of chapter tests, inventories, vocabulary, and practice exercises suggests that you follow a fixed pattern.

Chapter one

Since few general mathematics students hold mathematics as their major interest, a novel yet completely appropriate approach is called for. The study of symmetry, for example, will appeal to students, especially those whose interests center around art, architecture, design, and draftsmanship. Further, it provides a new start for pupils who are deficient in computation but who have spatial insights and artistic leanings. At the same time, they will find the work mathematically significant and challenging. In fact, many of their friends may be studying similar material in a geometry course.

The chapter introduces symmetry with examples from the students' environment. Simple constructions are used to develop a greater understanding of symmetry and an ability to utilize the basic ideas for the study of geometric shapes and figures. Geometric figures, together with a treatment of points and lines, are used as a convenient vehicle to review concepts of sets. You will find a number of words, terms, and mathematical expressions which are defined as they arise in the chapter. The chapter also contains an Inventory Test on the four fundamental operations, its accompanying Experts' Corner, and the typical three-part Chapter Test.

Page 1 It is essential to demonstrate that mathematics is not always confined to a textbook. To initiate the work, you might set up a bulletin board display to show symmetry in nature, in architecture, or in modern transportation equipment. Displays may also include Moiré patterns, advertising symbols, and magazine ads. You will find enough available material to be able to change the bulletin board frequently during the study of the chapter. Encourage the students to set up a bulletin board committee and to contribute material themselves.

"Words to Watch For" is a list of terms that might be unfamiliar to the class. In most cases, the term is defined where it is first introduced in the chapter. Examine the list with the class, and ask such questions as: "How many words do you know?" "Which words are not familiar?" It is not necessary to define all the words at this time. Use the discussion to:

1. Identify familiar words. (For example, set)
2. Explore possible meanings of words the students use and understand, but which have unique meanings in mathematics. (For example, regular as in regular polygon; intersect or union as in sets)
3. Alert students to the unfamiliar words.

Developing a knowledge of the language of mathematics is important. Time studying vocabulary is time well spent. Use the "Words to Watch For" during the study of a chapter to remind the students of the importance of proper use of mathematical terms. Part Two of the Chapter Test is usually devoted to the use and understanding of vocabulary. We recommend that this section be used for learning rather than grading purposes. For grading purposes, you are encouraged to develop your own vocabulary tests.

Three types of questions lend themselves to testing students' understanding of words. These are:

1. The incomplete sentence. Example: A well-defined collection of objects is called ______.

2. The multiple-choice question. Example: Which of these is a well-defined collection of objects?

   a. an element **b.** a set **c.** an intersection **d.** a polygon

3. Matching questions. See examples on page 38.

Generally speaking, include at least eight words and/or definitions to be matched. List the words alphabetically. For

matching questions, you should have three or four more words than definitions.

Many designs and activities of the first pages can provide extended activity projects for interested students. The construction of the designs may encourage the class to make original designs of their own as extra-credit bulletin board displays.

Page 4 Encourage correct use of ruler and compass, and provide practice at this point. Proper use of these instruments will make later work easier and more interesting. Students enjoy constructions and may be encouraged to set their own standards of accuracy and neatness as a sense of pride and accomplishment develops. Many of their friends may be taking geometry, and for this reason the students may take a greater interest in their work as constructions are usually an integral part of that course, too.

Page 7 The word *set* is not defined in mathematics; instead, it is used as a basic term to define other words. Furthermore, *set* does not have a useful synonym. Therefore, an understanding must be gained through the use of a variety of examples. Point out how we commonly use the word *set* when we refer to a group; set of golf clubs, set of dishes, set of encyclopedias, set of silver service, set of book ends, and so on. As used in mathematics, a *set* may consist of a group of objects that have no similarity or relationship to one another. For example: The set consisting of {Mickey Mantle, a lemon, and a grand piano} is a perfectly legitimate set. A review of the ideas above would serve as a valuable reinforcement before beginning this group of exercises.

Page 8 The exercises are designed to provide practice on the steps listed here as well as to reinforce the ideas of sets. To prepare the class, you may ask such questions as:

1. Can you draw a picture to illustrate the data given?
2. Do you understand all the words?
3. Can you restate the problem in your own words?

The ability to solve problems is not achieved automatically by solving problems. The methods, procedures, and approaches need to be pointed out, discussed, and worked on before slower students can recognize the thought processes involved. Discuss and have the students illustrate the problem-solving steps on the page.

To show the value of tabulating data, ask the students to develop a formula that would apply to any number of points, n. Many students will draw seven non-collinear

points. See if they can get the 21 lines. In Exercise 7, the pattern may best be detected by listing the results of the page of exercises.

Points	2	3	4	5	6	7	8	9	10	n
Lines	1	3	6	10	15	21	28	36	45	$\frac{1}{2}(n)(n-1)$

Some in the class may notice that in the preceding column $3 + 3 = 6$ lines or $4 + 6 = 10$ lines, but if 100 points were given, the student would soon lose the patience needed to develop a chart of that size. Following Exercise 6 on page 21, you may wish to draw a similar conclusion, using angles. Angles formed by rays $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AD}$ will be three in number, $\angle BAC$, $\angle CAD$, and $\angle BAD$. Four rays $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$, and $\overrightarrow{AE}$ would determine six angles, and so on. The overlapping angles are difficult to identify.

Page 9

The key terms here are *axis of symmetry* and *corresponding points*. Their relationship should be clearly understood. The wide range of ability present in general mathematics classes makes discussion essential for understanding of unfamiliar terms.

Page 11

The terms *point* and *line* are not defined in mathematics. Illustrations are given and models pointed out to make these concepts clear, but no distinct definitions are attempted. These words are used to define other mathematical expressions. (Some of your class may wish to read a geometry text to show how undefined terms are used to build a logical system.) On the other hand, *ray* is clearly defined, using directly or implicitly, both point and line. Point out the way definitions build on one another. Beginning with point and line, the words ray and segment are defined. Ray is used to define angle and segment is used to define polygon, and so on.

The definition of an *angle* in Exercise 8 states that two rays having the same endpoint *form* an angle. In the study of polygons we will refer to two adjacent sides which *determine* an angle. The two sides of a triangle cannot *form* an angle according to the definition, and thus we use the word determine.

Page 13

After Exercise 9, you may wish to ask the class to develop a formula to find the number of axes of symmetry for a regular decagon (10) or dodecagon (12).

Page 14 The construction of a regular pentagon requires skill and accuracy in using the compass. The rationale for the construction is beyond the work of the course but you may wish to look at it again after studying the Pythagorean property in Chapter 5.

Page 20 This section will require discussion (1) to develop the vocabulary and (2) to clarify the symbolism. Students should read aloud and discuss statements such as those in Exercise 1. Read **1.a** as: Segment $\overline{AB}$ intersects segment $\overline{DB}$ to form what set (or at what point)?

Since this section defines a *segment* as a set of points and *intersection*, $\cap$, as a set operation, the solution to the exercise is a solution set and therefore is expressed in set language. Read **1.b** as: Segment $\overline{AC}$ intersects segment $\overline{BD}$, forming the set consisting of point F. When this is completely clear to the class, abbreviate the expression to: $\overline{AC}$ intersection $\overline{BD}$ is point F and write it as $\overline{AC} \cap \overline{BD} = F$.

Page 22 The exercises on these pages do not lend themselves to homework assignments. Exercise 1 reminds the class that radii of the same circle have equal measures. This fact furnishes the basis of the constructions. In Exercise 15, the class will notice that the obtuse triangle fits within a semi-circle, and the acute triangle has the center of the circle within the triangle.

Page 24 The construction illustrated here is a basic one in the chapter and has been used previously. In constructing a perpendicular to a segment, we actually construct a perpendicular bisector to a portion of the segment.

1. What is the axis of symmetry? $\overline{OP}$
2. What are the corresponding points? C and D
3. What segment connects them? $\overline{CD}$

The same type of questions may be continued to help students develop further insight into construction procedures.

Page 26 At this time, you may wish to assign someone to find out how the Babylonian numeration system based on 6 and 60 serves as a basis for our angle measure and time measure. This information can be found in most encyclopedias.

In Exercise 4, the pattern may be easier to detect if the students list the results in a table. (Have the class supply the information.)

Figure	*No. of sides*	*Sum of angles (S) degrees*	*Right angles*
Triangle	3	180	2
Square	4	360	4
Pentagon	5	540	6
Hexagon	6	720	8
Octagon	8	1080	12
Nonagon	9	?	?
Decagon	10	?	?

The general formula is $S = (2n - 4)$ right angles. The sum of a 12-sided figure is 20 right angles, or 1800°. The relationship may also be expressed as: $180° (n - 2) = S$.

Page 27 When the students have learned how to bisect an angle, give them an opportunity to practice so that they may develop more skill and make the procedure more automatic. Have the class draw 4 angles of random sizes and bisect them. To provide further practice in using the protractor and compass, you might assign the following problems.

1. Draw an angle of 127° and bisect it.
2. Draw an angle of 86° and bisect it.
3. Construct an angle of 45° without using a protractor.
4. Construct an angle of $22\frac{1}{2}°$ without using a protractor.
5. Bisect an angle of 13°.
6. Construct an angle of $56\frac{1}{4}°$.
7. Draw an angle of 158° and bisect it.

You can construct a $56\frac{1}{4}°$ angle by bisecting a 90° angle, bisecting the upper half to get $22\frac{1}{2}°$, and bisecting again to obtain $11\frac{1}{4}°$ angle. Thus $45° + 11\frac{1}{4}° = 56\frac{1}{4}°$. Using a colored pencil to outline the angle may make the construction easier to check.

Page 28 After doing this set of exercises, ask the class to determine, by construction, whether the 3 altitudes of a triangle meet at one point. They will discover that they do. The intersection points are located respectively: obtuse, outside; right, on the right angle; and acute, in the triangle. Compare these results with the results on page 22.

Page 31 The Inventory Tests do not have specified time limits; the time should be adjusted to the particular class. If the students place a folded sheet of paper immediately beneath the addition and subtraction exercises, the need for copying the figures onto paper is eliminated. This practice not only reduces the time required, but also improves accuracy by eliminating errors in copying.

Have the class exchange papers. Read the answers aloud while the students score the papers. Have the students return the scored papers and then ask: How many have one error? How many have two errors?

Then decide how many need to do the Practice Exercises. Ask each student to determine his source of difficulty before he turns to the Practice Exercises. To make this session worthwhile, make the purpose of remedial practice clear to the class. Remedial Exercises are provided to correct specific student difficulties. Practice Exercises are designed to provide the student with extra practice on difficulties revealed by the Inventory Tests. Students who score below an acceptable level on a given section of the test should work on the appropriate exercises, which are designed to correct the difficulty. After the students complete work on the Practice Exercises, they are ready to take the Practice Test. When a student decides he is ready for a test, he can move to a front seat and time himself. When finished, he can check his answers against the correct answer list and score his own paper. You can now compare his performance on the Practice Test with his performance on the Inventory Test. If little improvement takes place, you should provide remedial help at a mutually suitable time. The extra attention you give to the student can serve as a strong motivating factor.

Page 37 The chapter test may be assigned as homework. During the next class period, the pupils may exchange papers and mark them as you read the answers. Time should be provided for analysis of the results as it will help you to plan review work and adjust the pace of the class. NOTE: Since students have the opportunity to examine them in advance, tests in the text are not intended to be important for grading purposes. However, they will serve as models for your own tests that you can use for grading.

Chapter two

This chapter introduces the mathematical sentence, develops an understanding of its importance, and promotes facility in its use. It is used in problem solving as well as in expressing mathematical ideas in general. Also, the set of fractional numbers of arithmetic, F_a, is introduced, and computations with these numbers are designed to extend understanding of both the numbers and the operations.

The mathematical sentence is a key feature in developing an ability to solve applied problems. Expressed in symbols, it is a powerful tool of mathematics. In this chapter, the sentence is used (1) as a means of discovery and exploration and (2) as an instrument in problem solving. The mathematical

elements and relationships are expressed precisely and concisely without any of the distracting and irrelevant elements of the physical setting.

To use the mathematical sentence efficiently, the student should develop a competency in the use of symbols. Expressing relationships ($=$, $>$, $<$, $\ngtr$, and so on) and ideas with symbols requires regular practice to make the use of symbols automatic and, therefore, more useful to the student. Regular practice in expressing ideas with symbols and translating symbolic statements should be provided as a part of the class activity. This practice will help to increase the student's understanding of his work.

The effectiveness of these mathematical sentences is illustrated in several places in the chapter. Why division by zero is meaningless and the relationships between the fundamental operations and the meaning and properties of the multiplicative inverse, among others, are discussed. The outcome of such discussions may well be an understanding of what Bertrand Russell had in mind when he defined mathematics as "the science of necessary conclusions."

This chapter also introduces subscripts. The student may not be wholly familiar with *subscript* notations. Subscripts are used extensively in this text because they contribute to the clarity and economy of mathematical sentences. The addend-addend-sum relationships are identified by an $a_1 + a_2 = s$ notation. The means and extremes of the proportion are identified as m_1, m_2, e_1, and e_2. You should encourage the student to find other uses of subscripts and to discover other situations where they are useful.

Solution of equations in this chapter is limited to those that can be solved in one step. The solution should at first be derived from intuitive reasoning from the ideas expressed in the question. The equation $x + 17 = 23$ may be read "if 17 is added to a number, the sum is 23." Intuitively, the number is 17 less than 23, and if we subtract 17 from 23, the number x is this difference. A variety of such intuitive solutions will reveal that the procedure is based on using an inverse operation to "undo" addition. Multiplication is used to "undo" division. Formulas involving multiplication and addition provide equivalent equations that "undo" the original operation. For example:

Given	*Equivalent Equations*	
$x + y = a$	$x = a - y$	$y = a - x$
$ab = c$	$c \div a = b$	$c \div b = a$
$\frac{a}{b} = c$	$\frac{a}{c} = b$	$a = bc$

The language of sets introduced in Chapter One is extended to apply to the number system and the vocabulary is extended. The student is being prepared for the study of F_a, the fractional numbers of arithmetic, and later the set of rational and irrational numbers. Attention to symbolism and vocabulary will be essential; you may wish to review the material on sets in Chapter One.

Page 43 Reading these exercises orally in class will provide the surest way to associate the symbol with its verbal equivalent. To master symbolism, you should provide learning activities spaced over a period of time. At regular intervals, exercises of similar design should be used in class for short oral practice until you are certain that mastery is achieved. Notice, too, that Exercise 8 is false. You might call this to the attention of the class.

Page 48 The student should understand that he is learning to set up equations which constitute an important tool for obtaining answers to problems. You may usefully discuss setting up equations for the exercises assigned for homework.

Page 52 Although students' abilities vary widely, a time limit of seven minutes for each of the four sections is suggested. Instruct the class to go on with the next section if they finish one ahead of the time limit. They should understand that not all are expected to complete each section. At the end of each seven-minute interval, the directions should be: "If you have not already done so, go on to the next section." It is not necessary for the students to copy the addition and subtraction exercises. By placing a sheet of paper under the row of exercises, they can write the numbers and answers of the problems. As the first row is completed, the row of answers can be folded under and the operation repeated with the next row.

It would be a wise precaution to walk around the classroom during the test to see that students do not write their answers in the text. If such a tendency develops, it may be more useful to give the tests in mimeographed form.

Page 62 A ratio is useful in comparing two numbers in a given order. Order in a ratio is important and is a major source of confusion for the student. Often, students tend to express the ratio as a proper fraction, even if the number to which another is compared is the smaller of the two. Yet students will agree that $\frac{3}{7}$ is not equivalent to $\frac{7}{3}$.

To help the student determine which part of the comparison is the denominator of the ratio, ask: "Which is the number to which another is compared?" If you introduce the term *base* to denote *number to which another is compared*, the question may be shortened to "What is the base?" The *base* becomes the denominator of the ratio expressed as a common fraction. (When the ratio is a decimal, the base is 1.0. When the ratio is expressed as per cent, the base is 100%. These facts become important in later chapters.)

The use of the term *base* facilitates drill in expressing ratios. Sets of exercises may be used to answer the question "What is the base?" without stopping to solve the exercise. This will help the student adopt this technique in his own work.

Page 63 Call attention to the use of subscript notation.

Proportions can be used to prove that if two ratios are equivalent, then the following relationships are true. A numerical parallel is given to the right of each general case.

If $\frac{a}{b} = \frac{c}{d}$, then $\frac{b}{a} = \frac{d}{c}$	If $\frac{5}{8} = \frac{10}{16}$, then $\frac{8}{5} = \frac{16}{10}$
If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{c} = \frac{b}{d}$	If $\frac{5}{8} = \frac{10}{16}$, then $\frac{5}{10} = \frac{8}{16}$
If $\frac{a}{c} = \frac{b}{d}$, then $\frac{c}{a} = \frac{d}{b}$	If $\frac{5}{10} = \frac{8}{16}$, then $\frac{10}{5} = \frac{16}{8}$
If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{b} = \frac{c+d}{d}$	If $\frac{5}{8} = \frac{10}{16}$, then $\frac{13}{8} = \frac{26}{16}$

The class might consider whether

$$\frac{5}{8} = \frac{10}{16} \text{ also yields } \frac{5}{3} = \frac{10}{6}.$$

The use of numerical examples will clarify these relationships. Encourage the students to discover other valid arrangements. Let the rest of the class check the validity of the ones suggested. In general, if $\frac{a}{b} = \frac{c}{d}$ is a true proportion, then all proportions where $m_1m_2 = e_1e_2$ are also true. An interesting class discussion can be held by writing a true proportion on the board, identifying its elements, and setting up a series of true proportions by using these elements.

Page 64 You may wish to symbolize the multiplication of fractions using subscripts $\frac{n_1}{d_1} \cdot \frac{n_2}{d_2} = \frac{n_1n_2}{d_1d_2}$ and ask the class to describe what this symbolism represents.

Page 65 Understanding the multiplicative inverse is important for what follows. The inverse will be used in solving certain types of equations and also in dividing with fractions.

Some students may not follow the development as presented, so you may wish to approach this problem by using the equation $f_1f_2 = p$ in which f_1 and f_2 are multiplicative inverses. By definition then $p = 1$ and $f_1f_2 = 1$. By using the equivalent relationships, $f_1 = \frac{1}{f_2}$ and $f_2 = \frac{1}{f_1}$.

You should use numbers to provide plenty of concrete illustrations. Questions should be raised with numbers and the answers tested to see if the product is 1. If f_1 is 3, what is f_2? Are they multiplicative inverses? Such exercises in table form may be suitable for classwork.

Directions: If f_1 and f_2 are multiplicative inverses, write the missing factors.

f_1	f_2	p	f_1	f_2	p	f_1	f_2	p
6		1		2	1	$\frac{2}{3}$		1
$\frac{1}{6}$		1		$\frac{1}{2}$	1	7		1
	7	1	8		1	$\frac{5}{6}$		1
	$\frac{1}{7}$	1	$\frac{1}{8}$		1	12		1

Page 75 The method of rounding as shown here is commonly used except in the retail stores where fractions of a cent are rounded up and charged to the customer. Articles 2 @ 37¢ would sell for 19¢ each, 3 @ 88¢ would sell for 30¢ each. Why would articles on sale 3 @ 88¢, not sell for 29¢ each?

Page 78 A series of Problem Scales are included in the text to serve as the basis for identifying students' difficulties in problem solving and for planning special practice to correct weaknesses. These scales may be administered in a variety of ways, depending on the time available and the interest of the students. Some suggestions follow.

1. Assign the scale for homework and ask students to bring the completed assignments to class and score them by exchanging papers.
2. Use the scale for a class activity, collect the papers and return them the next day, either scored by the teacher, or scored by exchanging papers.

Each student should make his own diagnosis of his errors. The Steps for Solving Problems provide a useful basis for this purpose since they cover the most common types of errors. At the beginning of the year, the student may require some assistance in identifying his source of error in a given problem. After a little experience, the student should be able to make his own diagnosis; this ability will help him avoid making the error again. The student should hand in the correct solutions to the problems he missed and list his source of error in terms of the Steps.

If you find a general weakness in the class, provide practice on that problem step through the analysis of similar problems. Practice on a step does not require the complete solution of the problem. Almost any set of problems in the text can serve as a basis for class discussion.

Page 87 Notice again the convenient use of subscripts. The third proportion uses e_1 to represent earnings in the first situation and h_1 to represent hours spent in the first situation. You

may wish to ask the class the meanings of various subscripts on this page. The exercises in this section give the student an opportunity to set up his own notation to describe proportions that he uses to solve the problems. Some possible examples for page 91 are:

11. $\frac{p_1}{h_1} = \frac{p_2}{h_2}$ **13.** $\frac{m_1}{t_1} = \frac{m_2}{t_2}$ **15.** $\frac{n_1}{t_1} = \frac{n_2}{t_2}$

12. $\frac{d_1}{g_1} = \frac{d_2}{g_2}$ **14.** $\frac{f_1}{w_1} = \frac{f_2}{w_2}$ **16.** $\frac{l_1}{p_1} = \frac{l_2}{p_2}$

Many other variations are possible using the proportion ideas previously discussed.

Chapter three

This chapter has several interrelated purposes:

1. to provide an understanding of the set of rational numbers
2. to develop competence in addition and subtraction with emphasis on the use of these operations in equations
3. to extend the study of mathematical sentences to include equations with constants and variables on both sides of the equal sign
4. to continue the study of problem solving with emphasis on setting up and solving the conditional sentence (Step 5)
5. to stress the understanding of percentage situations

Page 95 Negative rational numbers are introduced here to complete the set of rational numbers. Even if you know the students have had previous experience, emphasize the fact that these numbers indicate direction as well as quantity. Using a variety of specific examples in class discussion will increase the students' understanding. For instance:

above and below zero temperature
traveling north and south
above and below sea level
debits and credits in accounting
traveling east and west
deposits and withdrawals
north and south latitude
east and west longitude

The students may suggest their own examples such as gains or losses from a line of scrimmage in a football game.

Many students of varying mathematical aptitude find it easier to understand abstract mathematical concepts if they can first have experience with concrete situations illustrating the concepts. Because of this and the number of examples you find it necessary to develop, it may take days for all students to learn to use signed numbers effectively.

Each example should be worked out on the number line to help them abstract the concept from the physical setting. For class reference, use adding machine tape and a magic marker to make a number line across the room, or refer to the number line on page 97. Treat addition and subtraction with signed numbers informally—intuitively without the use of rules—to encourage recognition of the addition and the subtraction situation.

Page 99 If the students have difficulty graphing the subtraction operation on the number line, direct their attention to a familiar setting. Using positive numbers, have them examine the details of the procedure and develop confidence in their work. As you have them label $s - a_2 = a_1$ for each problem, they should soon see that:

1. The arrow representing a_1 starts *from* a_2 and extends *toward* s.
2. The measure of the length of the arrow determines the absolute value of a_1.
3. The direction of the arrow determines the sign of a_1. Once these ideas involving positive numbers are understood, the students can apply them to signed numbers.

$$+7 - (-8) = +15$$
$$s - a_2 = a_1$$

Use these generalizations to discuss each Exercise. Emphasize that:

a. The arrow extends *from* a_2 *to* s.
b. Its length determines its absolute value.
c. Its direction determines its sign (to the right, positive; to the left, negative).

Provide class practice until the students can predict the length and direction of the arrow before it is graphed.

Some students always expect a sum to be greater than either addend. Discussing the series of addition exercises should help reduce confusion.

a_1	16	16	16	16	16	16	16	16	16
a_2	6	5	4	3	2	1	0	(−1)	(−2)
s	22	21	20	19	18	17	16	15	14

Use as many examples as you need to get the class to generalize: If a_1 remains constant, and a_2 decreases, s will decrease; and when a_2 becomes less than zero, s becomes less than a_1.

Then consider subtraction, generalized as $s - a_1 = a_2$:

s	25	25	25	25	25	25	25	25	25
a_1	5	4	3	2	1	0	(-1)	(-2)	(-3)
a_2	20	21	22	23	24	25	26	27	28

In this case, emphasize: As the subtrahend a_1 becomes less, a_2 increases, and as a_1 becomes less than zero, a_2 becomes greater than s.

Have the class discuss these generalizations before asking them to memorize any rules.

Page 101 In solving equations the student should become accustomed to seeing the variable isolated on the right as well as on the left of the equal sign. Consider Exercise 13 and others to see if a procedure could be devised for determining on which side the variable will be easier to isolate. You may wish to provide additional exercises in class such as:

1. $2x + 7 = 6x - 17$
2. $5x - 19 = 7x - 55$
3. $2x - 5 = 9x - 44$
4. $9x + 2 = 13x - 15$
5. $36 - 3x = 12 + 9x$
6. $47 - 5x = 15 + 11x$

Page 105 This problem set provides opportunity to review and to practice each of the Problem-Solving Steps.

These steps emphasize the student's need to improve his ability:

to read the problem
to understand the problem situation
to set up the mathematical sentence
to perform the necessary computations

At this time, give the class extra practice in reading problems to help them understand the problem setting.

Have the class discuss how to set up the equation needed for each exercise before assigning it for individual study. Many approaches may be considered. In exercises 2, 5, 6, 7, and 9, the sum of two numbers is given. In exercises 5 and 6, the sum of the measures of two sides would be one-half the perimeter which leads to the relationship $s - a_1 = a_2$. In Exercise 2, a total of 320 miles is driven, and if x represents the mileage of Harry's father, Harry drove $(320 - x)$ miles. This would lead to the equation

$$x + 30 = (320 - x)$$
$$(\text{father's mileage}) + 30 = (\text{Harry's mileage})$$

Also $x + (x + 30) = 320$ would be equally appropriate. In Exercise 5, the sum of the measures of two sides is 20.

If x represents the length, $(20 - x)$ represents the width. This would lead to the equation

$$\begin{aligned} x &= (20 - x) + 4 \\ \text{length} &= (\text{4 more than width}) \end{aligned}$$

In Exercise 5, $2x + 2(x + 4) = 40$ is equally valid. (A labeled sketch would help in this situation.) Similar reasoning may be applied to other problems in the set.

This activity leads to the realization that a variety of equations may be used to solve any given problem. After considering the alternatives, the student should choose the one that offers him the greatest understanding and the least chance for error.

Page 109 All students have studied per cent in previous years. It is re-introduced here as one of three ways in which any given ratio may be expressed. It is not necessary to drill on the fractional equivalents of per cents. The students will recall those that occur most frequently. As standard practice, when per cent is calculated, it should be expressed to the nearest tenth of one per cent written as a decimal, not as a common fraction. If working with decimals, express the decimal to the nearest thousandth before changing to a per cent.

Page 113 Have the class practice setting up the proportion needed to solve each statement. The question they should ask about each statement is, "What is the base?" Remind them that the base, in any ratio, is the number to which another is compared. If the base is unknown, as in some of these exercises, it is named by the variable. In a proportion dealing with per cent, the base is the denominator in one fraction and 100 is the denominator in the other.

Page 115 You will find it profitable to discuss the percentage statement for each problem before it is assigned for written work. Use this same procedure each day until it is clear that every student knows how to set up the percentage statement for a problem.

Setting up the percentage statement is Step 5 in a problem involving per cent. It should become standard practice for the student to write out the statement in the form displayed at the bottom of page 114 before trying to set up the proportion. The percentage statement, if properly expressed, defines the relationships among the numbers in the problem so that the base is clearly identified. This helps the students avoid the common error of assuming that the greatest number must be the base.

Page 121 Because you might want to discuss this in class, we do not show the grouping. Additional exercises of this type can prove to be a profitable class activity.

Page 130 It is recommended that Part Three of the Chapter Test be used as a Problem Scale. Discuss in class the use of the Problem-Solving Steps including setting up the equation (Step 5). After discussing the section, it can be assigned for homework and scored in class on the following day. Each student should then analyze his errors to determine the source of any difficulty he might have.

Page 131 This is the first Cumulative Test. Located after each third chapter, the test covers computation, information, and problem-solving techniques that have been emphasized in previous chapters. It is designed to reveal individual and group difficulties. Some review may be necessary before assigning the test.

Each Cumulative Test will probably take two days of class work. Administer it in class and score it yourself, or assign it as homework and then score it in class. You may want to design a Review Test patterned after the items on the Cumulative Test for measurement purposes.

Chapter four

Earmarking funds for future expenses is worth consideration by students as well as by the adults who manage family finances. Reasons for saving money and investing it intelligently are many and varied. Some of them are brought to the students' attention in this chapter. Analyzing and making graphs of statistical data, a natural outgrowth of the topic, is also a vital part of the students' work in this chapter.

Saving money should be recognized not as an end in itself nor merely a commendable social virtue, but rather as a means of deferring non-essential current expenditures in order to provide for a future need. To begin work on this chapter, you may wish to raise these questions:

1. What reasons can you suggest for saving part of one's income?
2. What savings plans and investment opportunities are available? What are the merits of each?

Special emphasis is placed on the need for saving to finance further education. These costs are steadily increasing, and the need to finance at least part of the expenses is more prevalent today than ever before. Some college preparation or post high school technical training are increasingly required to make a young adult an effective part of today's labor force. His lifetime earnings will reflect the training too. Managing family finances does not have the compelling interest now for the high school student that it will

have in later years. This is part of his present family life. It is also very much a part of the news. Antipoverty, OEO, Social Security, Medicare, retirement and investment opportunities receive constant publicity.

Family savings programs reflect a wide range of purposes. Naturally, financing high school and college education should be part of a family plan. Other parts might include:

A fund for vacations, recreation, and seasonal activities

A fund to meet emergencies, accident, illness, or interruption of family earnings or income

Provision for retirement

Consider the interests of your particular class in these discussions. What feelings exist toward further education—college or technical? Do the students know much about education or training costs? Have they considered how they can meet these costs? What other future expenditures may be covered by savings?

Page 137 The figures in the text are from the Bureau of Labor Statistics and also from the United States Department of Health, Education, and Welfare. You can get up-to-date figures regularly from these sources.

The figures on these pages need some interpretation. Often, dropouts are characterized by lack of normal drive, ambition, interest, or sense of purpose. These traits limit a person at any level of education and usually prevent him from gaining or holding a good position. When a student is properly motivated to acquire further education, he qualifies himself for a higher level of earning power.

The predicted shortage of technicians, engineers, and mathematicians has resulted in the setting up of a fund for helping high school graduates finance their further education. For instance, banks and other lending institutions across the country participate in guaranteed loan programs for college students. A program authorized by the 1965 Higher Education Act assists families which cannot bear today's heavy education expenses. Many colleges and universities have their own loan and scholarship funds. United Student Aid Funds, supported largely by private business, offers scholarships.

Your class may be interested in seeking information about these opportunities. School guidance counselors should be able to provide detailed information. Publications such as Lovejoy's College Guide are also sources of information.

Page 156 Since compound interest is treated informally (mainly to differentiate from simple interest), you may wish to provide extra practice here. The class might be interested to learn what procedures are used by banks in your area. Perhaps, you might consider asking a bank official to discuss matters of this type with the class.

A stock holder shares a company's profit. Some companies have good earnings but their stocks have rather low yields. They are mainly corporations which put a large portion of their earnings back into the business. In these cases, however, the value of the stock usually increases steadily. Increased productivity and a growing market also add to the value. Investors who do not need income from dividends and prefer to see the value of their investments increase often favor such "growth" stocks.

Over a period of years, the value of business enterprises in this country have been increasing about three per cent a year, which is somewhat above the rate of increase in the cost of living. Investors have been alert to that growth. More than 20 million families own stock in corporations listed on the New York Stock Exchange, and the number of families is increasing at a rate of about ten per cent a year.

If a person buys stock for investment, he selects those he expects to provide income and to increase in value over a period of years. He expects to hold his stock during periods of recession and diminishing values as well as during periods of boom and expansion. He is not concerned with the day-to-day fluctuation in the market value of his stock.

The speculator, on the other hand, tries to predict rises and declines in the market value of stocks. If he expects the value of a stock to go up, he buys it. When the value does go up, he sells the stock and makes a profit. If he expects a decline in value, he may sell the stock to avoid serious loss.

A person would not be inclined to speculate in stocks without being well informed on factors that affect stock prices. What factors need to be considered if one decided to speculate in a given stock, say steel?

We have tried to provide enough information for a general understanding of stocks and bonds. If the class is interested and time permits, you can initiate a number of interesting activities based on the financial pages of a daily newspaper or other publications obtained from individual brokers. You may be able to arrange to have an investment broker to speak to the class on types of stocks that are available. Some other activities might center about:

1. How stocks are bought and sold; what the New York Stock Exchange is; how transactions in stocks and bonds are supervised and regulated; and what the privileges are of membership on the Exchange. All this and more will be found in a booklet published by the Exchange and distributed by local brokerage firms: UNDERSTANDING THE NEW YORK STOCK EXCHANGE.

2. A study leading to familiarity with stock transactions as reported in financial pages of a newspaper. You might ask the class to pretend they have $10,000 to invest and keep a record of the fluctuations in the stocks they have selected. At the end of the term, they could determine if they would have profited or not. Also, they could consider whether keeping or selling their stock would be worthwhile.
3. Reading the financial pages of a newspaper may encourage the class to make studies of specific companies. They might note the goods or services produced; the various industries represented; new industries, such as plastics, electronics, automation, or space. Finally, the class might consider the merits of owning stocks in certain of the companies they study.

The following "Rules for Investors" were published by an authority. You may wish to discuss these in class.

a. A good distribution of investments is as follows: 10% in savings accounts; 30% in bonds; 60% in common stocks in a variety of basic industries.
b. Invest only in stocks and bonds of leading corporations in essential industries.
c. Invest only in stocks listed on the New York Stock Exchange.
d. Purchase only stocks that have an unbroken record of dividend payments for the past ten years.
e. Own stocks in at least five different industries.
f. Own a few low-yield stocks as a means of providing for growth of capital.

Page 166 Many students are apt to use pencil and paper for *all* computations, even multiplication and division by powers of 10. They fail to realize that with a little practice many computations can be done more easily and quickly "mentally." Practice in computation without pencil and paper is important for several reasons:

1. At Step 4 in the Problem-Solving Steps, the student needs an approximate answer to use as a check on his final answer.
2. Frequently, especially as an adult, he will find himself in a situation where he needs to solve a problem and has no pencil and paper.
3. Many problems require only an approximate answer. You do not need to know *exactly* what it will cost to fill the gasoline tank, but you need to know *about* what it will cost so that you can have enough money with you.

4. "Mental" computation gives a person additional insight into number relationships and problem-solving techniques that are neglected in the more mechanical pencil-and-paper computations.

Frequent oral practice in performing simple computations without pencil and paper can be stimulating and worthwhile to the class. Exercises may be selected from the text or written on the board.

Page 168 Have the class discuss and practice the Problem-Solving Steps before you assign the problems. Assign the second section for homework, or as a classroom test with a full period as a time limit. Have each student analyze his paper and, before he turns it in, note on it any steps on which he may have failed.

Page 170 A family plan for savings and investment reflects that family's feelings about savings. One family may consider it important to save systematically through a bank to provide protection against emergencies such as accident, illness, or death of a wage earner. Another family buys insurance to provide immediate protection against such risks.

Life insurance policies feature savings and protection in varying proportion. The purchaser needs to understand the differences to be able to select the kind of policy to fit his needs. The exercises in the text are intended to develop an understanding of how premiums are calculated, and also some of the factors that should be taken into account in selecting a policy.

Page 172 Some questions about insurance coverage are a matter of judgment, requiring the student to defend his answer. More than one answer may be defensible—the best answer is the one that has the best defense. If you can invite an insurance broker to visit the class, the discussion might be especially lively and stimulating.

Chapter five

This chapter deals with some of the most interesting and important concepts in mathematics. The triangle, for example, is used for ornamental purposes, for rigid construction, and for indirect measurement. A bulletin-board display of uses of the triangle can be highly interesting and instructive. In most bridge construction, the triangle is the central feature. Even though combined with the arch, or with suspension cable, the triangle usually remains as an essential feature in the finished structure. If bridges are not found locally, other constructions can be studied for similar patterns.

The class may be interested in viewing films on building bridges such as those described in *Educators Film Guide*. Check this source or audio-visual catalogs for further information.

The study of the Pythagorean relationships involves square roots and provides an occasion to introduce the set of irrational numbers. The trigonometric ratios provide a further extension of the set.

The study of trigonometric ratios affords an opportunity to explore two important areas of wide interest—surveying and navigation. From a mathematical standpoint, an understanding of trigonometric functions is basic to the study of more advanced fields of mathematics.

Page 179 Many of the students are no doubt already familiar with the Pythagorean relationships. If they now learn to recognize Pythagorean triples (whole numbers a, b, and c among which there is the relationship, $a^2 + b^2 = c^2$ and their multiples) they will save themselves much time and effort. Using squared paper or ruler and protractor, or even compass and ruler, the class might draw triangles to represent the triples.

There are said to be over a thousand proofs of the Theorem of Pythagoras. A special case is illustrated on page 180. Some members of the class may be interested in finding some simple proofs to present in class.

Page 181 It is important for the class to learn to use a table of squares, both for finding squares and for finding square roots. They should be encouraged to memorize the squares of whole numbers from 1 to 25. Occasional oral practice in class on such questions as "What is the square of 23?" and "What is the square root of 576?" will be helpful. For variety, you might ask, "What is the square root of 81 times the square root of 64?" or "What is the square root of 169 times 3?" and so forth.

Page 183 While the student should be skillful in using the table of squares, he should not be totally dependent on it. The method presented here for finding a square root can be carried to any required degree of accuracy. Once the student has a little practice with it, he will find it economical and convenient. A quick comparison of finding $\sqrt{5}$ in the tables and finding $\sqrt{5}$ by the "divide and average" method will show the value of the latter method. Two or three divisions will yield the approximation listed in the table.

An important outcome of the study of square roots, aside from its practical value, is the introduction of irrational numbers. Other members of the set will be studied later on in the chapter when the trigonometric ratios are used. The student is likely to feel that the irrational numbers are not real numbers, since they cannot be expressed in the form $\frac{a}{b}$.

Reference to π, with which he is already familiar, and to the length of the diagonal of a square with sides of unit length will help to clarify the reality of irrational numbers. Practice in locating points associated with $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, and so forth on the number line, as in Exercise 9, is especially useful for this purpose. Checking the art in the book with a ruler should enable the student to locate these points to the nearest tenth.

Page 195 This section is important to an understanding of the trigonometric ratios. The key point is: If one acute angle of a right triangle is known, the other acute angle can be determined. The students should be given plenty of oral exercises to develop this fact. Later we will suggest exercises that may be used to reinforce this idea.

Page 199 The fact that in similar triangles, measures of corresponding sides are in the same ratio is basic to the concept of trigonometric ratios. In the work on this section, considerable oral practice should be given to the study of right triangles with corresponding acute angles having the same measure, as in Exercise 7. You may consider having the class construct three right triangles whose sides are (3, 4, 5), (6, 8, 10), and (9, 12, 15). Have the class cut out the triangles and notice that the corresponding angles are equal. Thus the trigonometric ratios of a given angle measure are constant.

Page 203 Success in these topics depends upon the student developing: (a) ability to identify the trigonometric ratio to be used in the proportion; and (b) ability to set up and solve the proportion. Extensive oral classwork spread over several days will help the students a great deal.

1. Sketch right triangles on the board, labeled in various ways and in various positions, and ask the students various questions such as: What is tan $m\angle M$? What is sin $m\angle N$? Which ratio of $m\angle M$ is $\frac{m}{p}$?
2. Using sketches of ladders, heights, distances, and so forth, as in the problems in the text, ask the students to state the trigonometric ratio that can be used to find the solution. Solve the problems on the opening page of the chapter.

If the student is having genuine difficulty in identifying the proper trigonometric ratio for a given situation, a useful procedure is to have him use the numerical data and find his answer by making a scale drawing. He is already familiar with the scale drawing as a means for finding distances and other measures by indirect measurement. Given

the measurements necessary for a trigonometric solution, he can make the scale drawing. The results will have only a limited degree of precision, but by then he will have sufficient insight into the problem to identify the necessary trigonometric function, and can use it to refine his answer. NOTE: The hypotenuse is *never* used in the tangent ratio. It will be to the student's advantage to locate and label the hypotenuse before deciding on a ratio to use.

The sketch should remain as an integral part of the solution to a trigonometric problem, even for students who are not wholly dependent on it. Before attempting to solve each problem, the students should be required to draw a sketch approximately to scale, using ruler and protractor, lettering the key parts, and indicating by x or another variable the part to be found. The sketch serves to orient the student to the use of the proper ratio as well as to provide a useful check on the correctness of the answer, since even a rough sketch will provide a useful approximation. This technique also serves to detect arithmetic errors that may be made in solving the proportion.

The sketch also provides a useful guide in setting up the trigonometric proportion. Thus, the proportion in Exercise 1 is: $\frac{500}{n} = \frac{625}{1000}$. In Exercise 4, the proportion is $\frac{n}{1200} = \frac{625}{1000}$. This difference in order is evident from an examination of the sketch, but may readily be overlooked if the solution is pursued mechanically without a sketch.

The terminology in this chapter, "Sine of the measure of angle A, i.e., sin $m\angle A$" and so forth, is used to emphasize the facts that: (a) the trigonometric functions are ratios between measures, and (b) the ratios are determined by the measures of the acute angles of the right triangle. It is recommended that this terminology be discussed, together with the reasons for using it, and that the use be continued throughout Chapter 5. In Chapter 7, the more economical terminology, as sin A, is introduced, with an explanation.

Page 214 This section will provide an interesting class discussion. Some of the angles here are familiar to the students from experience with the isosceles right triangle, which has acute angles measuring 45°, and with the 30° − 60° − 90° triangles. The characteristics of the functions as the angles approach 90° and 0°, however, will not be familiar to the students. Point out that these characteristics are related to the fact that a triangle can have only one right angle, and see if the students can discover the relationships: sin 90° = 1, cos 90° = 0, and tan 90° is undefined.

Page 217 Review the Problem-Solving Steps before assigning this section as a Problem Scale. Indicate that making a sketch is included as a part of Step 1. The sketch should always be a required part of the solution and should also be used as a check on the accuracy of the solution.

Chapter six

This chapter extends the use of real numbers to the operations of adding, subtracting, multiplying, and dividing. Notice that the rules of signs in these operations have been developed intuitively and are not to be construed as a rigorous proof. They serve, however, to make the rules plausible and acceptable. The number system is further reviewed and amplified to reinforce the students' understanding of the various number systems and their subsets. Irrational numbers receive special emphasis here, along with further development and review of the commutative, associative, and distributive properties and their application to the real number system.

Further practice with linear equations in one variable, the backbone of elementary algebra, is carried on throughout this chapter. The work is systematically extended to consider a system containing two variables. Equations, inequalities, and their solution by graphing receive attention. (Students enjoy this part of elementary algebra. If student interest seems to justify it, further study of this topic may well be called for.) The sections on inequalities and the distance formula are offered as optional topics. Some students may show enough interest in conic sections to study them in depth. Many good references on this topic are readily available, so none will be listed here.

Note the cumulative test at the end of this chapter. If you find it helpful to give semester-type examinations, both as a learning incentive and also as a measure of the effectiveness of the presentations of previous topics, you may wish to use some parts of this test as a model for a semester test.

Page 220 Emphasize the relationship of the symbols $>$ and $<$, and numbers on the number line. It is common for students to quickly say that $-20 > -\frac{1}{2}$, or $0 < -3$. The idea that any number a to the right of b indicates that $a > b$ should be helpful. Also, $a < b$ may be used to remind students that "less than" will be represented as "left of" on the number line. The concept of absolute value can also be confusing unless handled with care. The absolute value of a number is its distance from zero on the number line, and can never be negative. The most common formal definition of absolute value is as follows:

$$|x| = x, \text{ if } x \geq 0; \ |x| = -x, \text{ if } x < 0$$

It may be well to give some students an opportunity to explain this definition in their own words. They should be able to illustrate the fact that $|x|$ can never be negative.

Page 221 Re-emphasize that dividing by zero is undefined. Therefore, rational numbers are not closed under division. Dividing any rational number by another produces a rational number, except when the divisor is zero.

Page 223 It should be pointed out in Exercise 1 that irrational numbers can be located on the number line in more than one way. For example, instead of finding $\sqrt{10}$ by using a right triangle of legs 1 and 3, we can use legs of $\sqrt{5}$ units each or even use $\sqrt{3}$ and $\sqrt{7}$ from previous determinations. NOTE: $\sqrt{3} + \sqrt{7} \neq \sqrt{10}$. Some students may wish to make a series of irrational lengths, using different approaches as suggested.

Page 225 As mentioned earlier, the rules of operation with signed numbers have been developed here intuitively. "$+f$" is a symbol used to represent a "positive factor" and should not be confused with an algebraic variable. The students should have little difficulty in accepting the plausible and consistent conclusions. A reasonable amount of drill should insure that they obtain correct signed answers in these operations. Shortcuts can be developed in class for problems involving multiplication and division of examples containing several factors. For example, if the product to be obtained contains an even number of negative factors, the final product will always be positive; and if the product to be obtained contains an odd number of negative factors, the final product will always be negative. Extra practice in class may lead students to these conclusions, while providing multiplication practice in a different setting.

Page 232 Emphasize the difference in meaning between such phrases as $2N$ and $(N + 2)$, and $\frac{N}{2}$ and $(N - 2)$ to help the student realize the need for careful reading. Solving first degree equations in one variable can be made relatively routine by discussing and developing the best procedures for isolating the variable. You may wish to re-emphasize that although the variable seems to end up on the left in most of the examples, it is just as proper, and many times easier, to isolate it on the right side. Having learned a given procedure, the students may profit from considering other valid approaches. Discussing alternate procedures and lines of reasoning, and encouraging their use not only lends variety to classroom activities but also leads to more insight into procedures. It also develops an attitude that mathematics is more than just working by rote. Having mastered the reasoning and operations illustrated in the examples, the class may find it interesting to discuss and experiment with solutions using

equivalent equations. (An equation is equivalent to a given equation if any term on one side of the equation is replaced with its additive inverse on the other side.)

Given Equation	*Equivalent Equation*
1. $x + 5 = 19$	$x = 19 - 5$
2. $3x = 2x + 9$	$3x - 2x = 9$
3. $6x + 5 = 20 + 3x$	$6x - 3x = 20 - 5$
4. $4x - 7 = 27 - x$	$4x + x = 28 + 7$

Students should become familiar with this operation and should understand how to obtain equivalent equations. Actually, any term of an equation can be "moved" from one side of the equation to the other, providing its sign is reversed. This information can suggest to students a way of by-passing the steps shown in the examples: They can isolate the terms containing variables on one side, and the constants on the other side, merely by shifting terms. If this approach is used, the student should be asked occasionally to justify the reasoning behind the "moving of terms."

Equations with fractions are of sufficient difficulty to most students to warrant special attention. You can work with the class to develop the concept.

"If each term of a given equation is multiplied or divided by the same non-zero number, the resulting equation is equivalent to the given equation."

Given Equation	*Equivalent Equation*	
$\frac{2x}{3} - 1\frac{1}{3} = 5\frac{1}{3}$	$2x - 4 = 16$	[Multiply by 3]
$2x - 4 = 16$	$2x = 16 + 4$	[Move (-4) from left to right]
$2x = 20$	$x = 10$	[Divide by 2]

Once these operations have been mastered and can be explained, the students should be allowed to use them if they prefer this approach to the one given in the Examples. This suggestion reflects a philosophy of offering alternatives and building a background from which the student can make a decision suited to his understanding and needs.

Page 238 Students invariably find so-called word problems much more difficult to solve than ready-made equations. Careful reading for comprehension is very important. Note that we suggest that the letter selected as the variable be assigned to the smallest unknown where feasible. This eliminates fractions in most cases and thus simplifies forming the correct equation. While other approaches are valid, this one may lead to the simplest solution and success on the part of the class.

Page 240 Some time and care should be spent in class discussion to illustrate the fact that the graph of a linear equation in two variables, which is a line of infinite length, contains only points whose coordinates are ordered pairs which satisfy the equation. This can be done successfully by examples. It is not necessary to use integers for ordered pairs and students should recognize that fractions, decimals, and other forms could be used but are not practical for graphing on the usual grid paper. Graphing is generally quite well received by the class. The fact that there is exactly one pair of values which will satisfy two *independent equations* (capable of solution) in the same two variables should be carefully discussed and illustrated. The types of linear systems are described in most algebra texts. In how many places can two lines intersect? *ONE.*

Most of the class will intuitively realize that the intersection of the two lines will indicate the solution set. Although it is true that many of the problems in two variables can be solved using only one variable (this is developed in Chapter 10), it should be pointed out that setting up a pair of equations in two variables is often simpler than using only one equation. The practice of using graphic technique for interpreting is the purpose of this section. Ordered pairs and graphing can certainly be compared to longitude and latitude on a map, and the concept of a one-to-one relationship between a point and its coordinates can be reinforced at this time.

Page 243 This section can be used to emphasize the fact that many situations have more than one answer. Here we merely separate the right from the wrong. The treatment in the text is brief and a thorough discussion in class is recommended, particularly in choosing which side of the line to shade. Note also that this unit is optional. You may wish to refer members of the class to a modern algebra text for further treatment of these concepts.

Page 244 The distance formula for two points on a graph is presented here as an optional topic. However, it presents a good opportunity to review the Pythagorean Theorem and irrational numbers. The analytic geometry form

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

is rather difficult. Using letter subscripts makes the procedure somewhat mechanical but more understandable. Also, in using this formula, the student will get an opportunity to use the fact that $-(-x) = x$. The graph will illustrate this

very clearly. The study of conic sections, arising from applications of the distance formula, is a fascinating one. Special reports by the class on the ellipse, hyperbola, and parabola can be very rewarding. Ballistics, optics, astronomy, space flights, and many other timely and practical topics are tied-in closely with conic sections. Your library should have ample references on all these topics.

Page 253 This is the second cumulative test, and will probably be given near mid-year. At this point it should be helpful to see what areas already covered need to be strengthened.

Chapter seven

The study of space figures is intended to familiarize the student with common figures seen in his every-day experience and to help him understand their use. He will learn formulas for computing volume and surface areas of these various figures and will become familiar with the units in which these quantities are measured. The chapter seeks to develop the student's skill in using and understanding the common measurement units.

Considerable attention is given to volume and capacity, with distinction made between the two concepts. Since the vocabulary may be new, you may wish to place special emphasis on the Words to Watch For.

Frequent references to special projects, reports, and investigations are included to emphasize space figures in the environment. These are enrichment activities and may be used during the course of the chapter. The very nature of the work with space figures and the computations required for finding capacities, areas, and volumes involve extensive work with multiplication and formulas. Therefore, this chapter will provide a considerable amount of practice in computation and with the algebra involved in formulas. Students' ability to work with multiplication should improve during the study of this chapter.

Page 256 The definition of *space figure* which states that: a space figure is determined by a set of points that forms a surface which completely encloses a portion of space, should be compared with the definition of *plane figure*. Models of plane figures and space figures can be exhibited, so that the pupil not only completely understands what is meant by a space figure, but also further develops his understanding of plane figure. A plane figure, of course, does not enclose any space, but is flat (two-dimensional). The faces of the most familiar space figure, the cube, are square surfaces.

It would be useful to have available a model of a cubic foot. Since a cubic foot has a spacious interior, it can be used as a storage box for models of cylinders, cones, and so forth.

Point out the number of cubic inches necessary to form one layer of a cubic foot (144 cu. in.). The *volume* of a space figure is constant for that figure and is measured in cubic units: cubic feet, or cubic inches. On the other hand, capacity is the amount of substance that can be placed or stored in a particular volume. We speak of the capacity of a car, a truck, or a box. The capacity is sometimes measured in cubic inches or cubic feet, but more often in gallons, bushels, or barrels.

The first set of problems is used to recall some of the familiar measurements used in daily life. Considerable time may be spent in connecting the concepts of space, volume, and capacity with measuring devices that the students are familiar with, such as teaspoons, cups, quarts, milk cartons, and so forth. At the same time, use opportunities to move from concepts that are known to those that are unknown. Consider, for example, Exercise 6, which asks, "What unit of measurement is used to measure water?" A common unit of measure is a gallon, and this is the best answer to that question. However, in climates where irrigation is essential to the growth of crops, a common measurement of water is the *acre-foot.* Thus you can develop a definition of acre-foot at this time.

Place a chart of conversion units at the front of the room for ready reference in class. The information for the chart may be taken from the tables in the text. These conversion units are used extensively in this chapter.

Page 258 The Exploration and Discovery section on Properties of Prisms is another set of exercises devoted to organizing the students' knowledge of space and space figures, by using new vocabulary and new shapes. The idea of sets is reintroduced, and their application to space figures is developed. The vocabulary of terms identifying the parts of a prism (lateral face, vertex, etc.) should be emphasized.

Page 260 The purpose of this section is to provide the background necessary for finding the volume of a prism. It is necessary, therefore, to reexamine the formulas and methods for finding areas of common plane figures. The base of a prism is a common plane figure.

In Exercise 4 it may be useful for the teacher to remind the students that a rectangle that is 9 units long and 7 units wide would have 7 rows of square units, each row containing 9 units. Therefore, the measure of the area may be obtained simply by multiplying 9 times 7, and attaching the appropriate symbol of measurement to the 63.

In Exercise 14 the relationship between the area of a trapezoid and the area of a triangle is asked for. The area of a trapezoid is expressed as $A = \frac{1}{2}(b_1 + b_2) \times h$. Consider the trapezoid whose top base becomes smaller and smaller until it is simply a point. At this point the trapezoid has become a triangle, and the top base has disappeared. Therefore, b_2 has become 0, and the formula becomes $\frac{1}{2}(b_1) \times h$ or $\frac{1}{2}bh$, which is the area of the triangle.

Use Exercises 17, 18, 19, and 20 to ask such questions as "What happens to the area of the figure when the base becomes larger?" "What happens when the base is doubled?" and "What happens when the base and altitude are both increased?" It is useful in the understanding of these areas to examine the answers to the various exercises, comparing the rectangle not only with rectangles of different sizes, but also with a parallelogram of the same size, or with a trapezoid of similar dimensions. The use of grid paper may clarify these concepts for students who are experiencing difficulty. See the illustrations on page 260, etc.

In the problems, we speak of the area of a rectangular "region" and the "region" enclosed by a square. Remember that the rectangle and the square are polygons, and a polygon is made of line segments. Therefore, we cannot find the area of a square or a rectangle, or more generally, of any polygon. These geometric figures are just sets of segments. What we can find is the area of the region *enclosed* by the polygon. To simplify our discussion, however, we will talk about the area of the square or rectangle, or whatever geometric figure we are considering at the time. But keep in mind that we mean the region enclosed by the geometric figure when we talk about area.

Page 262 The Exploration and Discovery Unit entitled Finding Areas Using Trigonometric Ratios may be omitted for classes that have difficulty in working with abstractions. This section provides a review and shows the application of trigonometry to area study, a topic usually covered in plane geometry. For some classes, time can be better spent on other topics.

Page 266 Exhibit the cubic foot and cubic inch. The cubic yard can be constructed from a large cardboard box. Commercial models are also available. See the NCTM 21st *Yearbook* for further ideas.

Understanding cubic units is difficult for many students. Frequent references to known space figures which resemble a cubic foot or cubic yard, such as fish tanks, aquaria, refrigerators, large packing boxes, and so forth, help the students

to become conscious of size. In Exercise 12 the relationship between cubic inches, cubic feet, and cubic yards is developed. Models of these figures would be helpful here.

Reference to cubic feet and cubic yards in construction (excavating for a basement or pouring concrete) helps to clarify these concepts. If possible, take a field trip or send students to observe some construction project.

Page 269 This inventory of fractions is useful at this point, since many formulas, computations, and measurements involve fractions.

Page 270 Specific gravity can be a very interesting topic. The relationship between the weights of various substances having the same volume has an inherent interest for many students. While the ordinary young man can easily lift a cubic foot of water in the form of ice, a cubic foot of gold would be unmanageable. Gold is extremely heavy, though it is not the heaviest substance known. The science-minded students in the class may know the reason for the difference in the specific gravity of ice and water, and why a lake doesn't freeze solid in winter.

Page 271 Note that the cylinder to be discussed here is a right circular cylinder which means that it will be perpendicular to the surface on which it stands. An example of a cylinder that is not a right circular cylinder might be something like the Tower of Pisa.

Various uses of the cylinder in industry (for canning food) and the home (for water heaters) are worthy of mention. The cylinder is also the basic figure in rockets, missiles, and space ships.

Exercise 5 asks for cylindrical shapes observed in nature. Some examples are stalks of bamboo or corn, tree branches, tree trunks, and ears of corn. In Exercise 9 the best explanation of why a cylinder is not a prism is that the bases are not polygons.

Page 272 The Exercises in the section "Volume of a Cylinder" use the symbol in a statement, $\pi \approx 3.14$. The symbol $\approx$ means *is approximately equal,* rather than *is exactly equal.*

Take this opportunity to extend the students' understanding of π and its approximations. Recall that π is an irrational number. We use a variety of approximations, the most common of which are 3.14 and $3\frac{1}{7}$.

Page 274 The pyramid and cone are common figures in our environment. However, students may not be familiar with the calculation of volume of these figures. Therefore, it will be helpful if the teacher has a cone, a cylinder, and pyramid.

Each should have the same size base and altitude. Show that the pyramid or the cone filled three times with sand will fill the cylinder.

Page 277 The sphere is another commonly used space figure. Nearly all team sports utilize a spherical ball. The discovery of the ball bearing was an important milestone in industrial progress, and the use of the sphere for the bearing is worth exploring for those students who are mechanically minded.

In Exercise 5 the question is asked about the relationship between two spheres, one having a radius twice that of the other. The answer is, of course, that the larger sphere has eight times the volume of the smaller. In this regard it might be useful to compute the volume of an orange. If a thick-skinned orange is used, measure as accurately as possible the thickness of the skin and compute what part of the volume of the orange is peel. For example, the volume of an orange with a radius of three inches is $113\frac{1}{3}$ cubic inches. Now if the peel of that orange is $\frac{5}{8}$ of an inch thick, the fruit's radius is $2\frac{3}{8}$ inches, the volume of the fruit of the orange without the peel is 56.1 cubic inches. Therefore, the peel is approximately $\frac{1}{2}$ the volume of the orange even though the peel is only $\frac{5}{8}$ inches thick and the orange, itself has a radius of $2\frac{3}{8}$ inches.

Page 278 This section is designed to promote a greater understanding of the common units of measurement, showing their similarities and differences.

Using the problems in the text as models, think of similar problems applicable to your particular community. Exercise 17 may be done in many ways.

Page 280 For many purposes the lateral area or the area of the outside of the figure is the important factor. For example, to the manufacturer of cardboard cartons or plastic containers, the volume is not as important as the amount of material needed to construct the actual figure. This is also true of painting or papering rooms, and coating the interior of tanks with non-corrosive or galvanized material.

Page 283 The Special Project answers are:

Can	*Volume*	*Total Surface*
A	1155	694 sq. in.
B	1155	782 sq. in.
C	1156	678 sq. in.

2. a. $1\frac{1}{2}$ **2. b.** $2\frac{1}{4}$ **2. c.** 1.2

Exercise **2. d.** can be used to point out that to increase the capacity of a cylinder, it is far more economical to increase its diameter than to increase its height. Your classes may wish to investigate this matter in greater detail. The prices of canned goods can be very misleading when cans of the same product are of the same height but different diameters. The "per ounce" price may vary a great deal. Naturally, quality, packaging, and other factors also enter into the determining of prices. As a project, ask students to investigate fruit juice cans used by various competing companies. The activity could be extended to include a wide variety of canned goods if the class shows a great deal of interest in this idea.

Page 286 As mentioned earlier, this chapter has included an extensive number of problems involving multiplication. These are sometimes complex. The students at this time should be completely ready to examine ways in which the multiplication process can be simplified. Use of the associative property is one of these methods.

Page 289 This unit provides practice in Problem-Solving Step 3, *identifying and solving the hidden question.* Go through these problems in class, using class discussion to identify the hidden question in each exercise. For example, in Exercise 1 the hidden question is: What is the volume of the excavation?

Another way to handle this section is to ask the students, for their first assignment, to go through the exercises, identify the hidden question, write it out in words, and solve the hidden question only. On another day, the students can complete the solution to the problem.

Page 290 The use of recreations to develop an interest in mathematics must not be jeopardized by forcing students who are not interested in a particular recreation to spend much time working on it. While puzzle problems have important uses, rarely can recreations be justified on any practical basis; therefore, a student who is not interested in a form of recreation may find it interfering with his feeling towards mathematics rather than developing a positive feeling for it. The mathematical puzzles, requiring specific measurements with odd-size containers, are basically a problem of division and identifying the proper remainder.

Page 291 The study of the problems in this section lends itself readily to class discussion. Having gone over the exercises and identified some clues, you may have some interesting class sessions in leading the class to identify the clues and to sense the relationships between the computations.

This chapter is concerned with mathematical concepts and their relation to the construction industry. This relationship can be useful as the student applies mathematical concepts to his personal environment. In most cases, the purchase or building of a home represents the largest investment that any individual or family ever makes. Acquaintance with typical problems and the possible decisions are of great importance as intelligent alternatives are considered.

The construction industry is a vital part of our economy. When activity in this industry increases or decreases, its effects are far reaching. The change affects the production of raw materials used in building, such as lumber, steel, cement, and other items that go into the completed house or office building. Employment and prosperity in various professions and crafts engaged in the industry, such as plumbers, bricklayers, are directly affected while banks, insurance companies, real estate agencies, public utilities, and many others, whose activities and prosperity depend on the well being of the construction industry are indirectly affected.

The student should be led to understand these facts and their impact on him. However, he will probably be more interested in discussing the career opportunities of the industry. These range from the professional groups and indoor "white-collar" sub-professional, technical, and clerical positions to the skilled crafts of woodworking, plumbing, electrical, and masonry. A major attraction to the building industry may be financial, for the average hourly earnings of a skilled construction worker exceeds $4 per hour, whereas the average hourly wage of a factory worker is about $2.75. In addition the opportunity for starting one's own business should be considered. Most communities have their share of men who started in the trade as common workers and are now proprietors of a prosperous business.

For these reasons, a look at the construction industry is worthwhile. Some students may be interested in preparing first-hand reports for the class on the materials and the kinds of occupations involved in current construction projects in your area. Speakers from trade unions, apprentice training programs, real estate groups, and workers might be invited to speak to the class. The break in the routine, the first-hand experience of meeting community personalities, and the recognition of the many applications of mathematics will be both interesting and beneficial.

You may wish to assign another project related to this chapter. In an average year, over 1.2 million new houses are built. Major pieces of equipment are needed in all new houses. Ask the class to make a list of the products required. Some items are optional; what are they? What new developments have occurred in building and equipping new homes in the last 25 years? What opportunities exist for your class as sales personnel in all the aspects of construction, equipment, and home appliances? Consulting a catalog can provide prices etc. for other projects.

Page 296 No new techniques are presented here. It may be necessary to review areas, volumes, and scale drawing procedures. Caution the students on the necessity of reading the problem carefully and understanding thoroughly what is to be done. Many errors result from incomplete understanding. Point out to the class that nearly all basic arithmetic from addition to percentage and ratio is constantly used in this chapter.

Page 300 This Experts' Corner probably can be handled successfully by most of the class. Review and analysis of formulas for the cylinder and the prism will be worthwhile before undertaking this section.

Page 302 Calculation of board feet of lumber may be a new experience for most of the class. The process can be developed and summarized into the following rule:

> Find the product of the thickness of the board in inches, the width of the board in feet, and the length of the board in feet. The result is the number of board feet.

By discussing many examples in class, most students will probably discover this rule for themselves. If possible, ask the industrial arts instructor to share his opinion about the importance of knowing the information in this section.

Page 304 Stress the importance of a thorough understanding of the problem. What are the consequences in competitive bidding if a bid is too high because of an error in arithmetic, or because the estimator lacks an understanding of the job? If the bid is too low for the same reasons?

Page 306 Various units of volume are stressed here. It is important that the student be able to convert from one unit to another. Stress that volume formulas which usually include several different dimensions must be computed with all dimensions expressed in the same unit. You may wish to consider metric units of measure, too. Tables at the back of the book are provided. Review the effect on volumes if the dimensions are doubled, tripled, quadrupled, etc. For example, if you double the dimensions of a prism, the new volume is 8 times the old. If you triple the dimensions, the new volume is 27 times the old.

Page 310 This section summarizes the problems involved in planning a home. Point out the many different alternatives present for the owner: choice of plans, materials, costs, etc. Discuss with the class the need to consult professional help, such as an architect and building contractor. The exercises may raise

questions about procedure. Take for example Exercise 2, would tile be placed under built-in units? When would tile of different shapes vs. linoleum be considered for use in kitchens or playrooms? What quality factors should be considered? Local practices may vary, so expect different answers to questions such as these. Discussion will be most worthwhile.

Page 313 Stress accuracy throughout this chapter. Many errors occur because of incomplete understanding of the problem, mechanical errors in locating the decimal point, or performing the operation incorrectly. Most gross errors can be eliminated if the student has an approximate answer in mind. Errors in decimal location can certainly be detected in this way. Estimating answers should be emphasized continually and practice provided so that the students may acquire the habit in their own work.

Page 315 This section introduces common formulas which were not previously discussed in detail in the book. These formulas can be "derived" by intuition and discovery during class discussion. Again, having a reasonable answer in mind will help the student to apply these formulas properly.

Page 317 The typical mathematical puzzle provides a recreational activity which lends variety to the classwork and arouses interest in students who are bored with "just plain arithmetic." It is important to realize that mathematical puzzles offer opportunity to practice the procedures of exploration, testing of inferences, trial and error, and verification of conclusions that constitute the methodology of mathematical problem solving.

Mathematical problem solving, as distinct from solving applied problems, is directed toward developing new insights and discovery of new generalizations. While the student lacks the maturity required to explore the frontiers of science, he can still use the procedures in these simple situations. While the puzzle situation is not too difficult, the procedures are important. Helping the student to develop his independent abilities to explore and to discover and to acquire an experimental attitude determines the methods used most effectively in the classroom. The class routine should be sufficiently flexible to allow for individuals and to encourage individual initiative. It is important to emphasize that an unsuccessful attempt is not a failure, because it has eliminated one technique or procedure and consequently another can be explored.

The student should be encouraged to develop his intuitive procedures, and at the same time be made aware that there is a need for a systematic approach. While this approach is superior to a purely random attack, the first approach comes as a result of experience. Consequently, trial and error

is to be encouraged, and resulting dead ends are to be considered as part of the learning experience. Several generalizations will emerge with experience in puzzle problems. Some of them are:

1. Few problems are as difficult as they appear.
2. Clues will emerge as the data are studied and analyzed.
3. Organizing the data and results with tables, or some other device, to find a pattern is often quite useful.
4. Everyone has to explore at the outset.
5. Don't just sit there hoping for the solution to appear in a great flash of light. Try something!

All three authors of this text have made extensive use of puzzle problems confirming the ideas emphasized in the preceding paragraphs. Creating interest and enthusiasm for mathematics can be increased by using the many collections of good puzzles that are now available in most libraries and in paperback editions; it is our intention to encourage their generous use.

Chapter nine

This chapter is concerned with community services and activities and how they are paid for. As used here the term community encompasses three levels of government: local, state, and federal. Emphasis is placed upon the applications of mathematics in the financing of community activities.

Sudden increases in the cost of community programs create many pressing problems. Newspapers, periodicals, such as Time, Newsweek, and U.S. News and World Report and government publications contain up-to-date information relating to the topics of this chapter. Students should be encouraged to keep abreast of these matters and class time should be devoted to the study of the facts and opinions expressed in such sources. An up-to-date bulletin board featuring student contributions will help to keep interest at a high level.

Page 322 In a few years the students will be more concerned than they are now about the efforts of informed citizens to achieve an intelligent tax structure. Familiarity with the present structure, its problems, and existing areas of inequity will be a good base for that future interest. In handling the topic, the following matters should be emphasized:

1. *Community services are essential.* We depend on community services such as schools, roads, and public transportation. Many of these services can be provided more economically by our government than by individuals. Other services, such as public utilities, are sometimes provided by private companies. The importance and cost of these services have received more attention as the perplexing problems of our times are demanding solution.

2. *Community activities are carried on at three levels: local, state, and federal.* In general, a given activity is administered and supported at the level where its significance is felt most directly, and where it can be handled most efficiently. Thus, sanitation is a local responsibility, highway traffic control is a state responsibility, and national defense a federal responsibility. The situation is apt to change rapidly with respect to any one activity. Education was originally a local responsibility. Today local, state, and federal government are involved in varying degrees in the educational enterprise.
3. *Government activities are paid for primarily by taxes.* The cost of most services would be prohibitive for individual citizens. For that reason, taxes should not be considered as an evil, but as payment for services. A well known adage is, "What you receive for your tax dollar is the best bargain you get."
4. *The tax structure faces continual adjustment to current economic conditions.* It is obvious that an increase in expenditures imposes an increased tax burden of some form.
5. *Everyone, even students, pays taxes.* The various forms of taxation and the costs involved in collecting them should be everyone's concern. Some taxes are direct, that is, they are paid directly to the government by the taxpayer. Others, like taxes on gasoline or telephone calls, are paid to an intermediary who passes them along to the government. The indirect tax is not likely to be felt as much as the direct tax, since it is paid only a little at a time. Thus, the taxpayer needs to be reminded about the true amount of money he pays out in indirect taxes. Your students will be interested in this point and in discussing the questions "What indirect taxes have you paid lately?" and "How much do you personally pay in indirect taxes per year?" Students may enjoy discussing these matters at home.

Page 324 Several facts can be brought out in discussion. Identify the community services and discuss how we use them; which level of government provides the services and why, and which are provided by private agencies and which by public agencies.

The budgets shown here are for a certain city and a certain state and for the federal government, in a recent year. Comparing that data with current data will add interest to the topic.

Page 330 The procedures used here for breaking a cipher are typical of those used in mathematical problem solving; they provide a starting place from which ingenuity, imagination, and trial-and-error can take over. The distinction presented here between codes and ciphers is recognized in technical work, but not ordinarily in popular usage, or even in the dictionary. The Morse Code on the one hand, we should call a cipher since each element stands for a letter and is transmitted as related dots and dashes. An algebraic expression on the other hand is a code since each element stands for a word or phrase, as you will see if you translate $\frac{3x}{4} - 12$ into English.

You may be interested in introducing this topic of ciphers by using a *transposition* cipher. In this type the letters to be transmitted are arranged in a specified geometric pattern, which the receiver has to rearrange in order to determine the message. For example: TSCEA HDHCN EIEID BAMMM ELEFE STSON . . . may be deciphered by placing the words in column form and reading down the first column, up the second, down the third, and so on, and the message would read *The best laid schemes of mice and men* . . . By showing the class how to decipher such codes and then selecting a quotation to encipher they can see the techniques involved. This would prepare the class to consider the simplest type, the *substitution* cipher. This is discussed in the text.

```
TSCEA
HDHCN
EIEID
BAMMM
ELEFE
STSON
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The weakest points of a substitution cipher are letter frequencies and word grouping. These are the points to attack in the deciphering. The first step is always a count of letter frequencies. Because the frequencies are abnormal, this type of cipher is easily identified. We try the most frequently used letters, in turn, as representing *e*. In the text example, the first guess was correct, but in practice, it may take several trials to determine the letter.

The next step is to attack the word grouping. A single letter is usually *a*, though it could be *i* or *o* as illustrated in Exercise 5. We can assume, however, that unless we find it as a double, or an ending of a word, that a single letter is *a*.

The most common two-letter words are, in order of frequency: *of*, *to*, *in*, *it*, *is*, *be*, and *at*. Note that *t*, *n*, and *o* can be either the first or second letters of a two-letter word. Ordinarily the two-letter words will identify *t* or *o* or both.

The most common three-letter words, in order of frequency, are: *the*, *and*, *for*, *are*, *but*, and *not*.

Having proceeded this far, the next step is to examine the words that have been partly deciphered and to fill in the missing letters by trial-and-error. With practice the student can become very skillful in this.

Many topics are of special interest in this area: for instance, topics worth reporting on are: the military Playfair cipher; the double substitution ciphers with the Vigenere or Beaufort table; the Saint Cyr ruler; and the Baconian cipher. An encyclopedia is a useful reference for these topics.

Many interesting stories, both true and fictional, are built around ciphers. Among the fiction classics are Conan Doyle's ADVENTURE OF THE DANCING MEN, and Edgar Allan Poe's GOLD BUG. Among the true stories are:

Fletcher Pratt, SECRET AND URGENT, New York: Bobbs Merrill, 1939.
Helen Fouche Gaines, ELEMENTARY CRYPTOGRAPHY, Boston: American Photographic Publishing Company, 1939.
Lawrence Smith, CRYPTOGRAPHY, New York: W. W. Norton, 1943.

NOTE: The work on this section should be kept by the student as further investigation is provided in the next chapter.

Page 334 The purpose here is two-fold: To have the student realize (1) that taxes are an immediate, personal problem for him and (2) that he faces a great variety of taxes. These facts should be brought out in the discussion.

Page 335 A current federal budget should be brought in and analyzed in the same manner as the one in the text. Comparisons should be made between the budgets: What changes are evident in the budgets? Why were they made?

Page 336 You can obtain current tax forms—both 1040 and 1040A—from the post office. The apparent complication of the form and the operation of filling it out will disappear if it is seen and analyzed, as an attempt to adjust the amount of the tax to ability to pay. A study of the form from this point of view will make an interesting class discussion. A complete unit on Income Tax for use in schools is available through the Bureau of Internal Revenue.

Page 339 Withholding tax provides for collection of income tax at the source of income. Some of the students in your class have probably had experience with withholding tax during their summer employment. If so, their experiences will add to the reality of the discussion.

Page 341 Discussing different kinds of taxes affords an opportunity for setting up criteria for judging the merits of various taxes and for deciding which is a good tax and which is not. Some criteria worth considering are these: The tax burden should be adjusted to ability to pay. A tax should be economical to collect. The taxpayer should be aware of how much taxes he is paying. Taxes should be solely for the purpose of raising money, not for controlling behavior.

Some examples of different kinds of familiar taxes are indicated in the table at the top of the next page.

Tax	*Federal*	*State*	*Local*
Property	No	All states	Yes
Income	Yes	36 states	200 local gov'ts
Sales	Yes (Excise)	42 states	2000 local gov'ts
Inheritance	Yes	49 states	No
Gift taxes	Yes	12 states	No
Liquor	Yes	50 states	No
Cigarettes	Yes	49 states	No
Gasoline	Yes	50 states	No
New cars	Yes	36 states	No
Tires, tubes	Yes	(Sales taxes)	(Sales taxes)
Auto tags	No	49 states	Counties, in Hawaii
Drivers' license	No	49 states	No
Air travel	Yes	No	No
Telephone	Yes	Yes	No
Mortgage recording	No	No	Most counties

Which taxes meet most of these requirements? Which meet the least? Can you think of other criteria? It is recognized that each person brings an attitude toward taxes to the situation. Discussion may clarify one's stand on the matter.

Page 342 States vary widely in the kinds of programs they support and in the methods of taxation. Analyzing a current budget following the pattern of the text will provide an interesting comparison for discussion.

The interstate highway construction program, initiated and largely paid for by the federal government, is being completed. A report on the cost, how it is being met, and how it relates to the local area, will be of interest here. Why is it a responsibility of the federal government?

Page 344 It is important for everyone to be familiar with the operation of the property tax because of growing criticism of its impact, and also because of increasing rates. An important question for discussion is this: Is ownership of real estate a measure of ability to pay taxes?

Consider some special cases: (a) Harry Evans owns a nursery, taxed as farmland. As the city grows, he finds that his property is being taxed at city rates. He has to sell and go out of business. (b) Arnold Newman is planning for retirement. By that time, he expects the taxes on his home to be so high that he will not be able to pay them on his limited retirement income. So he plans to sell his home, buy a less desirable one, on which the taxes are less, and invest the difference in corporation stocks, on which only the dividends are taxable.

Page 349 The Chapter Test, Part Three, provides another opportunity for a review of the Problem-Solving Steps, and a self-diagnosis by the student of his ability to use them. After classroom practice on each of the steps, the section may be assigned as homework. The papers may be exchanged and scored in class. In turning in his corrected problems, the student should list the steps he failed to perform successfully.

Chapter ten

This chapter is designed both to maintain the algebraic skills developed earlier in this text and to extend the students' knowledge of algebra. The authors have examined various civil-service and similar occupational examinations and picked out topics that are frequently covered in these examinations. While some of the topics are somewhat sophisticated for beginners and may be rather difficult, a knowledge of how to analyze them is helpful. This is particularly true of the work and mixture problems. Quadratic equations and conic sections can also be rather complicated if not introduced with care. Therefore, we suggest that these topics be considered optional, depending upon the ability and interest level of the class. However, we do strongly encourage class participation in them if at all practical, particularly in view of the fact that they are included on many job examinations.

Preparing for employment examinations

The Practice Examinations in this section consist of problems similar to those most frequently found in the civil-service examinations that would be required of applicants for federal, state, or local positions. One difference in format should be noted. Civil-service examinations, and other employment tests, commonly use the multiple-choice type of question, in which the student selects his answer from a number of alternatives. Machine scoring can then be used. In testing computational skills, however, use of multiple-choice questions is open to criticism because the students can often find the correct answer by a process of checking, rather than by using the procedure the question is intended to test. In response to this criticism, we have eliminated multiple-choice questions in these Practice Examinations and have utilized types of questions that require the student to work out the answers.

Students who are particularly interested in preparing for civil-service examinations may wish to undertake further study, individually or in groups, along the lines indicated in the Practice Examinations. Students should study these topics carefully and thoroughly.

The special topics commonly dealt with in the problems in the civil-service examinations fall under these headings:

Geometric series	Taxation	Mixtures
Work problems	Series	Retail transactions
Distance-rate-time		

Other topics include *Practice in the Operations.* They are:

Fractions	Decimals	Ratio and proportion
Equations	Interest	Squares and square roots
Algebraic expressions (including formulas)		

Of the above topics only two are not covered in this text. They are:

1. *Retail Transactions.* This topic is not included in this text but is thoroughly covered in *General Mathematics, Book 1,* Chapter 10. There the student can become thoroughly familiar with problems dealing with margins, profit and loss, overhead, commissions, and discounts.
2. *Geometric Series.* Problems involving series are occasionally found in civil service examinations. The geometric series is one in which the ratio of each term to the one which precedes it remains constant. An example is 1, 2, 4, 8, ... In this series, the ratio is 2. Although the geometric series is not included in the present text, the students can readily find information about it in a college algebra text.

Special attention should be given to the topics *Interest* and *Arithmetic Series,* not treated until Chapter 11, the next chapter in this text.

If the need for more practice on the above topics is revealed by the Practice Examinations, the Applied Problems and Inventory Tests throughout the text and the Review Practice section at the end of the text provide the necessary additional practice.

These practice examinations are intended to provide practice in taking an examination in problem solving, to reveal the importance of systematic procedures in attacking a problem, and to disclose students' weak spots.

Following is advice to the students, which they should consider carefully.

A student taking an important examination is likely to be handicapped by two difficulties aside from those presented by the problems in the examination: (a) becoming jittery; and (b) wasting time. Both can be avoided if the student has mastered a systematic procedure for problem solving. Such a procedure is outlined, with ample material for practice, on page 40 in this text and in more detail in Book 1 on pages 54 and following. Students can use several special devices in applying the steps to certain types of problems.

If the problem relates to geometric figures, make a sketch. This method is used to solve problems about perimeters, areas, volume, similar triangles, and so on.

If problems that relate to mixtures, distance-rate-time, interest, and so on are confusing, make a tabulation of the data.

If the problem relates to the arithmetic or geometric series, the student should be able to:

1. Identify the type of series.
2. Find a given term in the series.
3. Find the sum of the series.
4. Find the average value of a term in the series.

Always look to see what kind of conditional statement is to be set up. If it is to be a percentage statement or other proportion, what are the ratios? Be sure to set them up in the same order. If it is to be a general equation, proceed carefully in expressing the unknown quantity or quantities in terms of one variable, as explained with practice material on page 238 of this text.

You can administer these examinations by writing the questions on the board or by duplicating them. The outline on this page should be added to the bottom of each examination and should be filled in by the students after you have scored the examinations. You can adjust the intervals for administering these examinations to provide an opportunity for effective individual remedial study between times.

I had errors in problems number:	I should study these pages in my textbook:	I should practice problems on pages:
________	____________	____________
________	____________	____________
________	____________	____________

EXAMINATION I

Name..Date.............Score........

Answers

______ 1. Mr. Jacobson purchased four \$1000 bonds of the Standard Oil Company, which pay interest at the rate of $5\frac{3}{4}\%$ annually, payable each six months. How much will he receive for each payment?

______ 2. The assessed valuation of property in Oakdale is \$75 million. The budget for next year is \$3,750,000. How much will the property tax be on Mr. Jenkins' hardware store, which is assessed at \$35,000?

______ 3. On a road map a segment of 2.5 inches represents 35 miles. How many miles does a segment 6 inches long represent?

______ 4. In triangle ABC, $m\angle C = 90°$, $AC = 10''$, $BC = 24''$. Find $m\angle A$ to the nearest degree. You may use the table in your text.

______ 5. The diameter of an automobile wheel measures 30 inches. How many revolutions (to the nearest revolution) will it make in traveling a mile? Use $\pi \approx 3.14$.

______ 6. Mr. Swensen left an estate of \$96,000 to his four sons. His eldest was to receive $\frac{1}{3}$ of it, the second $\frac{1}{4}$, and the third $\frac{1}{5}$. The youngest was to receive the rest. How much did each receive?

______ 7. Express as a decimal the difference between $\frac{2}{5}$ and $\frac{3}{4}$.

______ 8. How many pounds of coffee worth 40¢ per lb. must be blended with coffee worth 50¢ per lb. to prepare a mixture of 50 pounds of coffee worth 44¢ per lb?

______ 9. A truck driver left Stillwater for Duluth at 8 AM driving at 40 miles an hour. At 10 AM Jim left Stillwater on the same road to Duluth driving at 60 miles an hour. At what time will he overtake the truck?

______10. Jane can type 20 pages an hour. Mabel can type 36 pages an hour. They are asked to type a manuscript 1120 pages long, working together. How long will it take them?

______11. The larger of two numbers is three times the smaller. Their sum is 168. What are the numbers?

______12. One side of a triangle is 4 inches longer than the second side. The third side is 6 inches longer than the first side. The perimeter of the triangle is 62 inches. What is the measure of each side?

______13. A solution of salt water is 15% salt, by weight. How much water must be added to 10 gallons (83 pounds) of the solution to dilute it to a 10% solution?

______14. The two legs of a right triangle measure 15 feet and 18 feet respectively. What is the measure of the hypotenuse, to one decimal place? (Do not use a table.)

15. Find the value for x in each equation:

______**a.** $\sqrt{x} + 9 = 14$ ______**c.** $\frac{x}{5} + 3 = 6$

______**b.** $18 \div x = 3$ ______**d.** $\frac{x}{3} + 6 = \frac{x}{4} + 9$

EXAMINATION II

______ 1. The annual income of the Hardy family is $7200. Of this amount, 0.11 is used to meet the expenses of the family automobile. How much is used in this way?

______ 2. After driving 189 miles, Jim found it took 12 gallons of gasoline to fill the tank. At that rate, how much gasoline would he use on a trip of 630 miles?

______ 3. A board $5\frac{5}{8}$ inches wide is to be trimmed to a width of $3\frac{3}{4}$ inches. How much is to be cut off?

______ 4. A real estate company sold a lot for $8500, and received a commission of $425. The commission was what per cent of the selling price?

_______ 5. The assessed valuation of property in Ellensberg is $32 million. The budget is $800,000. What is the tax on Mr. Henderson's Pet Shop, which is assessed at $25,000?

_______ 6. Jim can paint the garage in 3 days, while it takes Harry, his younger brother, 4 days. How long will it take them if they work together?

_______ 7. A tank can be filled if the intake valve is opened for 4 hours. The drain pipe will empty it in 6 hours. If by mistake both pipes are left open, how long will it take for the tank to be filled?

_______ 8. Mike left the house at 9 AM walking at the rate of $3\frac{1}{2}$ miles an hour. Two hours later his brother Jim followed him on his bicycle at the rate of 14 miles an hour. How long will it take Jim to overtake Mike?

_______ 9. A farmer wants to mix two kinds of feed to secure a 200-pound mixture worth 90¢ per lb. How many pounds worth 45¢ per lb., and how many pounds worth $1.20 per lb., should he mix?

_______10. How much pure anti-freeze must be added to 5 gallons of a 15% mixture to raise the concentration to 25%?

_______11. How much water should be boiled out of 3 gallons (25 pounds) of a 4% salt solution to raise the concentration to a 10% solution?

_______12. A square is circumscribed around a circle whose radius is 4 inches. What is the area of the square?

_______13. A contractor charges $2.50 a cubic yard for excavation. What should he charge for digging a trench 6 feet deep, 5 feet wide, and 36 feet long?

_______14. Bill paid $13, including sales tax, for a new driver for his set of golf clubs. The sales tax is 4%. How much sales tax did he pay?

_______15. At a sale where all TV sets were sold at a reduction of 15%, a color set sold for $276.25. What was the regular price of the set?

_______16. Mr. Jensen used 1800 rods of fence to enclose his south pasture. It is 3 times as long as it is wide. What is its area in square rods?

EXAMINATION III

_______ 1. In putting up book shelves in his room, each 3 feet 3 inches long, Jerry cut them from a 14-foot board. How many shelves did he get from the board?

_______ 2. Mike left a can of paint thinner standing open for two days. On the first day a fourth of the contents evaporated. On the second day half of the remainder evaporated. What fraction of the original contents remained?

______ 3. A rectangle is 12 feet wide and 20 feet long. If each of the dimensions is increased by 20%, by what per cent is the area increased?

______ 4. After he had been working in the Supermarket for 6 months Jim received a raise of 8% of his former salary. His raise amounted to $6 a week. What had he been receiving?

______ 5. Mr. Jones earns a salary of $600 a month. The average expenses of the family are: rent, 25%; food, 30%; clothing, 12%; automobile, 8%; household, 15%. Of the remainder, 50% is put into savings. How much is saved annually?

______ 6. A 4-gallon mixture of 16% antifreeze is diluted with 1 gallon of water. What per cent of concentration is the resulting mixture?

______ 7. The measure of one of the acute angles in a right triangle is 20° less than that of the other. What is the measure of each angle in the triangle?

______ 8. The length of the shortest side of a scalene triangle is $\frac{1}{4}$ that of the longest side. The third side is twice as long as the shortest side. The perimeter of the triangle is 42 inches. How long is each side?

______ 9. During the first 35 minutes of its flight the pilot of an airplane found it had traveled 385 miles. At that rate, how long will it take for the plane to travel 1980 miles?

______ 10. Two truck drivers started driving toward each other from cities 360 miles apart at 8 AM. They met at noon, and had lunch together. One had been traveling at an average rate of 40 miles an hour. At what rate had the other been traveling?

______ 11. The distance in feet that an object falls in t seconds is found (approximately) with the formula $s = 16t^2$. How many feet will an object fall in 10 seconds?

______ 12. Use the formula in problem 11 to find how long it will take an object to fall 256 feet.

______ 13. A rectangular garden is 50 feet by 20 feet. Mr. Adams offered to put a cement walk 30 inches wide around the outside of the garden at $2 a square yard. What would the walk cost at that rate?

14. What is the value of x in each equation?

______ a. $x^2 - x - 12 = 0$

______ b. $\frac{\sqrt{x}}{5} = 1$

______ c. $5\sqrt{x} = 1$

______ d. $\frac{8}{x} = \frac{32}{20}$

______ e. $x \div \frac{5}{9} = 45$

______ f. $\frac{7x}{3} + 5 = \frac{2x}{3} + 20$

________15. Jim can spade the garden in 6 hours. It takes Eric 4 hours. One day after Jim had been working on it for an hour, Eric helped him to finish it. How long did it take them to finish it?

________16. Joe used 44 yards of fence to enclose his circular garden. What is its area in square feet? Use $\pi \approx 3\frac{1}{7}$.

ANSWERS

EXAMINATION I **1.** \$115 **2.** \$1750 **3.** 84 mi. **4.** 67° **5.** 673 **6.** 32,000; 24,000; 19,200; 20,800 **7.** 0.35 **8.** 30 lb. **9.** 2 PM **10.** 20 hr. **11.** 42 and 126 **12.** 16″, 20″, 26″ **13.** 5 gal. or $41\frac{1}{2}$ lb. **14.** 23.4 ft. **15. a.** 25 **b.** 6 **c.** 15 **d.** 36

EXAMINATION II **1.** \$792 **2.** 40 gal. **3.** $1\frac{7}{8}$″ **4.** 5% **5.** \$625 **6.** $1\frac{5}{7}$ **7.** 12 hr. **8.** 40 min. **9.** 120 lb. at \$1.20; 80 lb. at \$.45 **10.** $\frac{2}{3}$ gal. **11.** 15 lb. or 1.8 gal. **12.** 64 sq. in. **13.** \$100 **14.** \$.50 **15.** \$325 **16.** 151,875 sq. rd.

EXAMINATION III **1.** 4 **2.** $\frac{3}{8}$ **3.** 44% **4.** \$75 per week **5.** \$360 **6.** 12.8% **7.** 35°, 55°, 90° **8.** 6″, 12″, 24″ **9.** 3 hr. **10.** 50 mph **11.** 1600 ft. **12.** 4 sec. **13.** 83.33 **14. a.** $\{-3, 4\}$ **b.** 25 **c.** $\frac{1}{25}$ **d.** 5 **e.** 25 **f.** 9 **15.** 2 hr. **16.** 562 sq. ft.

Page 353 This review of linear equations in one variable gives the student practice with parentheses and fractions. It would probably be advisable to discuss the techniques for removing parentheses by using the Distributive Property and by multiplying both sides of the equation by a suitable number in order to eliminate all fractions in a given equation. For example, $\frac{3x}{4} + 2 = x + \frac{5}{6}$

The lowest common multiple of 4 and 6 is 12

$$\begin{aligned} 12\left(\tfrac{3x}{4} + 2\right) &= 12\left(x + \tfrac{5}{6}\right) \\ 9x + 24 &= 12x + 10 \\ 14 &= 3x \\ 4\tfrac{2}{3} &= x \end{aligned}$$

You may wish to provide additional practice on this topic before proceeding.

Page 354 We have included a rather thorough review of algebraic phrases and word problems. Previous success and recall by the class should guide you in the amount of time to spend on this topic. To focus attention on data, emphasize the fact that each unknown should be represented in terms of some variable before the equation is written. Also, encourage the students to check their answers.

Page 358 This Inventory Test should be used to insure that the rules of signs are firmly within the grasp of all the students.

Page 359 It will undoubtedly be useful for students working this section to review the discussion of cryptography on page 320 of the previous chapter before trying to decipher the exercises. They will find the list of the most common two- and three-letter words helpful.

They may also find it interesting to encode messages, using a list that you prepare or have approved, and to exchange messages with their neighbors. Directions for this procedure are given in the previous chapter. We suggest that some restriction be placed on the content of the message. Messages might be limited, for example, to common sayings or proverbs, such as "There's many a slip 'tween the cup and the lip." The minimum length of the message should be ten words.

Page 362 Work problems of the type discussed here have always been a source of difficulty to beginning algebra students. Consequently this section might well be considered optional unless interest is high. The procedures to follow have been carefully developed, and we recommend that they be followed closely. Fractions will dominate this unit, and drill on eliminating fractions in equations will be helpful.

Exercise 11:

You can call an interesting pattern in work problems to the attention of the students by asking this question: If a complete task is to be done, given certain rates, can we detect a relationship between the given rates and our answers? We find the answer as follows:

If two rates are used, the answer will be the product of the rates divided by the sum of the rates. In general, if the rates are a and b, the time taken together, T, will be $\frac{ab}{a+b}$. This can be shown and understood if many numerical problems are first listed in a table.

$rate_1$	$rate_2$	*time taken together*
2	3	$\frac{6}{5}$
6	8	$3\frac{3}{7}$
3	4	$\frac{12}{7}$
5	8	$3\frac{1}{13}$

Proof:

1. $\frac{T}{a} + \frac{T}{b} = 1$	What does the equation say?
2. $ab(\frac{T}{a} + \frac{T}{b}) = ab(1)$	Why multiply by ab?
3. $bT + aT = ab$	
4. $T(b + a) = ab$	Is this equation the same as 3?
5. $T = \frac{ab}{b+a}$	Use $f_1f_2 = p$ relationship.

This procedure will *not* work for three given rates, say a, b, and c. The relationship in this case is interesting; the answer is the product of the rates divided by the sum of ab, ac, and bc.

By following the procedure of the first example, we can show this in general (after many numerical examples).

$$\frac{T}{a} + \frac{T}{b} + \frac{T}{c} = 1$$

$$abc\left(\frac{T}{a} + \frac{T}{b} + \frac{T}{c}\right) = abc(1)$$

This equation is equivalent to the preceding one. Now use the $f_1 f_2 = p$ relationship.

$$Tbc + Tac + Tab = abc$$

$$T(bc + ac + ab) = (abc)$$

$$T = \frac{abc}{ab + ac + bc}$$

In general, given four rates the answer would be

$$T = \frac{abcd}{abc + abd + acd + bcd}$$

Someone in class may be able to verbalize this situation.

In like fashion five examples may be derived. If the chapter on Probability in Book One was studied, the denominator is seen as directly related to the topic of combinations.

$$(2 \text{ rates}) \ \frac{\text{product}}{{}_2C_1}; \ (3 \text{ rates}) \ \frac{\text{product}}{{}_3C_2}; \ (4 \text{ rates}) \ \frac{\text{product}}{{}_4C_3}, \text{ etc.}$$

Page 365

Students should be encouraged to use the format shown in the examples on page 365. After all the spaces are filled in, it should be relatively easy to write the equation. Problems of this type involving motion can usually be separated into two types. They can be summarized as:

1. If the moving objects travel in opposite directions, the sum of the distance each travels is equal to the total distance apart. $D_1 + D_2 = D_{\text{total}}$
2. If the two objects travel in the same direction (one overtakes the other), the distance one travels equals the distance the other travels. $D_1 = D_2$

Page 367

Mixture problems are another source of difficulty for beginners. It has been our experience that using the diagram shown in the example on page 367 is quite helpful. It is important that the student understand that the sum of the values of all the ingredients will equal the value of the mixture. This principle is used in formulating the equation. Please note that we recommend avoiding decimals by representing costs in cents. This unit is optional but can certainly be attempted with many classes.

Page 369

Concentration problems often appear in civil-service and job examinations. A clear understanding of the definition of concentration is necessary, because it is used in most cases to state the equation. Decimals are commonly used here, so a drill on decimals will probably prove helpful to the student.

Note that the example on page 369 shows setting up the equation in the form of a proportion, which may be preferred by some students.

Page 372 This is a logical follow-up of the solution of equations in two variables as presented in Chapter Six. The methods of elimination and substitution should be within the ability of most classes. The text carefully develops both methods, and we encourage you to follow it closely. Show the students that they can solve any similar problem by using any of the three methods. They must choose which method is the best for any particular problem. Again, the importance of a clear statement of what each variable represents should be emphasized. Although we have not included such a unit, many teachers will probably review the graphical method of solving equations in two variables, and this is to be encouraged. It will also serve to give students a good idea of what an algebra course is concerned with.

Page 378 These pages present very briefly the difficult topic of quadratics. We encourage you to try it even though it is optional. Quadratics open up an entirely new aspect of problem-solving ability. We have kept the problems relatively straightforward so that even the slower students can have some degree of success. The question may arise about the situation in which the graph does not intersect the x-axis. Certainly it is proper to point out the nonexistence of any real solution.

Page 380 This brief introduction to conic sections is optional and not necessary for continuity. However, it is quite interesting to motivated students, and we have had many successful experiences with its use. The various applications of conics are so numerous that individual study and reports by interested students can be very rewarding.

Page 386 Solving quadratics by factoring is another topic difficult for the beginner. We have attempted to develop this procedure slowly and carefully to give the student a feeling of success. The problems have been kept within the range of abilities of the class. It is important that the students not be discouraged early in this topic, because the reward of comprehending the quadratic equation is great. If this unit is well received, you may decide to expand on it. This is to be encouraged. Certainly the quadratic formula would be a logical addition.

$$\text{If } ax^2 + bx + c = 0, \text{ then } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The solution set could also be indicated as

$$\left\{\frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right\}$$

The use of this formula provides a solution to any quadratic, not just to those few that are factorable. It will also provide valuable review in the use of formulas and rational and irrational numbers. Being able to recognize a, b, and c is vital in this type of work. Most of us have had the experience of finding factoring too difficult to consider and have fallen back on the formula, thus helping the slower student achieve some success with quadratics. This topic may occur at the time of year when the student is deciding whether or not to continue into an algebra course; his work on this chapter may influence his decision.

Chapter eleven

This chapter is organized to inform the student about practices in installment purchases and loans, how additional expenses are incurred by the merchant or lender in extending consumer credit, and how these expenses are passed on to the consumer in installment transactions. Mathematically, the emphasis is directed to the adaptation of the interest formula to the installment situation. The variables are defined in terms of this situation. The problems used are genuine and except where the student is told otherwise represent acceptable practices in reputable institutions.

From an informational as well as a mathematical standpoint, the content of the chapter is important to the adult and the teen-ager alike. The consumer is under continual pressure, through advertising in newspapers, billboards, radio, and TV, to open a charge account, use easy credit terms, or exercise his "borrowing power." With the consumer debt steadily rising and approaching the magnitude of the national debt as well as the size of the total national income of families, recognition is widespread that unless it is wisely used, consumer credit can become an economic hazard. Wise use of credit requires informed consumers. Otherwise one can be lured into extravagance, can overextend his credit, freeze his budget with installment payments so that he cannot buy what he *really* wants, and become vulnerable to emergencies that render payments impossible. The government is concerned for the consumer. Congress has considered the "truth in lending" bill which would reveal interest charges to the buyer. The class should be aware of the contents of the bill.

In recent years, these exhortations to use "borrowing power" have been extended to teen-agers, who need to consider the pros and cons with special care. In view of the wide use of credit today, both by the businessman and by the consumer, it is unrealistic to advise the teen-ager never to use credit. The problem here is to teach him to use mathematics to explore the consequences and learn some guidelines that will help him manage his

finances both now and later, as an adult. He needs to learn the wise use of credit, to know how credit transactions are carried out, especially on the installment plan, and to analyze the advantages and disadvantages of using credit.

The necessity for intelligent use of consumer credit is recognized in current efforts at state and national levels to pass so-called "truth-in-lending" legislation that would require disclosure of the cost of installment credit in two ways: as the annual interest rate on the outstanding balance and as the total cost of credit in dollars as measured, for example, by the difference between the cash price and the total cost of installment payments.

The largest single type of transaction affected by such legislation would be auto loans, now totaling over $30 billion. Also affected would be about $20 billion in personal loans, about $20 billion in second mortgages, $19 billion in loans for appliances and other consumer items, and $3.6 billion in home repair and modernization loans.

Thus, a lender financing purchase of a $2200 automobile on the installment plan in payments of $75 a month for three years would have to tell the purchaser that the payments included a $500 financing charge and that he is paying interest at an annual rate of about 16 per cent.

While such legislation will be helpful, intelligent use of consumer credit calls for information and understanding that can only be developed through the kinds of experiences discussed in this chapter.

Page 395 A reassuring feature of the installment debt owed by consumers is that it does not represent careless spending but, rather, conservative expenditures. In the year for which figures are given in the text, for example, the $246 billion installment debt was incurred as follows:

On home mortgages	$186 billion
On autos	25 billion
On personal loans	16.2 billion
On home repair and modernization	3.6 billion
On other installment purchases	15.2 billion

Page 396 This chapter provides many opportunities to discuss the conveniences and dangers of installment buying. The class may be interested in preparing an analysis of the advantages and dangers of installment buying. This summary may be drawn up and presented as a finished product of the class.

Some of these suggestions may be appropriate.

1. The cash purchaser does not pay for bad debts, accounting, or other expenses created by those buying on credit.
2. The installment purchaser has the use of the merchandise while he is paying for it.
3. One might never save up in advance enough cash to pay for expensive items as a house, a car, a refrigerator, or a color TV.

4. Installment buying is one way of budgeting one's income.
5. The merchandise purchased on installments costs more.
6. One is tempted to purchase on installments what he cannot afford.
7. Installments "freeze" a portion of the income. This diminishes flexibility in the budget that is needed for emergencies, unexpected recreational opportunities, attractive bargains, or investment opportunities.
8. The buyer does not own merchandise purchased on installments until payments are completed. In case of default, the merchandise can be reclaimed and sold by the seller. If what it brings on resale is not enough to complete the payments, the original purchaser may in some cases be required to make up the difference.

Not all of the above considerations are of equal importance. Each should be carefully discussed before being accepted. The students will contribute other ideas.

Page 397 The merchant selling for credit has certain expenses that are not incurred in selling for cash. These can readily be brought out in class discussion. It is largely to avoid these expenses that the trend toward cash-and-carry shops and supermarkets has become so strong in recent years.

The carrying charge is the means by which the cost of credit is paid for by the one who uses it. It is the difference between the cash price and the installment price of the merchandise. It is important that the student be able to identify the carrying charge in any transaction, since this will be called the *interest* when he comes to calculating the rate of interest on the installment purchase.

Page 403 The interest formula is to be used for calculation of rate of interest on an installment purchase. However, the variables need to be defined in terms of the installment situation. Here the student is given practice in use of the interest formula, especially in the form convenient in finding rate.

Page 405 The outstanding balances on an installment debt constitute an arithmetic series, since they are reduced by regular payments. Thus, the principal on which interest is to be charged is not the initial amount owed when the purchase is made. It is (approximately) the average of the initial and final balances. This fact provides an opportunity for the student to learn something about the characteristics of the arithmetic series and to learn how to find the average of an arithmetic progression.

Page 407 On which transaction is one paying a higher rate of interest, paying \$120 interest on the purchase of a \$1500 car by installments over a period of a year or paying \$5 interest on a purchase of a \$50 radio over a period of 6 months? It is clear that we need to take into account not only the amount of interest but also the size of the debt and the time it runs. Actually, the interest rate on the car is 8%, whereas the interest rate on the radio is 40%.

To calculate the interest rate on installment credit, we use the interest formula, $r = \frac{i}{pt}$. However, we need to define the variables carefully. Since the principal is steadily reduced by the installment payments, the average principal is slightly over one half the initial principal.

In learning this topic the student should understand that he is applying the familiar interest formula, $r = \frac{i}{pt}$, to a special situation that requires definition of the variables. Thus:

Interest. Everything in excess of the cash price is interest because it is *payment for use of credit,* which is the definition of interest.

Principal. Because regular payments are to be made, the principal is steadily reduced. We must therefore find the average amount owed over the period the installments are being made. Accordingly, we are concerned with three principals:

1. The initial principal, p_i, which is cash price minus down payment. If you were to borrow from the bank to pay the rest of the cash price, this would be the amount you would need. It would be the principal of your loan. It is logical to say that this is the initial principal of your debt on the installment purchase.
2. The final principal, p_f, is the payment on principal in the final installment. In order to determine what this is, it is important at the outset to set up a statement to show how each installment payment is divided between interest and payment on the principal.
3. The average principal, p_a, is the average amount owed over the period during which installment payments are being made. It is found by averaging the initial and final principals: $\frac{p_i + p_f}{2}$. This is p_a.

Page 409 Here we develop a systematic procedure for calculating rate of interest. Because these problems are often complicated, it is important for the student to develop the practice of following a fixed procedure in each problem. He should set down specific data for each problem:

1. Set up the statement to show how each installment is divided between principal and interest (as in the example).
2. Set up the value for each variable (encourage use of subscripts):
   p_i—initial principal (cash price minus down payment)
   p_f—final principal (equal to the regular payment on principal)
   p_a—average principal (becomes p in the interest formula)
   i—interest (total installment price minus cash price)
   t—time (fraction of a year over which payments are made)

Page 411 These examples provide an opportunity to clear up any difficulties the students may be having in setting up the values for the variables to be used in the interest formula. It will be helpful if in class discussion of each problem, the variables are listed, each with its value, before the formula is used for the solution.

Studying an alternative method of solution is useful, not only to lend variety to the discussion, but also to keep the solution from being mechanical. This alternative method of solution can be explored during the discussion.

1. Determine how much interest would be paid in a year. Call it i_a. For example, in Exercise **1. d.**, multiply the interest per week by 52.
2. When p_a is determined, write the percentage statement:

$$i_a \text{ is } r\% \text{ of } p_a$$

Then r is the rate of interest. This can be demonstrated by comparing it to the standard solution.

It is easy to see that this solution uses the interest formula in the form $r = \frac{i \div t}{p}$, instead of $r = \frac{i}{pt}$. The two expressions are equivalent.

Once they have tried this variation, the students can use whichever method they prefer.

The method used in this chapter for calculating rate of interest on installment credit is an approximation called the *constant-ratio* method. It gives a result close to the exact rate as calculated by the *actuarial method*. The latter, which requires the use of tables, takes into account the varying proportion of interest and payment on principal included in each installment payment. As payments are made and the balance is reduced, the amount of interest included in the installment is less. Since the installment remains the same, the payment on principal increases. Thus, the portion of the

installment directed to reducing the principal is least at the outset and greatest at the end. This is illustrated in the calculations of the Donaldsons' installment mortgage (page 418).

In using the constant-ratio method, we assume that the interest is the same fraction of the installment payment in the first payment as in the last, and that the principal is constantly reduced by the same amount. The balance owed, consequently, constitutes an arithmetic series. The first term of the series is the initial principal, and the last term is the final principal, which is identical to the payment on principal in each installment. The average principal is the average of the initial and final principals.

The constant-ratio method is legally acceptable; it is commonly used by business firms in computing rate of yield on installment notes and by the government in fixing rates for FHA loans. Since the initial balance is underestimated, the interest rate determined by the constant-ratio approximation is a fraction of a per cent higher than the exact rate as determined by the actuarial method. In the exercises in the chapter, the interest rates are computed to the nearest whole per cent for this reason.

In popular discussions of rate of interest in installment buying, the constant-ratio method is commonly presented with the formula in this form:

$$r = \frac{2mi}{B(n+1)}$$

where

r is annual rate of interest,

i is interest as defined above,

m is number of payments per year (12 if monthly, 52 if weekly, etc.),

B is initial principal,

n is number of payments called for.

That this is the constant-ratio formula is evident from this comparison:

Average principal is $\dfrac{B + \frac{B}{n}}{2}$

Time is $\frac{n}{m}$

Then, by the interest formula, $r = \dfrac{i}{\frac{1}{2}(B + \frac{B}{n}) \times \frac{n}{m}}$

$$= \frac{i}{\frac{nB+B}{2n} \times \frac{n}{m}} = \frac{i}{\frac{nB+B}{2m}} = \frac{2mi}{B(n+1)}$$

While the students may not be able to follow the derivation of the formula, they can use the formula once they understand that it is merely a variation on the interest

formula. It is important that they do understand, however, that it is an adaptation of the interest formula to the conditions of installment buying and not merely a rule of thumb.

Page 415 The constant-ratio method is useful also for determining the rate of interest on personal loans of the installment type. Personal credit differs from commercial credit in that the latter tends to be self-liquidating. The merchant who borrows money to purchase a supply of merchandise expects to realize sufficient profit from the transaction to pay the loan with interest and still have a profit. There is no similar feature in a personal loan.

Just as the merchant incurs additional expense and greater risk in selling on installments rather than for cash, so the banker incurs greater risk and expense in personal loans as compared to commercial loans. Until recently, the commercial banks had little interest in personal loans, and the field was left to the "loan sharks," who, by adding special fees and penalties, were able to exact exorbitant rates of interest on personal loans. To protect the borrowers from such operations, most states have enacted "small-loan" laws that establish legal interest rates at around $2\frac{1}{2}$ per cent a month (30% a year).

To make the topics real to the student, you should encourage the class to find advertisements of personal-loan agencies and of merchandise advertised for sale on the installment plan. These should be brought to class, and the more interesting ones can be posted on the bulletin board so that the class can calculate the rate of interest being charged. It will be found that in many cases some factor will be missing so that the rate cannot be determined. It may be stated that the loan or installment price is to be "repaid in 18 easy payments" or in "monthly payments of only $10," without the additional data to determine what the total is. The students will be interested in examples of these oversights. There have been requests for national legislation to require complete data to provide such information.

Chapter twelve

The automobile is a symbol of modern America; its use by our citizens from the age of 16 to past retirement is well known. Perhaps no other topic is more practical to study. The automobile has become one of the most expensive habits of Americans; at the present time, only food and shelter cost the American family more than operating their automobile. The study of this topic can clarify for young people the variety and amount of expense involved in owning and operating a car and can identify the ways in which the car

owner can effect savings. It has frequently been stated that in an average family with a youngster of college age, the youngster has a choice between owning a car and going to college. Only a youngster who makes unusual effort or whose family has an income above the average can do both. Most high school students cannot maintain a car and good grades, according to several published surveys.

The extensive use of graphs in this chapter also provides excellent practice for interpreting data presented in graphical form. This is frequently used as material on civil-service examinations.

Page 423 The first set of problems deals with the number of automobiles that exist and the sheer volume of traffic. Attention is directed to the related problems: fuel, smog, and highways.

Page 426 This topic introduces the cost of operating a car for the average owner. The use of graphs here is more revealing than the problem stated in words. The fixed expenses are those expenses that occur even when the car is not used. The variable expenses are those which develop as a consequence of driving the car. Class discussion will be needed to clarify the distinction between fixed and variable expenses. The cost of depreciation, which is a fixed expense, is probably the greatest of all these costs. This is further developed later.

Page 429 The data used in this section are as up to date as possible. They may vary from one locality to another, and changes in interest rates are frequent. Therefore, it may be useful to ask students to find the current rates in your community and to use them in place of the ones that are given.

Page 431 This may be an appropriate time to ask an insurance agent in the community to speak to the class on the cost of insurance and to identify the reasons for the excessive rates for drivers under the age of 25. Insurance companies have an abundance of data on this subject and can provide films that will develop the idea further.

Page 433 Many car owners keep careful account of expenses on their automobiles. Thus, you may be able to find current data from your own locality. It is frequently more stimulating to students to use local information.

Page 436 Much information is available on highway construction; these data, however, vary from community to community because of the nature of the materials used. An interesting class project would be to investigate the cost of building highways in your community. Community-resource people who are informed on this issue can provide information on either county or city highways.

Page 440 At this time, the use of airplanes in carrying freight is increasing rapidly. It may be an interesting class project to visit local airline offices and obtain data on costs of carrying freight and the extent to which local industries are utilizing air freight and air transportation.

Page 442 The graphs are used in this section not only to extend the students' idea of mathematical graphs but also to develop further the idea of safety in automobile travel. The data on automobile travel are widely available, and local information can be used here.

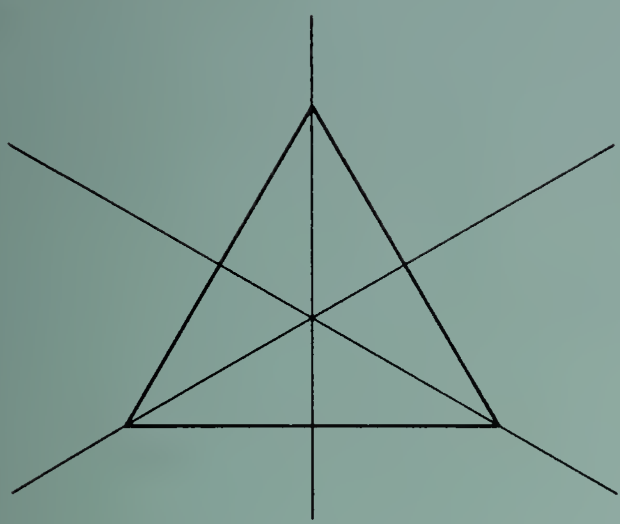
The causes of accidents are identified in several of the exercises. It is well known, however, that the driver who drinks is a frequent cause of auto accidents. It may be useful to explore the relationship between drinking and automobile safety. Most communities can provide resource people and data on this issue.

Page 444 This section is devoted to improving problem-solving skills. As in other units of this type, class discussion will be very useful in helping individuals evaluate their work.

Local communities and individual families may have typical trips that they take. It will create more interest to develop problems around local situations and to utilize the distance rate and time formula in solving these problems. For example, people in communities within a hundred miles of a city such as Chicago make frequent trips to the big city. Produce is hauled to the city, and buses and trains go there. Therefore, a variety of problems interesting to the students can be developed revolving around this distance.

Page 450 This can be an interesting unit if the Morrisons' trip to the national parks is followed carefully. It may be useful for the teacher to set up a map and a chart and discuss in some detail the trip that this family is taking. The discussion can be extended to the kinds of country and scenery that are seen by the travelers. It is possible that some students in the class have also taken this trip and can add to the discussion. The discussion may also furnish a few guidelines to direct students toward helping plan summer vacations for the family.

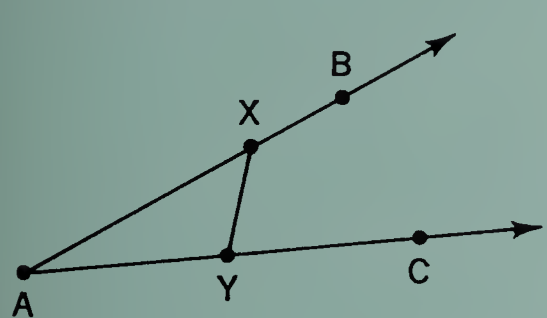
Page 2 **3.**



Page 3 **9.** vertical axis of symmetry: A, B, C, E, H, I, M, O, T, U, V, W, X, Y, Z; horizontal axis: D, H, I, O, X; both: H, I, O, X **10.** A plane figure is said to have line symmetry if it can be divided by a line into two identical parts, each of which is a mirror reflection of the other.

Page 10 **9. f.** Yes; if the figure were folded along $\overleftrightarrow{CD}$, point A would fall on point B.

Page 12 **12. 13.**



Page 16 **12.** A plumb line is used in sounding and in determining vertical direction.

Page 23 **14.** To circumscribe a circle around a triangle, construct the perpendicular bisectors of two of the sides. The point at which the bisectors meet will be the center of the circle. The radius will be the distance from the center to one of the vertices. **15.** If a triangle contains an obtuse angle, the center of its circumscribed circle lies outside the triangle. If a triangle contains a right angle, the center of its circumscribed circle lies on the hypotenuse of the triangle.

Page 24 **3.** C and D are corresponding points. The axis of symmetry is perpendicular to a line connecting corresponding points. **4.** Using C and D as centers, strike arcs meeting above $\overline{AB}$. O is on the axis of symmetry since it is equidistant from corresponding points C and D. $\overline{OP}$ is on the axis of symmetry.

Page 28 **5.** Using P as the center, draw arcs crossing the line. Label the points of intersection C and D. Using C and D as centers, draw arcs intersecting below the line. Label the point of intersection O. Draw $\overline{OP}$; this is the perpendicular to the line.

Page 29 **15.** The bisector of the vertex angle is also an altitude of an isosceles triangle. By vertex angle we mean the angle formed by the sides of equal measure.

Page 32 **1.** The area of the square is 64 square units; of the rectangle, 65 square units. The pieces in the rectangle are not a precise fit.

Page 33 **4. f.**

+	=	÷	×	−
×	−	+	=	÷
=	÷	×	−	+
−	+	=	÷	×
÷	×	−	+	=

Page 34 7. Newspapers and magazines are sources for designs having symmetry.

Page 43 The symbols $\nleq$ and $>$ have the same meaning. The symbols $\ngeq$ and $<$ have the same meaning. **11.** $(4 + 6)$ is not less than $(3 + 7)$. **12.** $(8 - 2)$ is not greater than $(10 - 3)$ **13.** $(27 - 13)$ is not equal to $(9 + 5)$ **14.** $(16 + 3)$ is equal to $(38 - 19)$ **15.** $(6 + 5)$ is equal to or less than $(28 - 17)$ **16.** $(3 + 8)$ is equal to or greater than $(5 + 4)$ **17.** $(56 - 48)$ is equal to or less than $(14 + 4)$ **18.** $(7 + 9)$ is not equal to $(25 - 11)$ **19.** $(15 + 5)$ is not greater than $(29 - 15)$ **20.** $(2 + 15)$ is equal to $(36 - 18)$ **21.** $(8 + 7)$ is less than $(20 - 3)$ **22.** $(29 - 5)$ is less than $(25 + 4)$ **23.** (5×6) is less than or equal to 31 **24.** $(20 \div 4)$ is greater than or equal to 6 **25.** $(7 + 6)$ is greater than (2×6) **26.** $(18 + 6)$ is equal to (4×6)

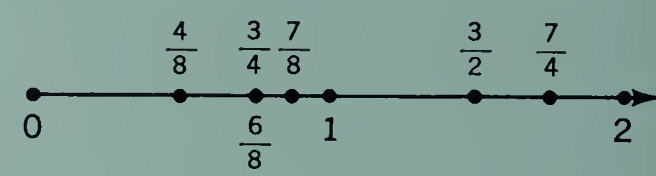
Page 44 **1.** {5, 4, 3, 2, 1, 0} **2.** {13, 14, 15, 16, 17, 18} **3.** {1, 2, 3, 4} **4.** {2, 1, 0} **5.** {7, 6, . . . , 0} **6.** {9, 10, 11, . . .} **7.** {1, 2} **8.** {3, 2, 1} **9.** {10, 11, 12, . . .} **10.** {4} **11.** {6, 7, 8, . . .} **12.** {all whole numbers except 5} **13.** {3, 4, 5, 6} **14.** {5} **15.** {all whole numbers except 4} **16.** {6} **17.** {23} **18.** {4, 3, 2, 1, 0} **19.** {11, 10, 9, . . ., 0} **20.** {12, 13, 14, . . .} **21.** {4, 5, 6, . . .} **22.** {all whole numbers except 11} **23.** {4, 5, 6, . . .} **24.** {20, 21, 22, . . .} **25.** {4, 3, 2, 1, 0} **26.** {12, 11, 10, 9, 8, 7, 6} **27.** {20} **28.** {4} **29.** {7} **30.** {8} **31.** {7, 6, 5, 4, 3, 2, 1} **32.** {7} **33.** ϕ

Page 45 The first addend equals the sum minus the second addend. The second addend equals the sum minus the first addend.

Page 47 The first factor equals the product divided by the second factor. The second factor equals the product divided by the first factor.

Page 54 **6. e.** Numerals ending in zero whose tens digit is zero or names an even number name numbers that are divisible by 20. **f.** 320, 740, 1580, 1360, 3540, and so on **g.** A number is divisible by 36 if it is divisible by 9 and 4. **h.** 504, 108, 216, 828, 936, and so on **i.** A number is divisible by 45 if it is divisible by 9 and 5.

Page 57 **1.**



Page 60 **2.** The sum of zero and any fractional number of arithmetic is the fractional number. **5.** If zero is subtracted from any fractional number of arithmetic, the result is the fractional number. **11.** The value of the numbers named by the digits is 2000, 0, 50, and 0. The zeros indicate that the hundreds and units places are empty.

Page 73 **1. a.** 5×10^3, 8×10^2, 3×10^1, 6×10^0 **b.** 7×10^1, 6×10^0 **c.** 8×10^3, 3×10^2, 5×10^1, 9×10^0 **d.** 1×10^4, 6×10^3, 2×10^2, 5×10^1, 7×10^0 **e.** 8×10^4, 2×10^3, 3×10^2, 5×10^1, 9×10^0 **f.** 4×10^4, 0×10^3, 0×10^2, 0×10^1, 5×10^0 **g.** 1×10^5, 2×10^4, 3×10^3, 5×10^2, 0×10^1, 9×10^0 **h.** 2×10^5, 0×10^4, 0×10^3, 3×10^2, 1×10^1, 2×10^0 **i.** 2×10^6, 0×10^5, 3×10^4, 8×10^3, 0×10^2, 1×10^1, 0×10^0

Page 74 **5. a.** 5×10^2, 9×10^1, 8×10^0, 6×0.1, 2×0.01, 3×0.001, 5×0.0001 **b.** 1×10^4, 0×10^3, 0×10^2, 7×10^1, 5×10^0, 4×0.1, 3×0.01, 2×0.001 **c.** 1×10^5, 8×10^4, 7×10^3, 6×10^2, 3×10^1, 5×10^0, 0×0.1, 2×0.01, 5×0.001

Page 75 2. In 1900 the population of the United States was 76 million. By

1960 the population had increased to 179 million. The sentence in which the numbers were rounded to the nearest million is easier to understand. **7.** To round a number to the nearest hundred, examine the tens digit. If it is 4 or less, the hundreds digit remains the same. If the tens digit is 5 or greater, round to the next hundred above. To round a number to the nearest thousand, examine the hundreds digit. If it is 4 or less, the thousands digit remains the same. If it is 5 or greater, round to the next thousand above.

Page 88 $\frac{h_1}{m_1}$, the ratio of the time to the distance in the first situation; $\frac{w_1}{p_1}$, the ratio of the number of games won to the number of games played in the first situation; $\frac{e_2}{h_2}$, the ratio of the amount of earnings to the time in the second situation

Page 91 **5. a.** 4×10^4, 0×10^3, 7×10^2, 0×10^1, 5×10^0 **b.** 3×10^5, 8×10^4, 9×10^3, 7×10^2, 6×10^1, 5×10^0 **c.** 2×10^6, 0×10^5, 5×10^4, 0×10^3, 7×10^2, 0×10^1, 6×10^0

Page 96 **3.** The magnitude of the number determines the distance from zero to the point associated with it on the number line. The sign determines the direction to the left or right of zero.

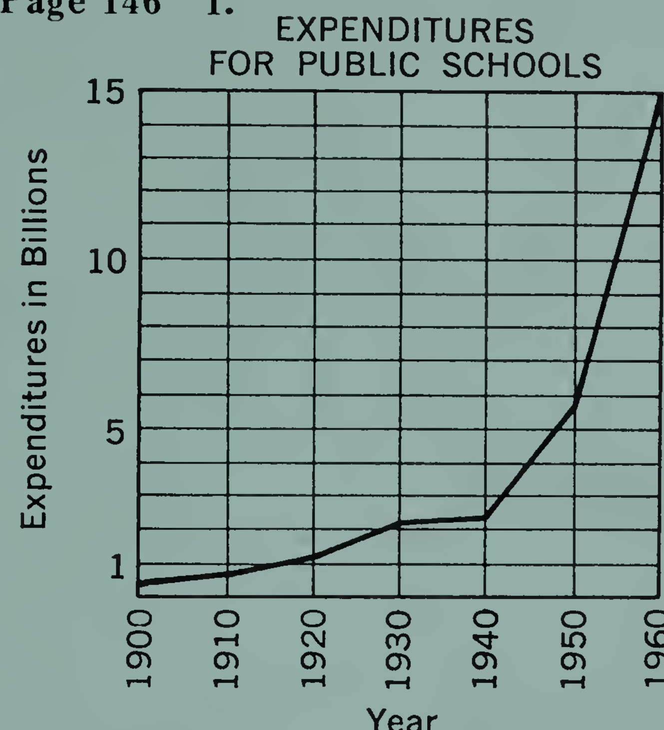
Page 108 **15. a.** 1×10^1, 2×10^0, 3×10^{-1}, 7×10^{-2}, 5×10^{-3} **b.** 2×10^1, 2×10^0, 2×10^{-1}, 2×10^{-2}, 2×10^{-3} **c.** 1×10^2, 8×10^1, 3×10^0, 0×10^{-1}, 7×10^{-2}, 0×10^{-3}, 2×10^{-4}, 5×10^{-5}

Page 143 **4.** The number of industrial laborers was 3,625,000 in 1900, 3,894,000 in 1950, and 3,675,000 in 1965. The number remains about the same. The number predicted for 1975 is 3,823,600 which is slightly greater than the number of industrial laborers in 1965.

Page 144 **4.** In 1900, 5% of the college-age youth were in college. By 1940, the enrollment increased by 12%. After 1940, the rate of increase in college enrollments became sharper. In 1960, 40% of the college-age youth were in college.

Page 145 **3.** 58% or 29,500,000 in 1900; 55% or 34,764,000 in 1910; 52% or 36,596,000 in 1920; 45% or 42,689,000 in 1930; 38% or 48,608,000 in 1940; 34% or 50,029,000 in 1950; 26% or 66,192,000 in 1960

Page 146 **1.**



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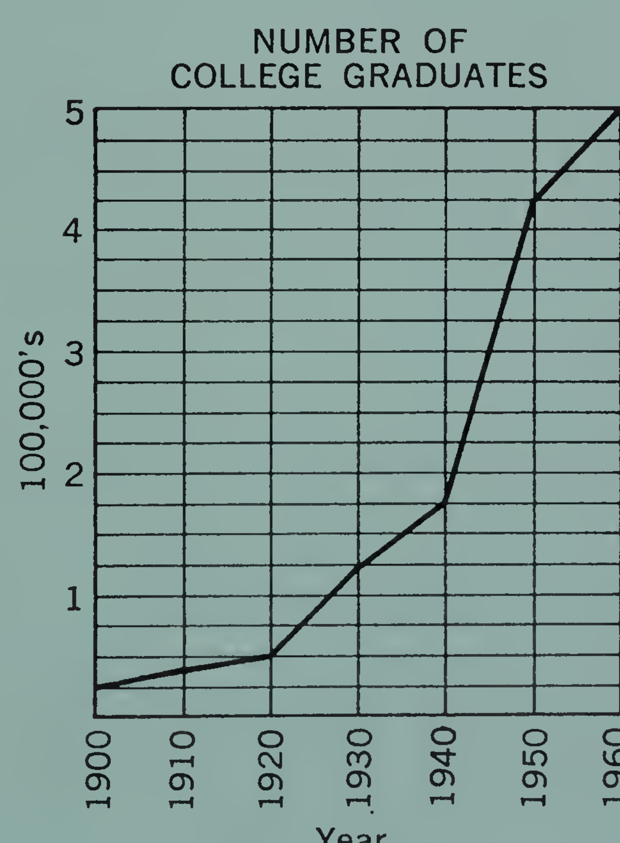

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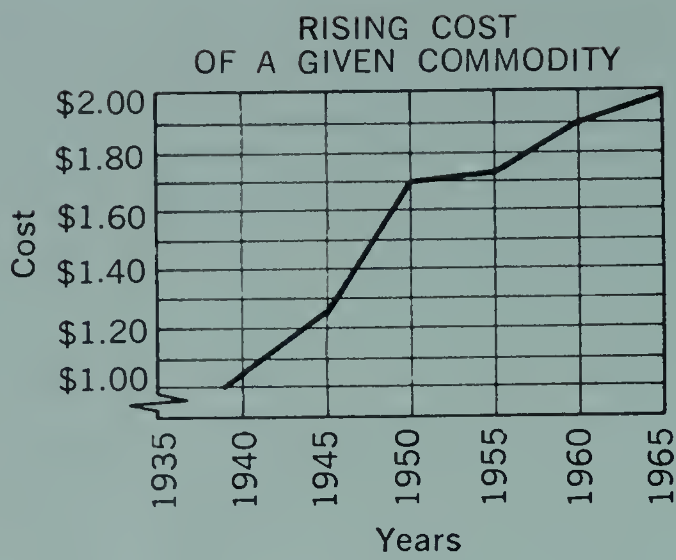

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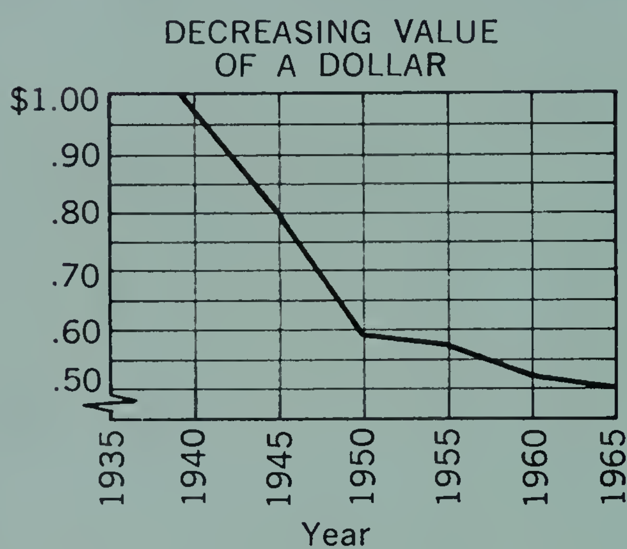
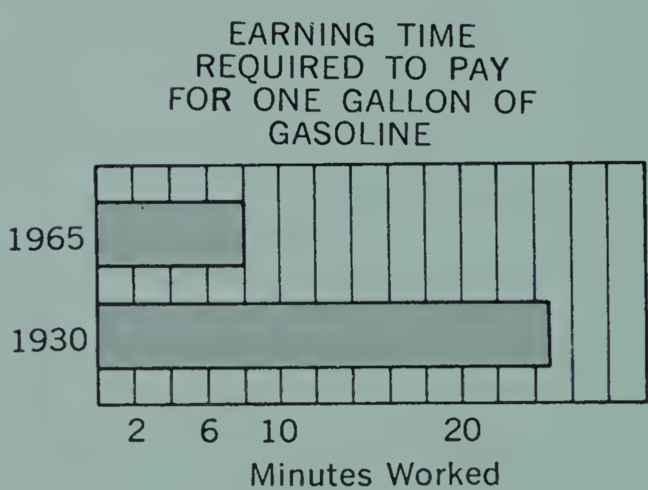
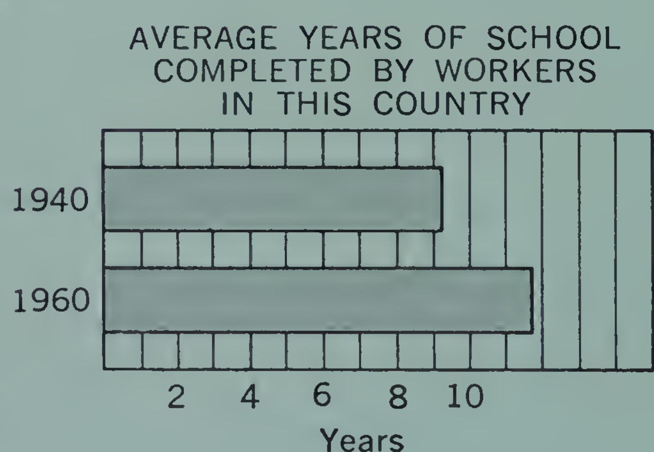


Page 148 1.



Page 149 5.



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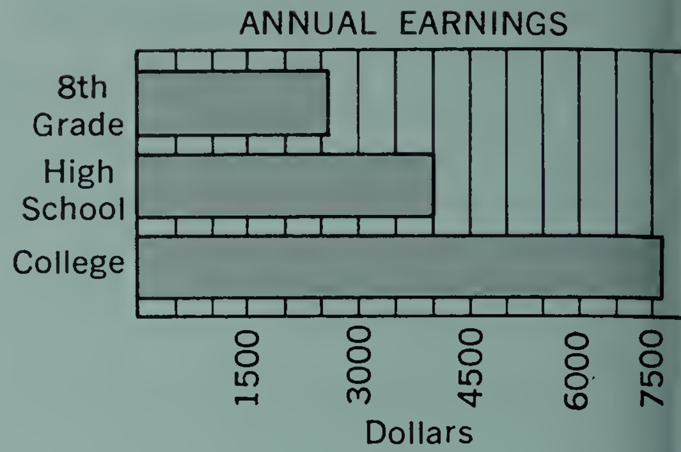

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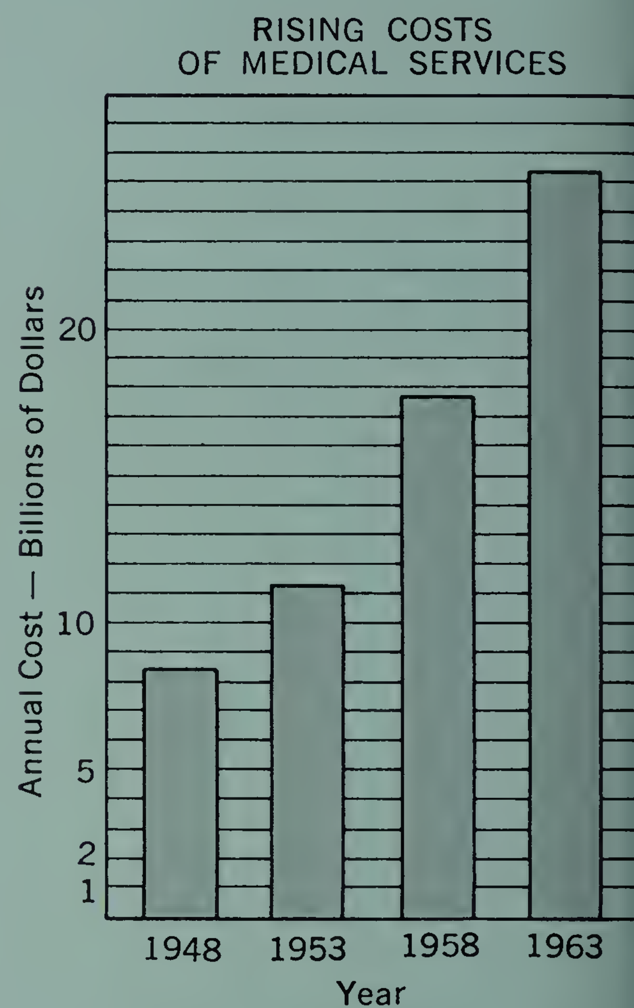

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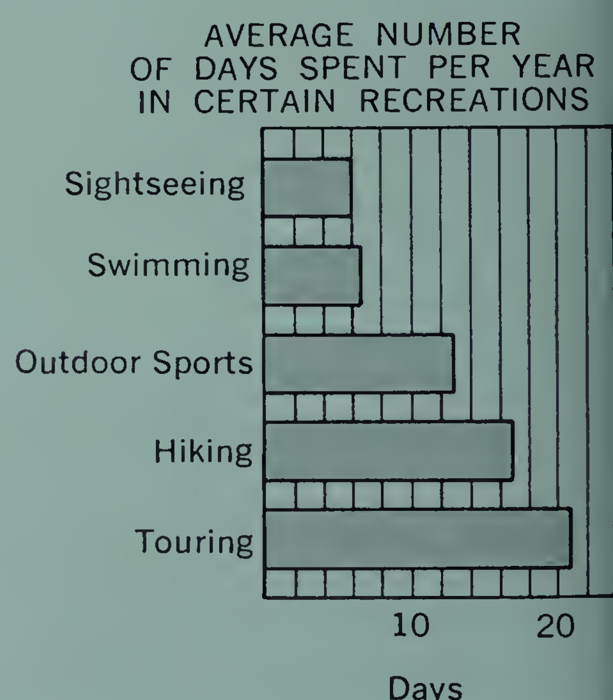


Page 150 2. room and board, 130°; tuition, 184°; other expenses, 47°

Page 151 5.

Item	*Cost*	*Per cent*	*Degrees*
Room and board	\$900	56	202
Tuition	350	22	79
Other expenses	350	22	79

Page 155 4. Clerical, operatives, craftsmen, services, laborers, and executives receive wages or salaries. Services and professionals receive fees. Farmers and proprietors receive profits.

Page 168 3.

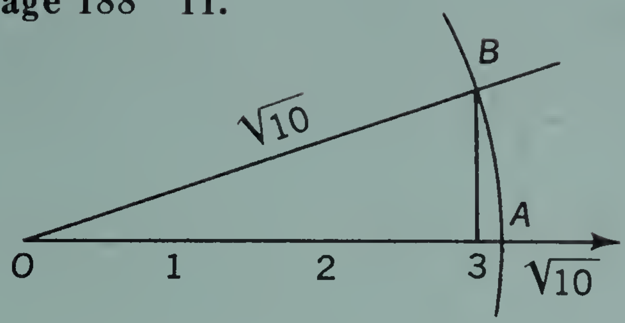
3 Mo. Period	*Principal*	*Interest*	*Total*
1	\$200.00	\$2.00	\$202.00
2	202.00	2.02	204.02
3	204.02	2.04	206.06
4	206.06	2.06	208.12

Page 173 10. A 20-payment life insurance policy is recommended for Mr. Arnold since he needs no saving and should pay for protection while earning. He can buy \$33,000 worth of insurance. **11.** Mr. Smith should buy \$35,000 of 30-payment life insurance and put \$25 a month in his savings account.

Page 180 2. $a = 6, b = 8, c = 10, 6^2 + 8^2 = 10^2, 36 + 64 = 100; a = 24, b = 7, c = 25, 7^2 + 24^2 = 25^2, 49 + 576 = 625; a = 5, b = 12, c = 13, 5^2 + 12^2 = 13^2, 25 + 144 = 169$ **6.** $13^2 + 84^2 = 85^2, 169 + 7056 = 7225$

Page 188 10. The decimal 3.3 terminates; the decimal 3.3 . . . does not terminate but there is no repeating cycle of digits; the decimal $3.\overline{3}$ does not terminate but the digit 3 repeats endlessly.

Page 188 11.



$OA^2 = OB^2 = 3^2 + 1^2 = 10$
$OA = OB = \sqrt{10}$

Page 200 6. $\frac{BC}{EF} = \frac{AC}{DF}$; $\frac{CB}{BA} = \frac{FE}{ED}$; $\frac{BA}{CA} = \frac{ED}{FD}$

Page 201 3. If one pair of acute angles is equal in measure, the other pair of acute angles will also be equal in measure.

Page 212 7. **b.** $\tan 35° = .7; \frac{a}{b} \approx \frac{7}{10}$ **c.** $\tan 58° = 1.6; \frac{a}{b} \approx \frac{8}{5}$ **d.** $\tan 70° = 2.75; \frac{a}{b} \approx \frac{11}{4}$

Page 215 21. As $m\angle A$ approaches 90°, a approaches c; therefore, a is increasing and b is decreasing. Thus, the ratio $\frac{a}{b}$ is increasing. Since $\tan (m\angle A) = \frac{a}{b}$, $\tan (m\angle A)$ increases rapidly as $m\angle A$ approaches 90°. **23.** As $m\angle A$ approaches 90°, a becomes more nearly equal to c. Since $c^2 = a^2 + b^2$, $b^2 = c^2 - a^2 = 0$, and b must be 0. **24.** Since $\sin 90° = 1 = \frac{a}{c}$, a must be equal to c. **25.** Since $\tan 90° = \frac{a}{b}$ and $b = 0$, $\tan 90° = \frac{a}{0}$; but, $\frac{a}{0}$ is undefined. Therefore, $\tan (m\angle A)$ is undefined at 90° when $b = 0$.

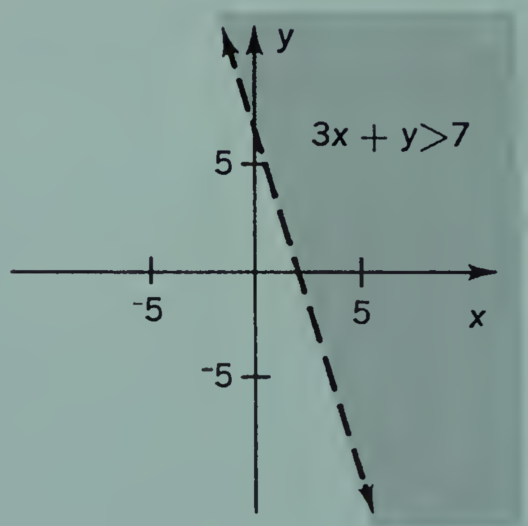
Page 221 2. When $b = 1$, the numbers $\frac{a}{b}$ have been called integers. Integers are rational numbers because they can be expressed as $\frac{a}{b}$ with a and b integers and $b \neq 0$. The set of rational numbers is closed under addition, subtraction, and multiplication. The set of rational numbers is not closed under division because division by zero is undefined.

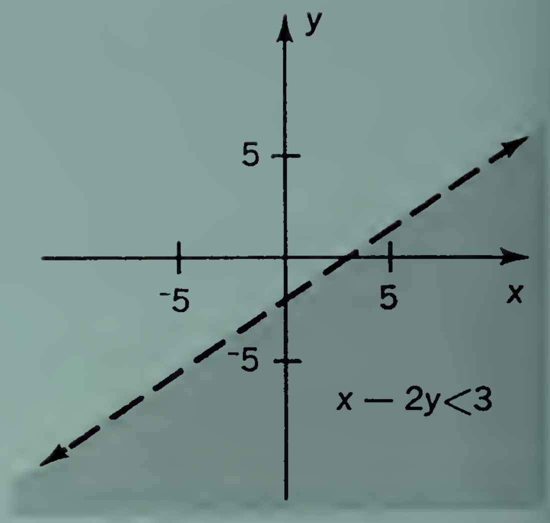
Page 223 **1.** The measures of the legs of the right triangles should be as follows: **a.** 2 and 1 **b.** 3 and 1 **c.** 3 and 2

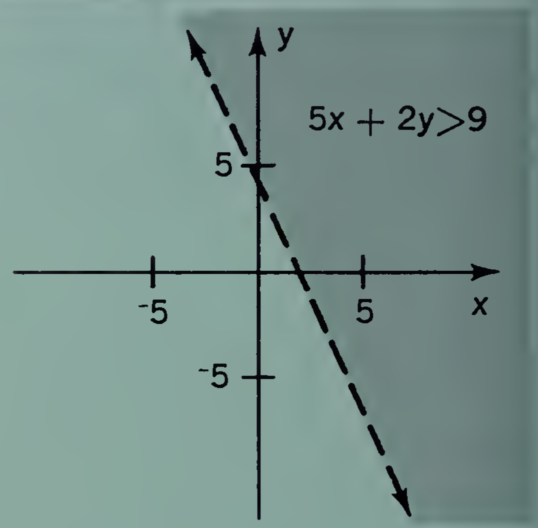
Page 233 **1.** twice a number increased by eleven **2.** three times a number decreased by two **3.** seven increased by five times a number **4.** eleven decreased by three times a number **5.** three times a number increased by twice another number **6.** eight times a number increased by five **7.** one-half of a number decreased by three **8.** fourteen decreased by one-third of a number **9.** three times a number decreased by twice a second number **10.** eight times a number increased by three times another number **11.** five times a number decreased by seventeen **12.** one-third of a number increased by one-half a second number **13.** eight times a number increased by one-half of another number **14.** three times a number decreased by eighteen **15.** eighteen decreased by three times a number

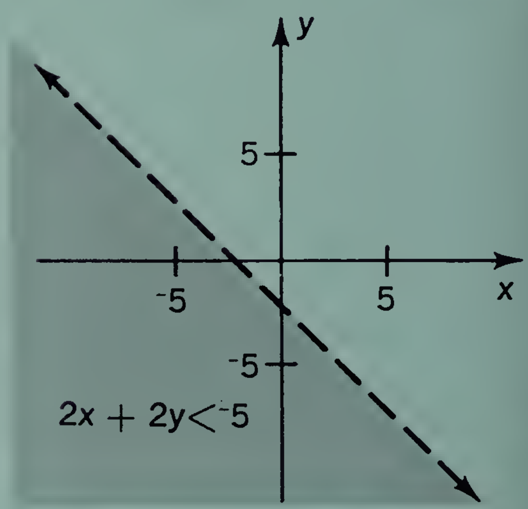
Page 242 **10.** To graph the equation $y = 2$ or $x = 7$, move two units above the x-axis and draw a line parallel to the x-axis then move seven units to the right of the y-axis and draw a line parallel to the y-axis.

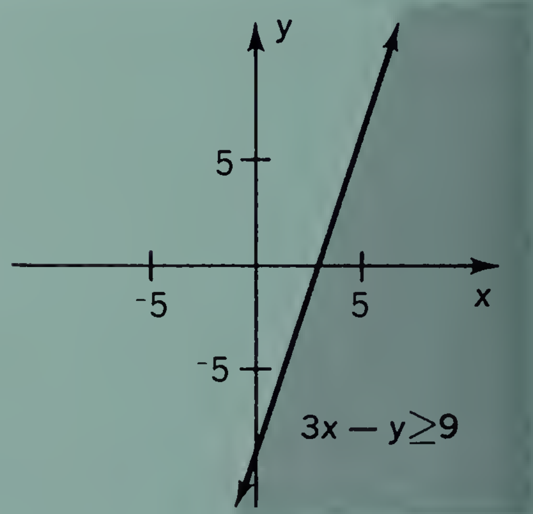
Page 243 **1-6.**

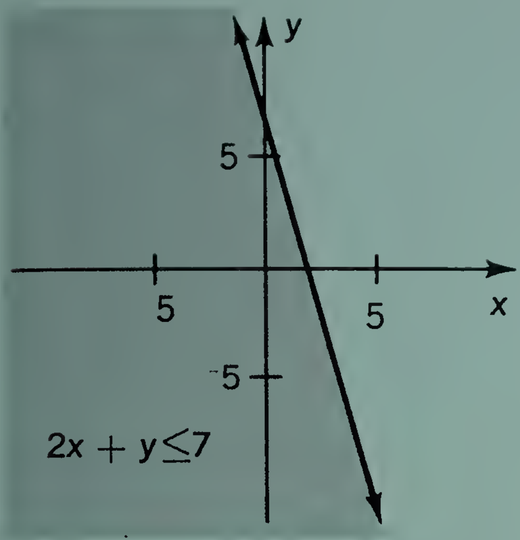








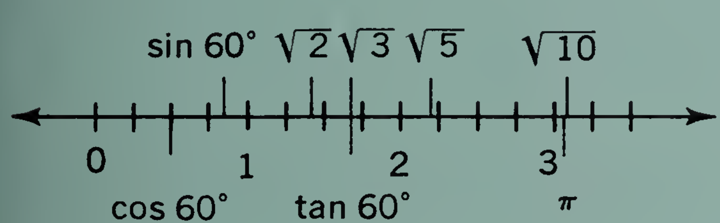




Page 247 1. C, $(3, {}^{-}2)$; r, 2 2. C, $({}^{-}7, 0)$; r, 4 3. C, $({}^{-}2, 3)$; r, 6 4. C, $(5, {}^{-}7)$; r, 5 5. C, $(0, 0)$; r, $\sqrt{17}$ 6. C, $(0, 2)$; r, 1 7. C, $(0, {}^{-}7)$; r, 8 8. C, $(11, {}^{-}11)$; r, 12 9. C, $(7, 0)$; r, $\sqrt{5}$ 10. C, $({}^{-}2, 7)$; r, $\frac{1}{2}$

Page 250 **D.** 1. a number increased by nine 2. the square of a number increased by seven 3. three times the square of a number decreased by seven

Page 251 **B.**



Page 261 13. A is the measure of the area in square units; h is the measure of the altitude in linear units; b is the measure of one base in linear units; b_2 is the measure of the other base in linear units. 14. The diagonal of a parallelogram divides it into two congruent triangles. Therefore, the formula for finding the area of a triangle is $A = \frac{1}{2}bh$.

Page 262 17. A, 63 sq. in.; B, 111 sq. in.; C, 33.95 sq. ft.; D, 126 sq. rd.; E, 152 sq. yd.; F, 3528 sq. in. 18. A, 1440 sq. ft.; B, 272 sq. yd.; C, 96.48 sq. ft.; D, 275.2 sq. in.; E, 1200 sq. rd.; F, 4920 sq. rd. 19. A, 600 sq. ft.; B, 400 sq. yd.; C, 900 sq. in.; D, 960 sq. rd.; E, 1192.5 sq. ft.; F, 450 sq. yd. 20. A, 216 sq. ft.; B, 500 sq. ft.; C, 52 sq. in.; D, 63 sq. in.; E, 36 sq. yd.; F, 160 sq. rd.

Page 264 1. $\tan 25° = \frac{x}{80}$; 37 ft. 2. $\sin 36° = \frac{x}{60}$; 35 ft. 3. $\sin 40° = \frac{x}{18}$; 12 ft. 4. $\sin 45° = \frac{x}{25}$; 18 ft. 5. $\sin 60° = \frac{x}{6}$; 5.2 in. 6. $\tan 30° = \frac{x}{10}$; 6 ft. 7. $\sin 65° = \frac{x}{30}$; 27 ft. 8. $\sin 38° = \frac{x}{25}$; 15 ft. 9. $\cos 15° = \frac{y}{120}$; 116 ft. 10. $\sin 39° = \frac{x}{15}$; 9 ft. 11. $\sin 56° = \frac{x}{18}$; 15 12. $\tan 28° = \frac{y}{100}$; 53; $\cos 28° = \frac{100}{x}$; 113.2; $\tan 15° = \frac{m}{113.2}$; 30.3; $\cos 15° = \frac{113.2}{n}$; 117.2

Page 265 1. 113.4 sq. ft. 2. 3355 sq. ft. 3. 3240 sq. units 4. 280.6 sq. ft. 5. 557 sq. ft. 6. 1595 sq. units 7. 452 sq. ft. 8. 1039 sq. ft. 9. 3760 sq. ft.

Page 267 7. The number of linear units in the height is equal to the number of layers of cubic units.

Page 282 7. To find the total surface of a cylinder, add the area of the rectangular piece that goes around the can to the total area of the bases. $2\pi r^2 + \pi dh = 2\pi r^2 + 2\pi rh$. Applying the distributive property, we get $2\pi r(r + h)$; therefore, $2\pi r(r + h)$ represents the total surface of a cylinder.

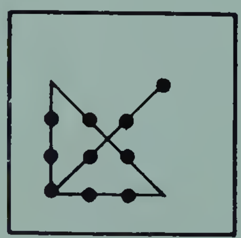
Page 290 1. The 8-quart pail was filled and poured into the 5-quart pail. This left 6 quarts in the 8-quart pail. 2. In Step 1, the 8-quart pail is filled from the 5-quart pail. In Step 2, 3 quarts are poured into the 8-quart pail leaving 2 quarts in the 5-quart pail. In Step 3, the 8-quart pail is emptied and the 2 quarts in the 5 quart pail are poured into the 8-quart pail. 3. As soon as 4 quarts are left in the 5-quart pail, we can get 7 quarts in the 8-quart pail by filling the 8-quart pail and then filling the 5-quart pail from it. 4. In Exercise 2, we are seeking a multiple of 5 which, when

divided by 8, leaves a remainder of 2. **5.** Fill the 3-quart pail from the 5-quart pail. Empty the 3-quart pail and fill it with the two remaining quarts that are in the 5-quart pail. Fill the 5-quart pail and there are 7 quarts of water. **6.** Fill the 7-quart pail twice from the 4-quart pail. Empty the 7-quart pail and fill it with the remaining quart that is in the 4-quart pail. Fill the 4-quart pail.

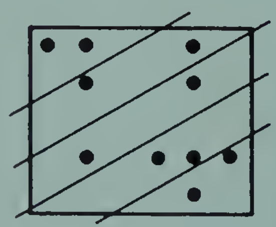
Page 292 **1. a.** $A = s^2$; 64 sq. ft. **b.** $A = \pi r^2$; 50.24 sq. in. **c.** $A = \frac{1}{2}bh$; $20\frac{1}{4}$ sq. ft. **d.** $A = \frac{1}{2}bh$; 1.6 sq. in. **e.** $A = bh$; 50 sq. ft. **f.** $A = bh$; 180 sq. in. **g.** $A = \frac{h}{2}(b_1 + b_2)$; 36 sq. yd. **h.** $A = lw$; 6.3 sq. in. **2. a.** $V = lwh$; 120 cu. in. **b.** $V = s^3$; 27 cu. in. **c.** $V = \frac{1}{3}\pi r^2 h$; 134 cu. in. **d.** $V = \pi r^2 h$; 502 cu. in. **e.** $V = Bh$; 108 cu. ft. **f.** $V = Bh$; 168 cu. in. **g.** $V = Bh$; 800 cu. in. **h.** $V = Bh$; 1280 cu. in. **3. a.** $T = 2lw + Ph$; 148 sq. in. **b.** $T = 2lw + Ph$; 54 sq. in. **c.** not to be done **d.** $T = 2\pi r(r + h)$; 352 sq. in. **e.** $T = 2B + Ph$; 186 sq. in. **f.** $T = 2B + Ph$; 339 sq. in. **g.** $T = 2B + Ph$; 586 sq. in. **h.** $T = 2B + Ph$; 848 sq. in.

Page 312 **14.** The measures of the central angles are as follows: carpentry, 71°; excavation, 89°; lumber, 62°; plumbing, 87°; hardware, 40°; miscellaneous, 11°.

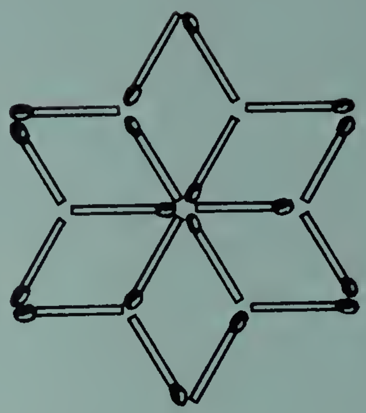
Page 317 **1.**



2.

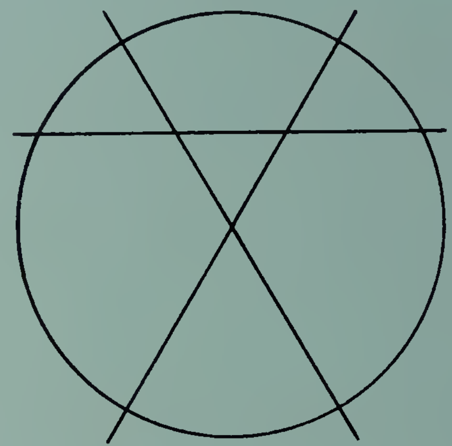


Page 318 **4.**



5. Arrange the remaining 24 matches so that no side is shared.

7.



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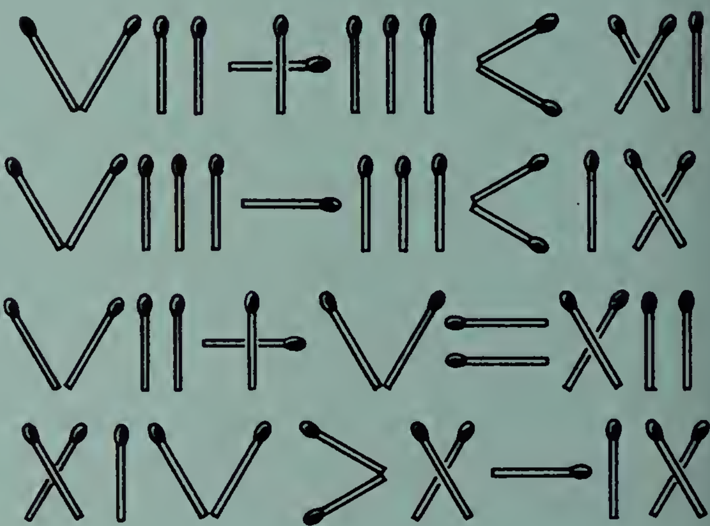


Page 324 **10.** national parks, national resources; foreign aid, international finance; farm aid, agricultural aid; interest on savings bonds, interest on debt; veterans' hospitals, veterans' welfare; highway appropriations, commerce and housing

Page 327 **12.** The measures of the central angles are as follows: education, 135°; police, 84°; streets and highways, 78°; parks, 12°; general government, 11°; miscellaneous, 40°.

Page 332

This golden moon that
shines for us again;
How oft hereafter will she wax
and wane,
How oft hereafter, rising,
search for us
Through these same branches
And for one in vain.

Omar Khayyam

1. Four score and seven years ago our fathers brought forth upon this continent a new nation, conceived in liberty and dedicated to the proposition that all men are created equal.

Abraham Lincoln

2. The vastly increased importance of mathematics in our time makes it all the more necessary that each of us should know something of the nature and uses of mathematics. While the deeper and more complex uses are not always understood, the essential nature of the subject is readily appreciated.

Morris Kline

Page 333 **3.** It is always a poor idea to give advice and it is fatal to give good advice.

Oscar Wilde

4. A drunken man was lying in the road with a bleeding nose upon which he had fallen, when a pig passed that way. "You wallow fairly well," said the pig, "But, my fine fellow, you have much to learn about rooting."

Ambrose Bierce

5. The educated southerner has no use for an *R* except at the beginning of a word.

Samuel Langhorne Clemens

Page 342 **3.** highways, $82 million; general government, $20.5 million; institutions, 36.9 million; police, $12.3 million; other, $20.5 million. **5.** both state and federal: schools, highways, general government; state: police, institutions **6.** individual income, $1.68 billion; motor fuels, $3.12 billion; other, $3.568 billion, auto licenses, $1.52 billion; corporation income, $1.088 billion

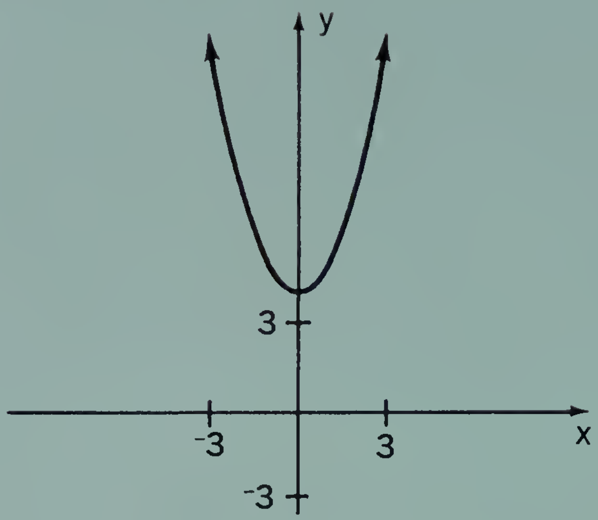
Page 343 **12.** $.50 to $.83, $.02; $.84 to $1.16, $.03; $1.17 to $1.49, $.04; $1.50 to $1.83, $.05; $1.84 to $2.16, $.06; $2.17 to $2.49, $.07; $2.50 to $2.83, $.08; $2.84 to $3.16, $.09; $3.17 to $3.49, $.10; $3.50 to $3.83, $.11; $3.84 to $4.16, $.12; $4.17 to $4.49, $.13; $4.50 to $4.83, $.14; $4.84 to $5.16, $.15

Page 344 **1.** schools, $384,000; hospitals, $240,000; health, $120,000; police, $108,000; streets, $72,000; general government, $96,000; miscellaneous, $180,000

Page 361 **1.** The man who is afraid to make a mistake can achieve little. **2.** It is of no use to argue with the inevitable; when winter comes, put on an overcoat. **3.** A little learning is a dangerous thing, but none at all is fatal. **4.** The only thing worse than making a mistake is not trying at all. **5.** Geography is about maps, biography is about chaps, and cryptography is about traps. **6.** Is it progress if a cannibal uses a knife and fork? **7.** He who has not lost his head over something has no head to lose. **8.** The best qualification of a prophet is to have a good memory. **9.** The pleasantest laughter is at the expense of our enemies. **10.** I am unable to understand how a man of honor could take a newspaper in his hands without a shudder of disgust. **11.** Marriage, in life, is like a duel in the midst of a battle. **12.** The desire to take medicine is perhaps the greatest feature which distinguishes men from animals. **13.** Speak roughly to your little boy and beat him when he sneezes. He only does it to annoy because he knows it teases.

Page 364 **11.** The product of the given numbers divided by their sum gives the same answer. In Exercises 2 and 7 the relationship is $\frac{a \cdot b \cdot c}{ab + ac + bc}$.

Page 382 **2.**



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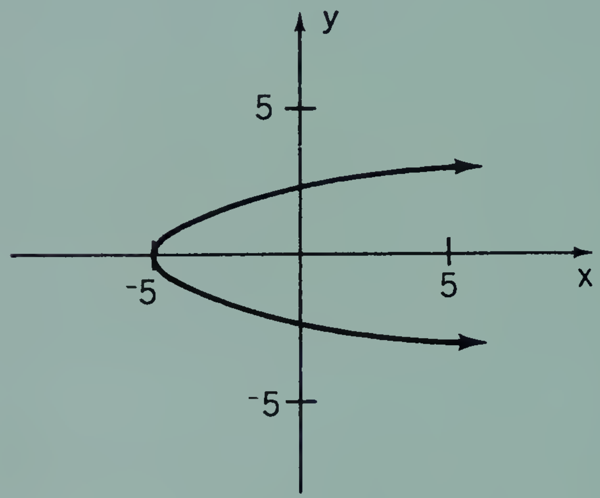
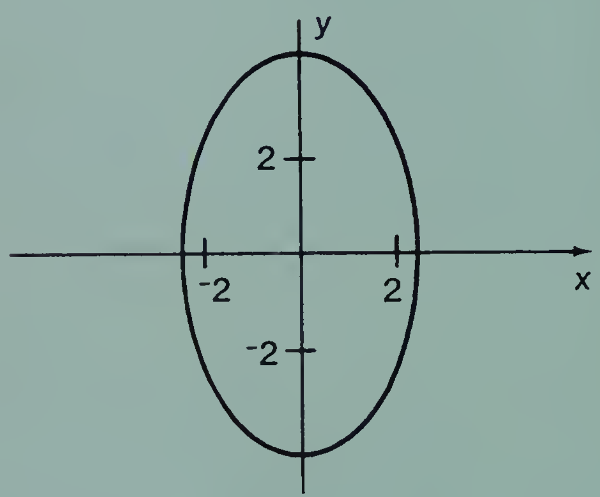
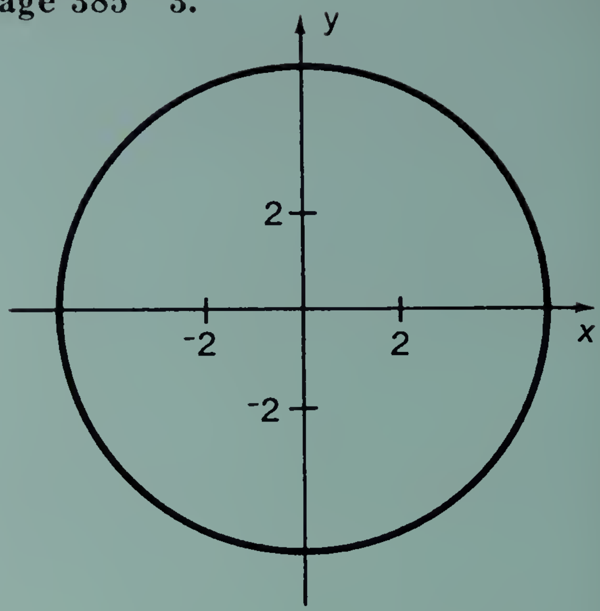


Page 383 **6.**



Page 385 area of the equilateral triangle, approximately 979 sq. yd. (using $24\sqrt{3}$ as the height) or 999 sq. yd. (using 41.6 as the height); area of the square, 1296 sq. yd.; area of the rectangle, 1152 sq. yd.; area of the circle, approximately $\frac{5184}{3.14}$ sq. yd. or 1651 sq. yd.

Page 385 **3.**



Page 395 **5.** automobiles, 65%; refrigerators, 53%; T.V. sets, 54%; furniture, 47% **6.** repairs and services, \$3 billion; personal loans, \$17 billion; automobiles, \$25 billion; other, \$17 billion

Page 397 **1.** Some of the extra clerical work involved are investigating credit ratings, calculating balances, recording payments, and sending notices of defaults.

Page 408 **7.** a = \$228.50; l = \$12.03; d = \$12.03; n = 19

Page 411 **3. e.**

Month	*Balance*	*Interest*	*Payment*
3	$38.00	$.57	$5.32
4	33.25	.50	5.25
5	28.50	.43	5.18
6	23.75	.36	5.11
7	19.00	.29	5.04
8	14.25	.21	4.96
9	9.50	.14	4.89
10	4.75	.07	4.82

Page 412 **7. d.**

Month	*Balance*	*Interest*	*Payment*
3	$120	$1.20	$21.20
4	100	1.00	21.00
5	80	.80	20.80
6	60	.60	20.60
7	40	.40	20.40
8	20	.20	20.20

Page 415 3.

Month	*Balance*	*Interest*	*Payment*
2	$90	$2.25	$12.25
3	80	2.00	12.00
4	70	1.75	11.75
5	60	1.50	11.50
6	50	1.25	11.25
7	40	1.00	11.00
8	30	.75	10.75
9	20	.50	10.50
10	10	.25	10.25

9.

Month	*Balance*	*Interest*	*Payment*
1	$150	$4.50	$14.50
2	140	4.20	14.20
3	130	3.90	13.90
4	120	3.60	13.60
5	110	3.30	13.30
6	100	3.00	13.00
7	90	2.70	12.70
8	80	2.40	12.40
9	70	2.10	12.10
10	60	1.80	11.80
11	50	1.50	11.50
12	40	1.20	11.20
13	30	.90	10.90
14	20	.60	10.60
15	10	.30	10.30

Total, $186

Page 421 **1.** The advantages for the customer are as follows: he is helped to save systematically to pay for the merchandise; he has the use of the merchandise while paying for it; he does not pay for credit unless he uses credit. The advantages for the merchant are as follows: the purchaser on credit pays for the expenses of clerical work and bad debits; he can sell for less to cash customers; trade is stimulated. **2.** The disadvantages for the customer are as follows: his budget may become so rigid that no allowance is made for emergencies and bargains; he may not be able to maintain payments; he may be tempted to purchase beyond his means. A disadvantage for the merchant is that trade drops off when the economy slows down more so than with cash sales. **3.** The cautions are as follows: reserve a margin for flexibility in the budget, do not purchase anything merely because you do not have to pay cash; always have a reserve in the savings bank to provide for payments when income is interrupted.

Page 424 **9.** construction and maintenance, $4.06 billion; interest, $322 million; local roads, $1.925 billion; police, $217 million; other, $476 million

Page 425 **10.** gasoline tax, $2.646 billion; borrowing, $1.59 billion; other taxes, $948 million; federal funds, $534 million; toll, $174 million; miscellaneous, $108 million

Page 426 **8.** depreciation, 3.20¢; liability insurance, 0.48¢; fire and theft insurance, 0.16¢; tires, 0.56¢; gasoline and oil, 2.64¢; maintenance, 0.80¢; license, 0.16¢

Page 430 **9.** first year, $960; second year, $480; third year, $240; fourth year, $240

Page 435 **10.** Some of the reasons why Jim's expenses were less are as follows: depreciation was 10% instead of 40%; Jim did not have fire and theft insurance; Jim's license fee was lower than the fee on a new car.

Page 439 **8.** The greatest number of fatal accidents occur later than the greatest number of all accidents due to dusk, crowded highways, tired drivers, and slower reactions. **9.** Although there are fewer drivers between midnight and 6 A.M., the per cent of all fatal accidents increases due to tiredness, slower reflexes, and greater speeds.

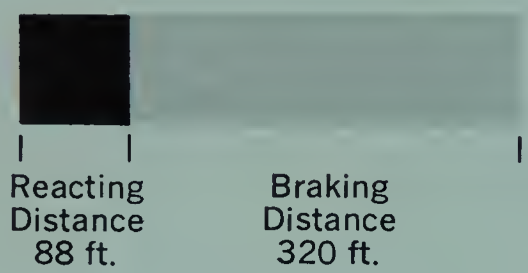
Page 440 1. 1940, 10%; 1945, 7%; 1950, 13%; 1955, 18%; 1960, 23%; 1965, 22%

Page 441 8.

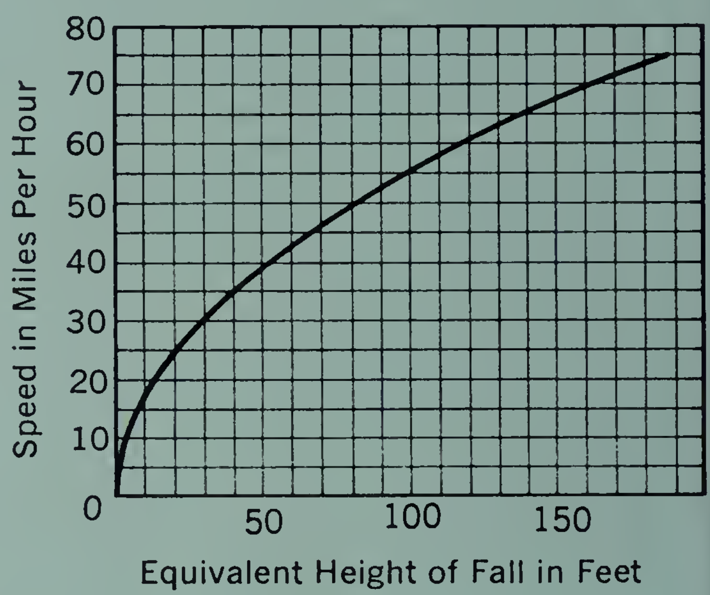
Year	*Trolley*	*Subway*	*Bus*	*Surface Railway*
1940	$\frac{1}{2}$ billion	$\frac{5}{2}$ billion	$\frac{9}{2}$ billion	$\frac{12}{2}$ billion
1945	$\frac{3}{2}$ billion	$\frac{6}{2}$ billion	$\frac{20}{2}$ billion	$\frac{19}{2}$ billion
1950	$\frac{4}{2}$ billion	$\frac{5}{2}$ billion	$\frac{19}{2}$ billion	$\frac{8}{2}$ billion
1965	$\frac{1}{2}$ billion	$\frac{4}{2}$ billion	$\frac{12}{2}$ billion	$\frac{1}{2}$ billion

9. 1940, $\frac{27}{2}$ billion; 1945, $\frac{48}{2}$ billion; 1950, $\frac{36}{2}$ billion; 1965, $\frac{18}{2}$ billion **10.** No, people are not traveling less, but they are using airplanes and automobiles in place of the transportation mentioned above. **11.** Airlines are not included in the graph because they are just beginning their career in public transportation.

Page 454 13.



Page 455 4.



BOOK TWO

General Mathematics

A PROBLEM SOLVING APPROACH

Lucien B. Kinney

Vincent Ruble

Gerald W. Brown

HOLT, RINEHART AND WINSTON, INC.
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BOOK TWO

General Mathematics

A PROBLEM SOLVING APPROACH

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Preface

GENERAL MATHEMATICS, A PROBLEM SOLVING APPROACH, Book Two is the second in a two-book series. Designed especially for students enrolled in general mathematics courses in the second, third, or fourth year of high school, it is intended to equip those students with the mathematics required for effective participation in modern society and to provide the foundation essential for further study of mathematics for those interested in such study.

Although this book is part of an integrated series, there is no necessary assumption that the student will have studied BOOK ONE in the series. In fact, the student may not have studied any mathematics since the eighth grade. While problem situations are somewhat more advanced than in Book One and mathematical concepts are treated at more mature level, all essential concepts and skills are fully developed from their beginnings.

The problem-solving approach used in this text provides an understanding and appreciation, not only of the role of mathematics in modern society, but also of the nature of mathematical thinking, of systems of numeration, and of computational processes as well. Mathematics began with man's efforts to understand and master nature, and further developed with his attempts to solve problems arising in science, business, and recreation. To be functional to the student, mathematics should be learned in the setting from which it was derived and in which it is to be used—in the solution of personal, vocational, and community problems.

By studying mathematics through problem solving, the student has an opportunity to see a wide variety of situations to which mathematics can be applied.

But dealing with mathematics in purely concrete terms is as incomplete an experience as dealing purely with its abstractions. Mathematical concepts are derived from concrete experiences only when they are explicitly examined and identified, their relationships explored, and the essential symbolism and vocabulary mastered. It is in this synthesis of the structure of mathematics, the abstract and the concrete, that the modern concepts, vocabulary, and point of view, sometimes known as "modern mathematics," make a useful contribution.

It is the hope of the authors that the students studying this book will come to see that one of the most interesting things in the study of mathematics is mathematics itself.

Contents

How to use the book

Mathematics is the language used to express functional ideas of science, technology, business, finance, sports, and recreation. Whenever you deal with a statement having to do with *how much* or *how many*, you are using mathematics. Today everyone needs an understanding of the nature and methods of mathematics to grasp what is going on in our rapidly changing and expanding society.

This textbook is designed to help you learn where and how to use mathematics. There are three major goals to keep in mind as you use the book:

1. To acquire skill in problem solving. Problem-solving skill can be developed through use of a systematic procedure. The text outlines this process and provides guidance for mastering it. You will find that the procedure encourages exploration of the data and originality in finding a solution. You can try out new ideas and look for patterns and clues without being afraid of making mistakes because you will learn a method to verify your results.

2. To develop accuracy in computation. Accuracy and speed in computation should be so automatic that you can give your undivided attention to solving the problem. Development of speed and accuracy is a relatively simple matter. Four points are essential: (a) identifying and correcting computational weaknesses; (b) knowing and practicing the correct techniques; (c) writing figures neatly and having an orderly ar-

rangement for your work; (d) checking your work. If you wish to improve your mathematics, you must identify and correct your computational weaknesses, *Inventory Tests* are provided to help you. Each test is carefully designed to show just which skill you need to develop. The *Practice Exercises* are keyed to the Inventory Tests, and provide just the practice you need. The correct form to use in practicing the computation is provided by showing you examples or referring you to a given topic.

3. To learn to understand and use the language of mathematics. Mathematics is a precise language. To express an idea precisely, your vocabulary must be precise. To master the vocabulary, note the list of *Words to Watch For* at the beginning of each chapter. These are key words and are explained and used in development of the chapter. As you study a chapter, identify the words and learn to use them correctly. A *Glossary* is provided so you can check the meaning of words that you have difficulty remembering.

Remember! Neatness in writing and in arranging your work is an effective key to problem solving. Even if your procedure is correct your answer will be wrong if you write a 4 that is mistaken for a 9 or 7. Your work should not be so disorganized that you lose track of your computations. In mathematics, clear and accurate recording of your work will avoid unnecessary mistakes due to carelessness. In the business world, misinterpretation and mistakes are often costly in terms of time, money, and energy to correct the harm done. In your work you should always strive to understand the mathematics as well as to get the correct answer.

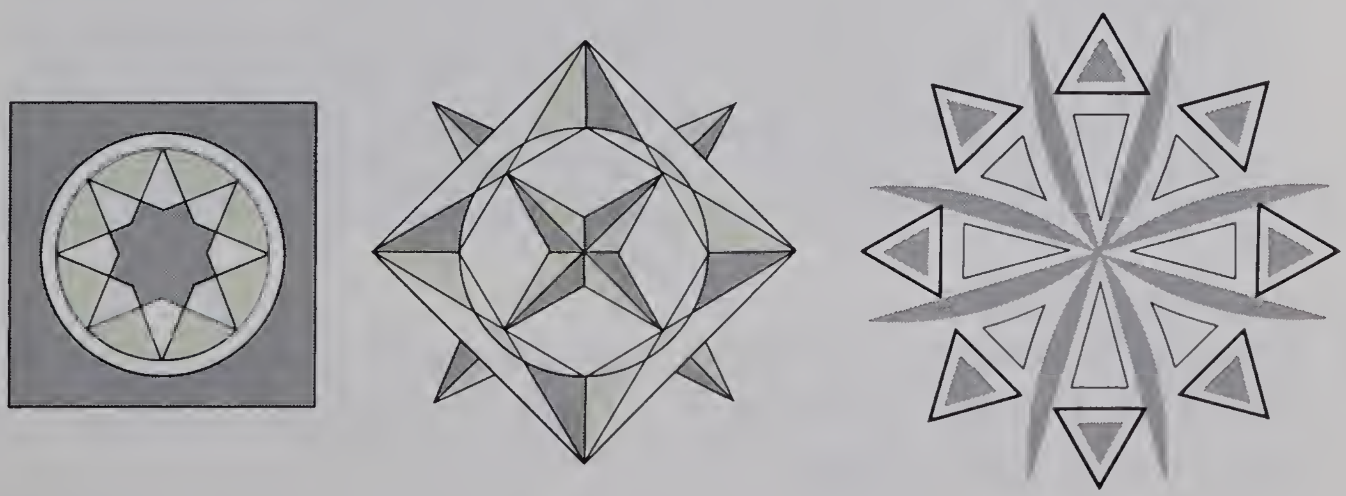
CHAPTER ONE

SYMMETRY

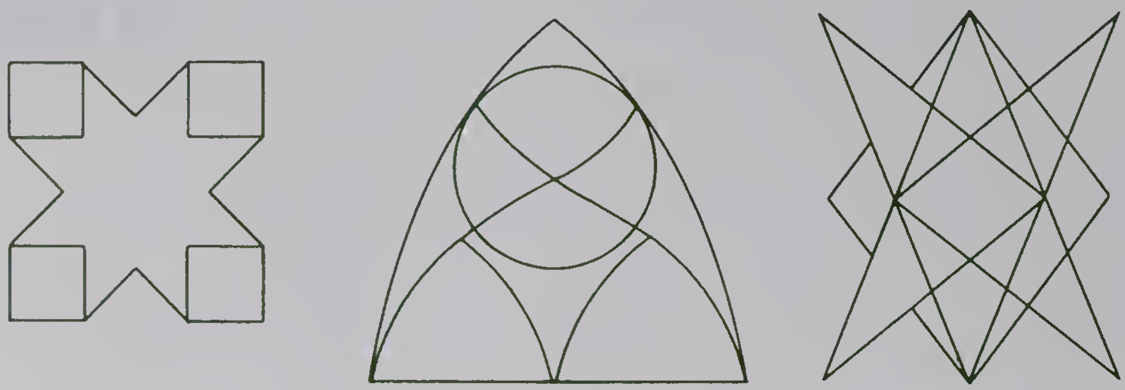
WORDS TO WATCH FOR

arc
axis of symmetry
chord
circumscribe
collinear
corresponding points
curved line
empty set
equidistant
equilateral
finite set
hexagon
horizontal
infinite set
inscribe
intersect
line symmetry
midpoint
oblique
parallel
perpendicular
point symmetry
polygon
regular polygon
set
skew lines
subset
symmetry
tangent
union
vertex
vertical

Symmetry is an interesting property of geometric figures, both from a practical and from a decorative point of view. The illustrations below provide examples of one type of symmetry in which each side of the figure is arranged about a central line, a line that divides the figure into two identical parts. That is, each side of the central line is like a reflection of the other side in a mirror. A figure that has this property has *line symmetry*.



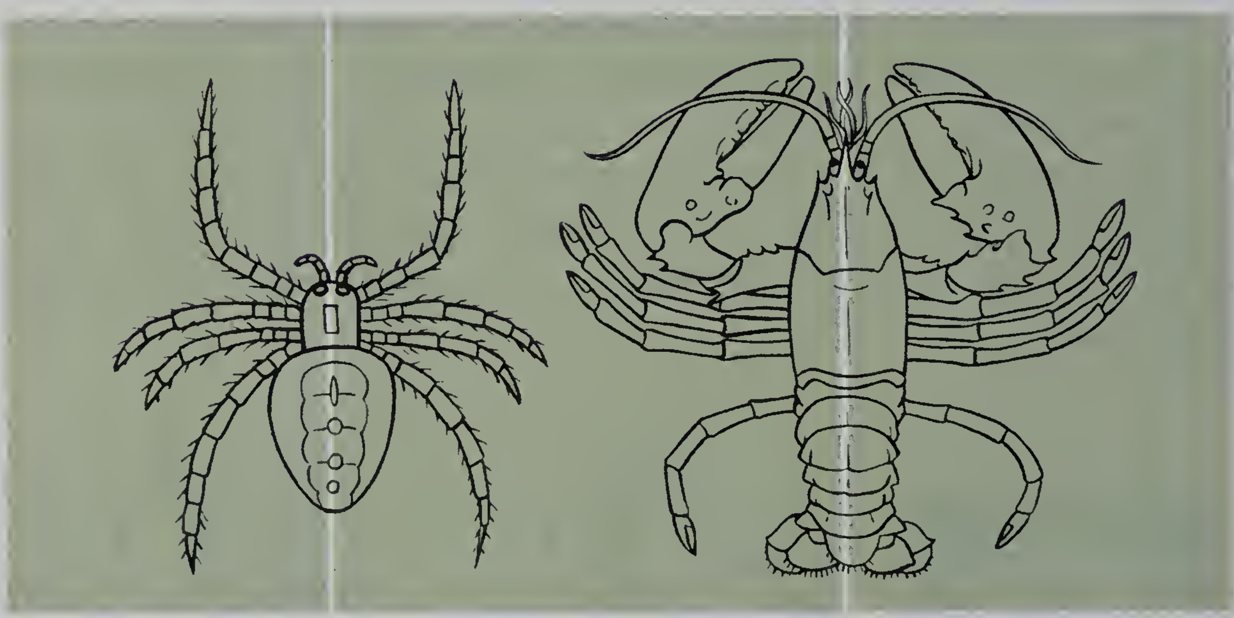
1. Copy one of the drawings below. Draw a line that divides it into two halves so that each is a reflection of the other. This line is an *axis of symmetry*. See if your drawing has more than one axis of symmetry.



2. Make some other drawings that have line symmetry. Draw, for example, a view of a speedboat from above. In each case show the axis of symmetry.

3. One example of a figure with more than one axis of symmetry is a triangle that has all sides equal in measure, that is, an *equilateral triangle*. Draw an equilateral triangle and see how many axes of symmetry you can draw on the figure. See front.

4. Many animals, birds, and fish as well as other living things have symmetric form. The front, dorsal, or ventral views, or a combination of these may reveal a natural symmetry. Draw or find a few examples of symmetry in living things and show the plane of symmetry.



5. Projectiles, aircraft, and racing cars are symmetric in design. Find a picture illustrating this fact. Draw the axis of symmetry and bring the picture to class.

6. Symmetric forms also occur among inanimate objects, especially in crystals. Snowflakes are examples of symmetric crystalline forms. Find some other examples of symmetric crystals.

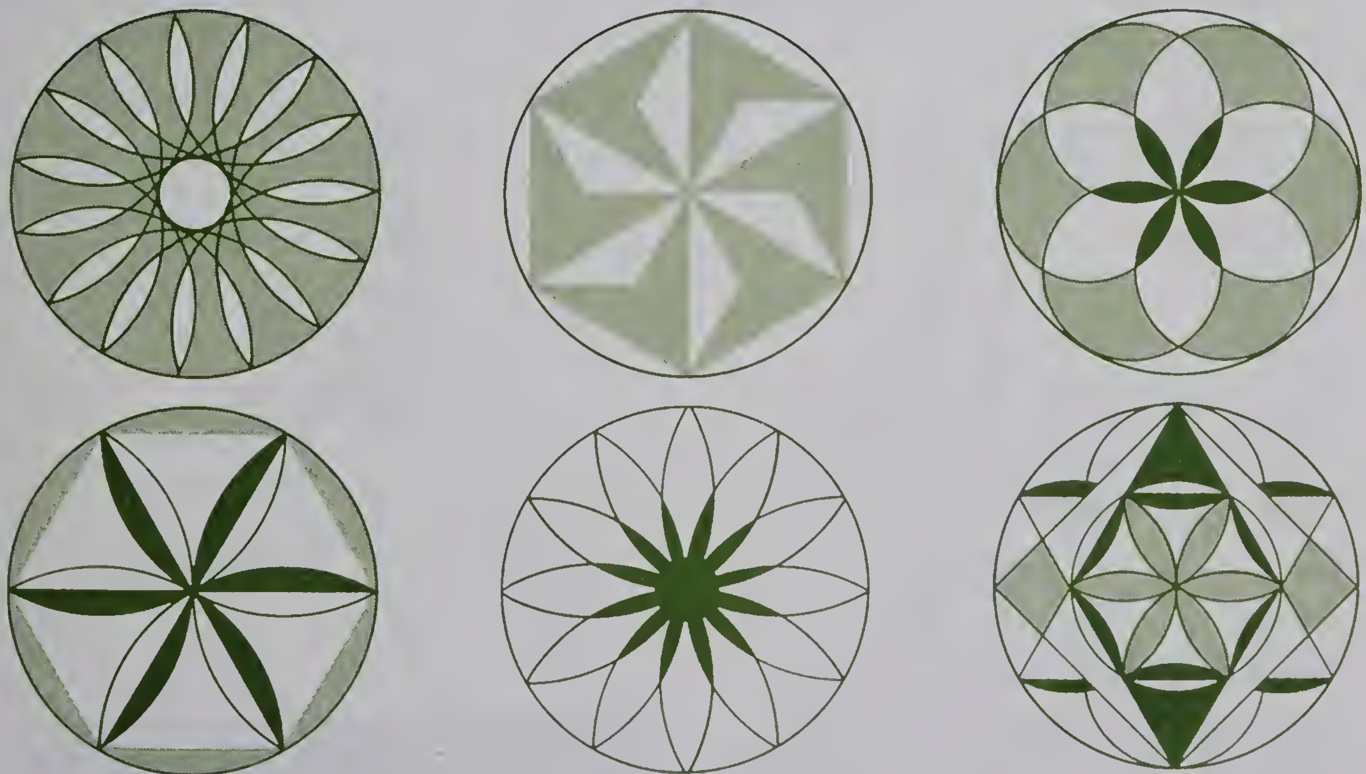
7. Snowflake designs are frequently used in the making of jewelry. Other designs that are symmetric are used. Find some pictures of symmetric designs which are used in jewelry and bring them to class.

8. In nature we find symmetry used as an efficient design of living and non-living matter. In general, symmetry is used by man for ornamental and decorative purposes and for designing objects that travel at high speeds. Find two examples of each of these uses.

9. Print the capital letters of the alphabet. Some of the letters have line symmetry. In the ones that do, draw an axis of symmetry. Which letters have more than one axis of symmetry? *See front.*

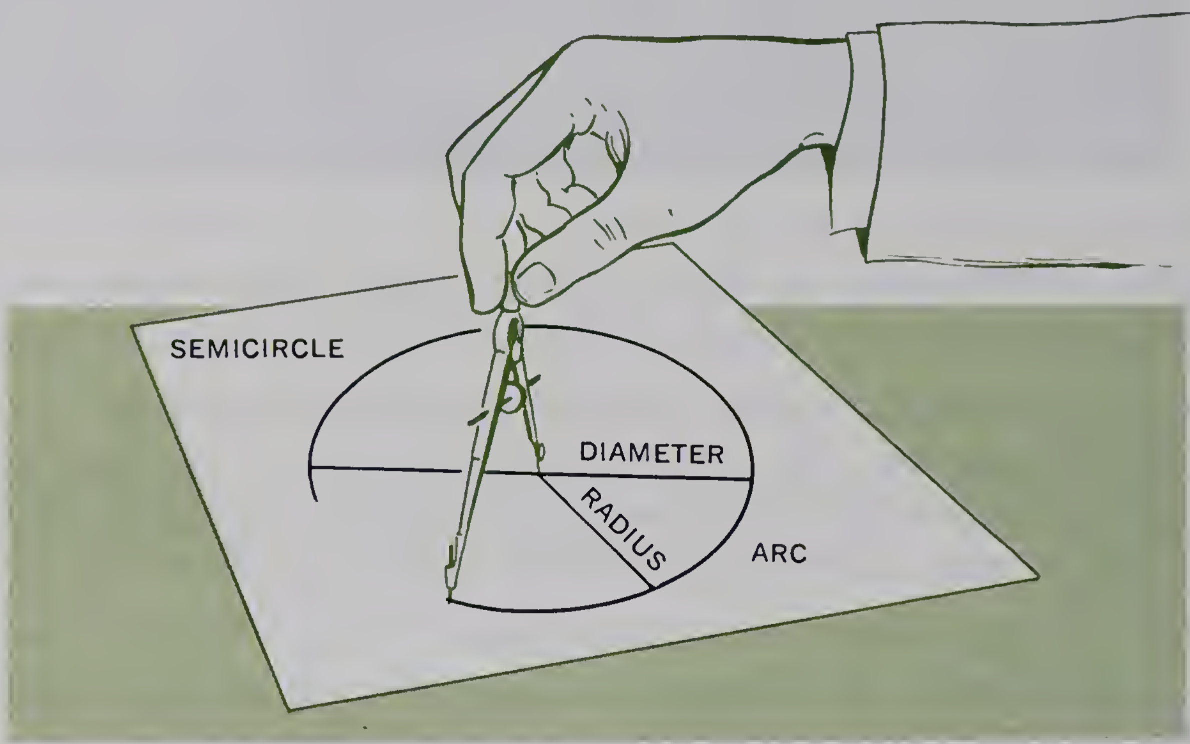
10. Draw a symmetric design. Thinking about what you had to do to make it symmetric, write a definition of line symmetry. *See front.*

11. Construct the following designs.



DESIGNS BASED ON HEXAGONS AND CIRCLES

A *compass* is a useful instrument for drawing circles and for determining equal measures. You will be using it for both purposes in this chapter. It is important to use it correctly. You should use only one hand in drawing the circle. Grasp the stem between the thumb and forefinger and draw the circle with a single stroke.



1. In the figure below on the right you see the correct method for setting the compass for a radius of a given length. Before you use a compass set the points at the same level. For what radius is the compass set? *3 inches*

2. Use your compass to draw circles that have radii with measure:

a. $2\frac{1}{4}''$ **c.** $2''$

b. $3''$ **d.** $1\frac{7}{8}''$

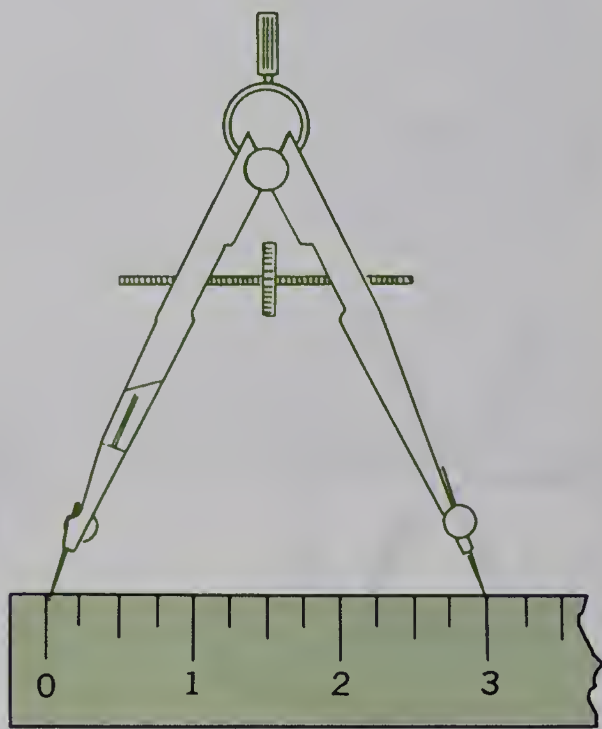
Now, use your compass to draw circles that have *diameters* with measure:

e. $1''$ **g.** $3\frac{1}{2}''$

f. $2\frac{3}{4}''$ **h.** $5\frac{7}{8}''$

Write a rule to describe how you would determine the radius of a circle if the measure of the diameter is given.

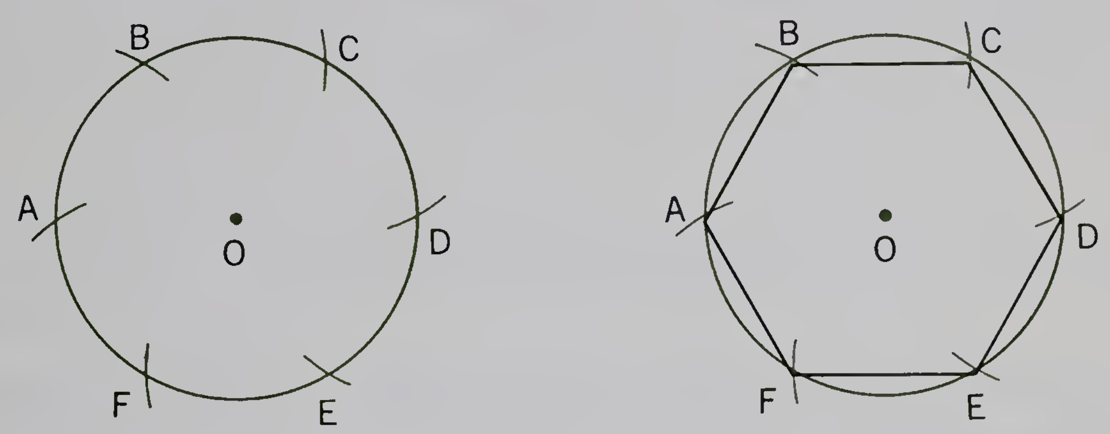
radius is $\frac{1}{2}$ the diameter



3. If you can use your compass accurately, you will find it easy to draw a regular hexagon. A *hexagon* is a six-sided geometric figure. The adjective "regular" means that each angle is the same measure and each side is the same length. Set your compass at $1\frac{1}{4}''$, and follow these steps to draw a regular hexagon. Do *not* change the setting of the compass during the entire construction.

a. Draw a circle with the given measure for the radius.

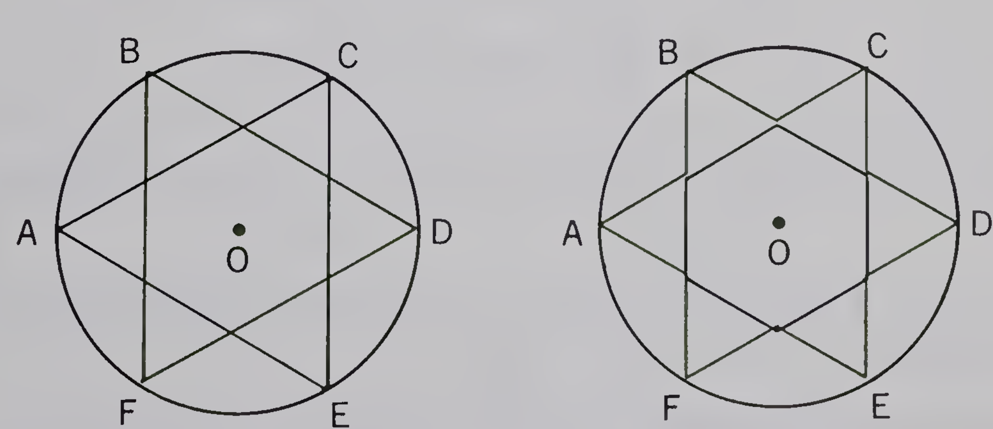
b. Set the point of the compass on the circle at a point A, and draw an arc intersecting the circle. Label this point of intersection B, and using B as center, mark and label point C; with C as center, mark and label point D; and so on around to locate F. See the left hand drawing below.



c. Connect the points as illustrated above at the right. Does this figure have the property of symmetry? *yes*

4. Draw two axes of symmetry on your hexagon. How many more can you draw? *four*

5. If you connect only the alternate points, say A, C, and E or B, D, and F, you will get an equilateral triangle. Construct an equilateral triangle in a circle with a $1''$ radius. Use the compass to check whether it is equilateral. See the figure below on the left.

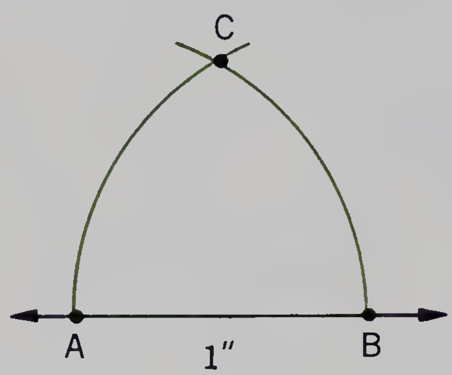


6. If you also connect the other three points on the circle you have a six-pointed star, as illustrated above on the right. Construct a six-pointed star in a circle with a $1\frac{1}{2}''$ radius.

7. The following steps show how to construct an equilateral triangle by a second method. We shall make each side 1 inch long. The compass setting will be 1 inch.

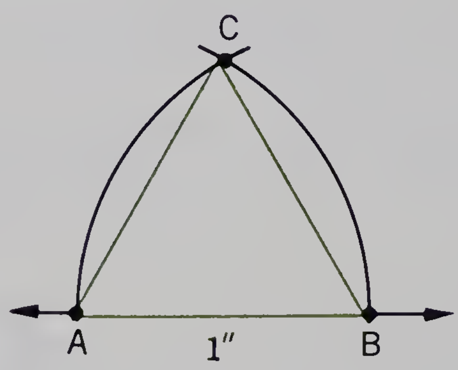
a. Draw a line and mark a segment of 1 inch and label the endpoints A and B. Remember? A *segment* is part of a line. It is a set of points consisting of the endpoints and all the points between them.

b. Place the point of the compass at A, and make an arc above side $\overline{AB}$. You may recall the bar above $\overline{AB}$ denotes a segment. Repeat this, placing the point of the compass at B. The two arcs should intersect as illustrated.



c. Label the point of intersection of the two arcs, C. Draw sides $\overline{AC}$ and $\overline{BC}$. Note: A set of arcs could intersect below the line segment, too. Then, we would obtain a triangle symmetric to the one we constructed.

Is this method of constructing an equilateral triangle easier than Exercise 5? yes



8. Construct equilateral triangles having a side of measure:

a. 2″ **b.** $1\frac{3}{8}$″ **c.** $2\frac{1}{2}$″ **d.** $3\frac{1}{8}$″

9. What is the perimeter of each of the above triangles that you have constructed? 6″, $4\frac{1}{8}$″, $7\frac{1}{2}$″, $9\frac{3}{8}$″

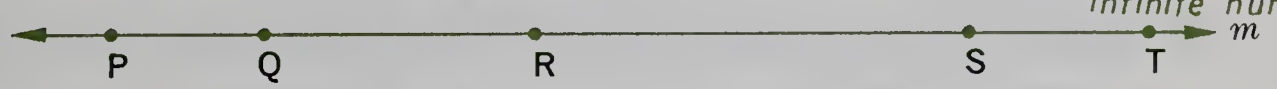
10. Construct an equilateral triangle that has a perimeter of: See front.

a. $2\frac{5}{8}$″ **b.** $4\frac{1}{2}$″ **c.** 9″ **d.** $7\frac{1}{8}$″

11. Describe a method to determine the measure of a side of an equilateral triangle when given the perimeter. $S = \frac{P}{3}$

We can get a better understanding of the figures in geometry by using the language of sets. You may have used the word set in your other mathematics classes in place of such words as group, family, or collection. A *set* is any well defined collection of things or elements. You can readily determine if an element is a member of the set or not. In your previous study of mathematics you have worked mostly with sets of numbers. In geometry we are usually concerned with sets of points.

1. Consider the line m. The arrow at each end indicates that the line extends indefinitely in each direction. A portion of the line is drawn here. How many points are in the set that makes line m? infinite number



2. Name three points of line m. P, Q, R

3. The elements of a set are said to belong to the set. Is point P an element of set m? yes

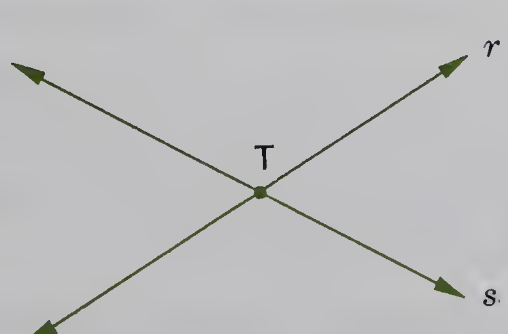
4. What other points are elements of set m? Q, R, S, T

5. A segment is a part of a line. It is the set consisting of two endpoints and all the points between these endpoints. Segment PQ (usually written as $\overline{PQ}$) is contained in line m. Every point of $\overline{PQ}$ is also a point of line m. Therefore, we say $\overline{PQ}$ is a subset of m. When each member of one set is a member of another set, we say the first set is a *subset* of the second. Name two other subsets of m.
$\overline{QR}, \overline{RS}, \overline{ST}$; $\overline{PR}, \overline{PS}, \overline{PT}$; $\overline{QS}, \overline{QT}, \overline{RT}$

6. Name a subset of $\overline{PR}$. $\overline{PQ}$, $\overline{QR}$

7. We say that $\overline{PR}$ is an infinite set of points. That is, we cannot count the number of points in the set. Is $\overline{PQ}$ an infinite set? yes

8. When two lines cross or meet, we say they intersect. In the drawing lines r and s intersect at point T. Can these lines also intersect at another point? no



9. Is the point T contained in line r? yes point T contained in line s? yes

Your ability to solve mathematical problems will be improved by special practice in giving attention to all the available facts, and the relationships among them. These exercises are designed to help you to see how this is done and how important it is in problem solving to give careful attention to the data provided with the problem.

STEPS FOR SOLVING MATHEMATICAL PROBLEMS

1. Understand the problem.
2. Analyze the data.
3. Discover new facts.
4. Follow up and verify promising leads.
5. Review your solution.

1. The intersection of two or more sets is a set consisting of all the elements that the two or more sets have in common. Is it possible to draw two lines that do not intersect? *yes, if they are parallel or in different planes*

2. When a set contains no elements, we call it an empty set. The intersection of two lines that never cross is an example of an empty set. Is it possible to have three lines that do not intersect? *yes*

3. Mark two points R and S on a sheet of paper. How many elements are in the set of all straight lines that contain both R and S? We say that two points determine one and only one straight line. *one*

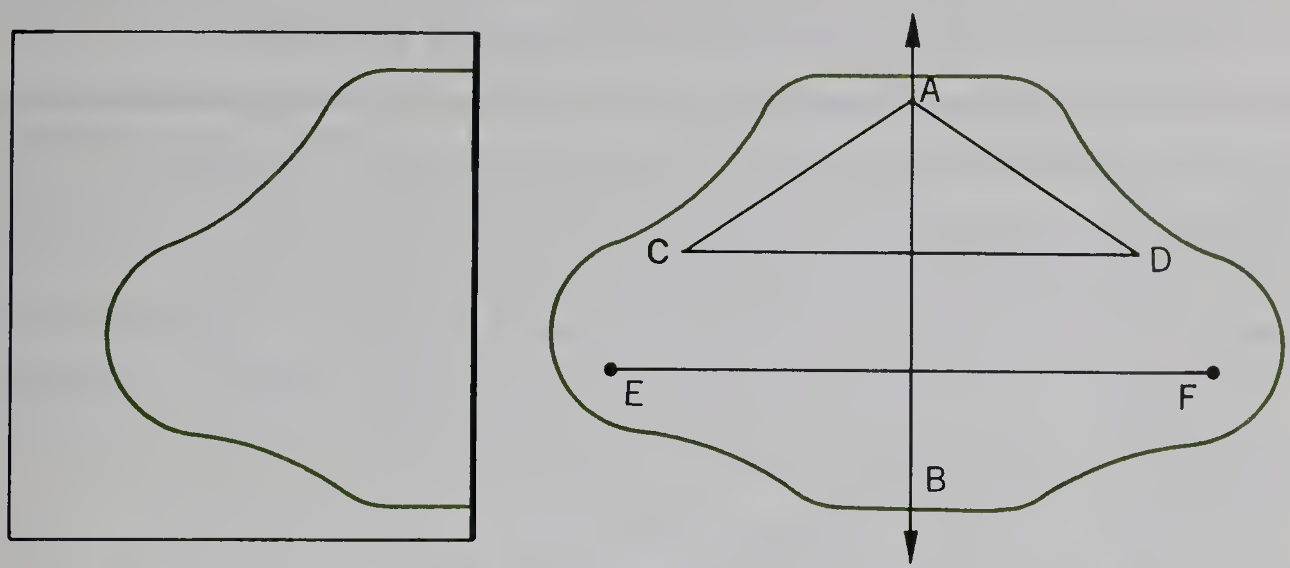
4. Mark three points M, N, and O on your paper so that the three points are not contained in the same line. How many lines are determined by these points? *three*

5. Mark four points on your paper so that no three of them are contained in the same line. How many straight lines are determined? *six*

6. Can you guess how many straight lines would be determined by five points placed so that no three are in the same straight line? *Noncollinear* points are not contained in the same straight line. On your paper, mark five points such that no three points are collinear. How many lines are determined? *ten*

7. Can you detect a pattern to help you predict how many straight lines would be determined by any number of points, say 10. If so, write a rule that you think will work and test it. $\frac{n(n-1)}{2}$, *where n is the number of sides*

1. Fold a piece of paper that measures about 2 inches on each side. Opposite the folded edge cut out a piece as shown. Unfold the paper and lay it flat. Label two points on the axis of symmetry A and B.



2. Refold the paper and push a pin or pencil point through both halves of the paper. Open the paper again, and label the holes C and D as illustrated. These are two corresponding points because each is a reflection of the other. Draw segment $\overline{CD}$. Label the point where segment $\overline{CD}$ crosses the axis of symmetry P. How do you know without measuring that the length of $\overline{PC}$ equals the length of $\overline{PD}$? Check with your ruler or compass to be sure. because P is on the axis of symmetry

3. What kind of angle is determined where $\overline{CD}$ crosses $\overline{AB}$? right angle

4. Refold the paper and push the pin or pencil point through both halves. Label these holes E and F. Draw the segment connecting E and F. What kind of angle is determined by $\overline{EF}$ and $\overline{AB}$? right angle

5. Does the axis of symmetry divide $\overline{EF}$ into two equal parts? yes

6. A line segment that is divided into two parts of equal measure is said to be bisected. The line that divides a segment equally is said to be a *bisector*. On your paper what segments are bisected? Which segment is the bisector? $\overline{CD}$, $\overline{EF}$; $\overline{AB}$

7. If a bisector is *perpendicular* and determines right angles to the segment it bisects, it is a *perpendicular bisector*. On your drawing, what is the perpendicular bisector? Of what segment or segments? $\overline{AB}$ $\overline{CD}$, $\overline{EF}$

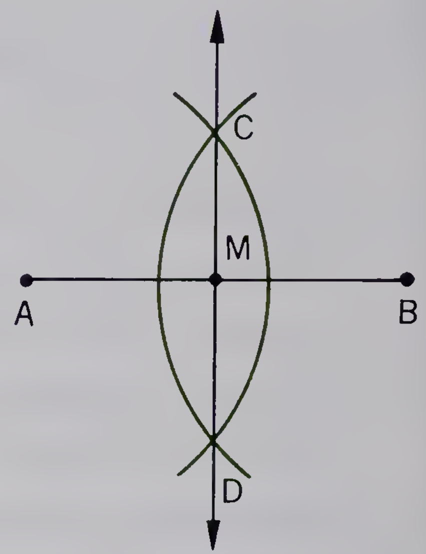
In line symmetry the axis of symmetry is the perpendicular bisector of any line segment joining two corresponding points.

---

8. From any point on $\overline{AB}$, draw segments to C, and D. How do the segments compare in length? Pick another point not on $\overline{AB}$ and draw segments to C and D, or to E and F. How do these segments compare in length? *equal ; not equal*

The segments drawn to two corresponding points from any point on the axis of symmetry will be equal in measure.

Later in the chapter you will learn the steps for using the axis of symmetry to construct the perpendicular bisector of a line segment. The figure shows how it is done. Set your compass at more than half the measure of the segment and strike an arc from each end of the segment to form two intersections, one above and one below the line. The line connecting these points will be the perpendicular bisector of the segment.



a. What is the line segment? *$\overline{AB}$*

b. What are the two intersections of the arcs? *C, D*

c. Compare the measure of $\overline{AC}$, $\overline{CB}$, $\overline{AD}$, and $\overline{DB}$. *equal*

d. What is the relationship of $\overleftrightarrow{CD}$ to $\overline{AB}$? *perpendicular bisector*

e. Is $\overleftrightarrow{CD}$ the axis of symmetry of the figure $ABCD$? *yes*

f. Are A and B corresponding points? *yes* How do you know this? *See front.*

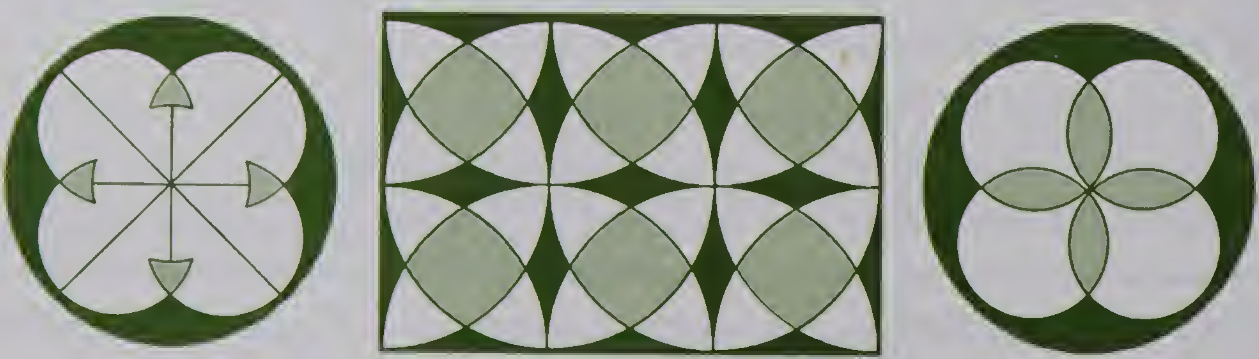
10. Draw a segment $1\frac{3}{4}''$ long and bisect it.

11. Draw a segment $2\frac{3}{8}''$ long and bisect it. Check with a ruler.

12. Draw a segment $3\frac{7}{8}''$ long and bisect it. Check with a ruler.

13. Draw a segment $4\frac{1}{8}''$ long and bisect it. Check with a ruler.

14. Construct the following designs.



CIRCLE AND SQUARE DESIGNS

In studying geometric shapes and figures, we are concerned with *sets of points.* In our discussion, "line" will always mean a straight line.

1. We cannot define *point* except to say that it has position but no dimensions. However, we may think of a small dot or the tip of a sharp instrument as the model of a point. Give some other models of points. intersection of two lines, corners in a room, point of a needle (answer not unique)

2. The examples given in Exercise 1 will serve as *models* of points. Imagine a small dot on your paper becoming smaller and smaller until it has no size at all but simply *determines a position.* This is a mathematician's idea of a point. How many points then could be placed on a dot the size of a period at the end of a sentence? infinite number

3. It may help you to answer Exercise 2 by imagining the period at the end of this sentence enlarged 1000 times. Would there then be space for many points? Since points have no size at all, when the period is reduced to normal size all the points that were on the enlargement can remain on the period. Can you erase a point? no

4. A line and a point are ideas rather than something that can be seen or touched. Is a line limited in length? Can a line exist without being drawn? no, yes

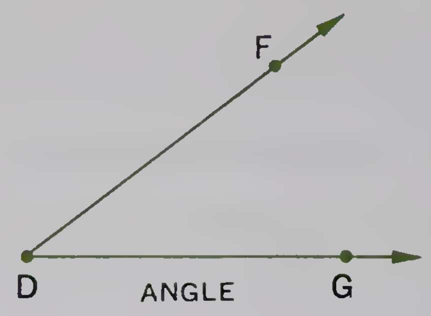
5. A line is thought of as a set of *points.* What then would be the width of a line? no width

6. A line has no endpoints. Notice the line illustrated below.

A and B are two points of the line. What tells you that the line extends indefinitely in both directions? The line above is "line $\overleftrightarrow{AB}$" or simply $\overleftrightarrow{AB}$. We use two points of a line to name it or sometimes we use a single letter, such as m. arrowheads

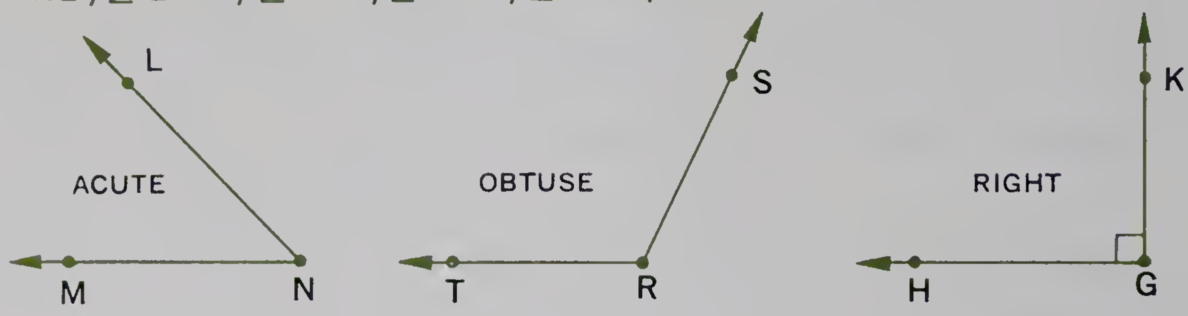
7. A geometric figure which has one endpoint and extends through another *point* and continues indefinitely in that direction is called a *ray.* What in the figure on the right tells you that the ray is extended indefinitely in only one direction? The ray is named ray $\overrightarrow{CE}$ or simply $\overrightarrow{CE}$. It can also be named $\overleftarrow{EC}$. What does the arrow over $\overrightarrow{CE}$ and $\overleftarrow{EC}$ mean? C is the endpoint of each of the rays.
the arrowhead

8. Two rays having the same endpoint but not on the same line form an *angle*. The point D in the figure on the right is the endpoint of what two rays? We call D the *vertex* of the angle. $\overrightarrow{DG}$ and $\overrightarrow{DF}$



An angle is generally named by using three letters, namely, a point on each of the rays and the point at the vertex. Thus the angle in the figure above is angle FDG, or $\angle FDG$ (or $\angle GDF$).

9. Using three letters, name the angles below in two ways.
$\angle MNL$, $\angle LNM$; $\angle TRS$, $\angle SRT$; $\angle HGK$, $\angle KGH$



TYPES OF ANGLES

10. As stated in Exercise 8, the common endpoint of the two rays is the vertex of the angle. What is the vertex of each angle in the figures of Exercise 8? N, R, G

11. If we now consider two points on a line, A and C, and all points of the line between these two points, we have a line segment AC, or simply $\overline{AC}$, which we discussed briefly on page 6. Name two other segments of line b. $\overline{AB}$, $\overline{BC}$

See front.

12. Mark two points on your paper and name them A and B. Draw ray $\overrightarrow{AB}$. Locate a point C such that a line cannot be drawn through A, B, and C. Thus we say that A, B, and C are not *collinear*.

13. If these points were located such that a line could be drawn through them, then they would be collinear. Draw three collinear points. Draw ray $\overrightarrow{AC}$. Locate a point X on $\overrightarrow{AC}$ and a point Y on $\overrightarrow{AB}$. Draw $\overline{XY}$. See front.

A *plane* is a set of points that make up a flat surface extending in all directions without limit. A plane can also be thought of as a set of non-skew lines. *Skew lines* are in three dimensional space and are neither parallel nor intersecting lines.

1. You can represent a plane with a sheet of paper and lines by pencil marks. Draw two lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, that intersect at a point O. Complete the statement: $\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = \square$. *O*
2. Explain the statement: Two intersecting lines determine a plane.
   Two intersecting lines lie in exactly one and the same plane.
3. Draw two parallel lines RS and TU. Do two parallel lines determine a plane? *yes*
4. A *polygon* is a plane closed geometric figure bounded by segments. A triangle is bounded by how many segments? *3*
5. Polygons are identified by their number of sides. The quadrilateral, pentagon, hexagon, and octagon have how many sides? *4,5,6,8*
6. A *regular polygon* is one that has all angles of equal measure, and sides equal in length. Are the polygons illustrated below models of regular polygons? *yes*

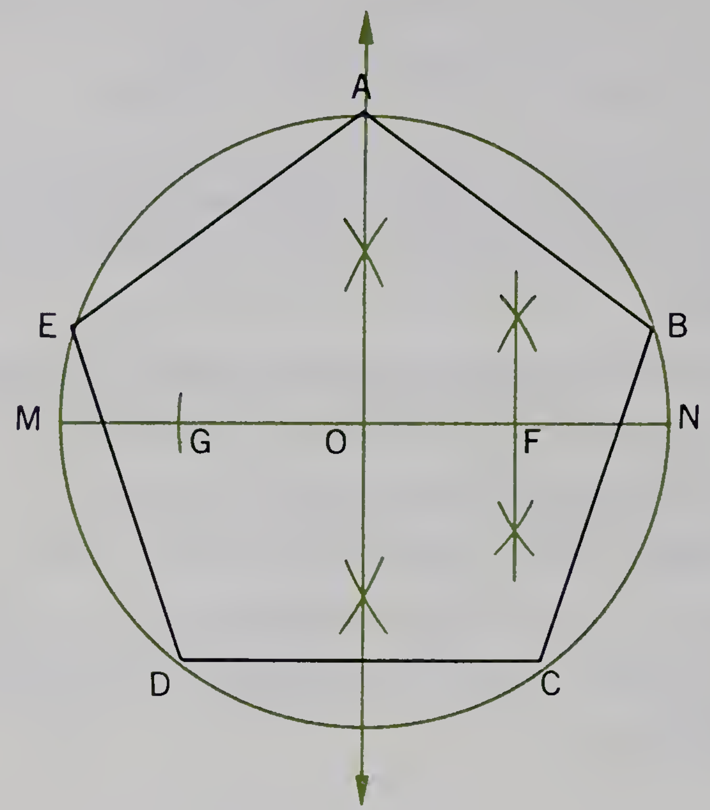
7. What is another name for a regular quadrilateral? *square*
8. Copy and complete the table:

Regular polygon	*No. of axes of symmetry*
Triangle	$\square$ 3
Quadrilateral	$\square$ 4
Pentagon	$\square$ 5
Hexagon	$\square$ 6
Octagon	$\square$ 8

9. All regular polygons possess symmetry. How many axes of symmetry does an equilateral triangle have? *3* a regular nonagon? *9*

In this chapter you have learned to construct a regular triangle, quadrilateral, and hexagon. Constructing a regular pentagon is a little more difficult, but you will find the construction interesting.

1. Study the figure below and follow the steps that describe how to perform the construction of a regular pentagon.



a. Construct a circle O of a convenient radius.

b. Draw $\overline{MN}$, a diameter of the circle.

c. Construct the perpendicular bisector of $\overline{MN}$ and label the bisector $\overline{OA}$.

d. Find the perpendicular bisector of $\overline{ON}$ and mark the midpoint F.

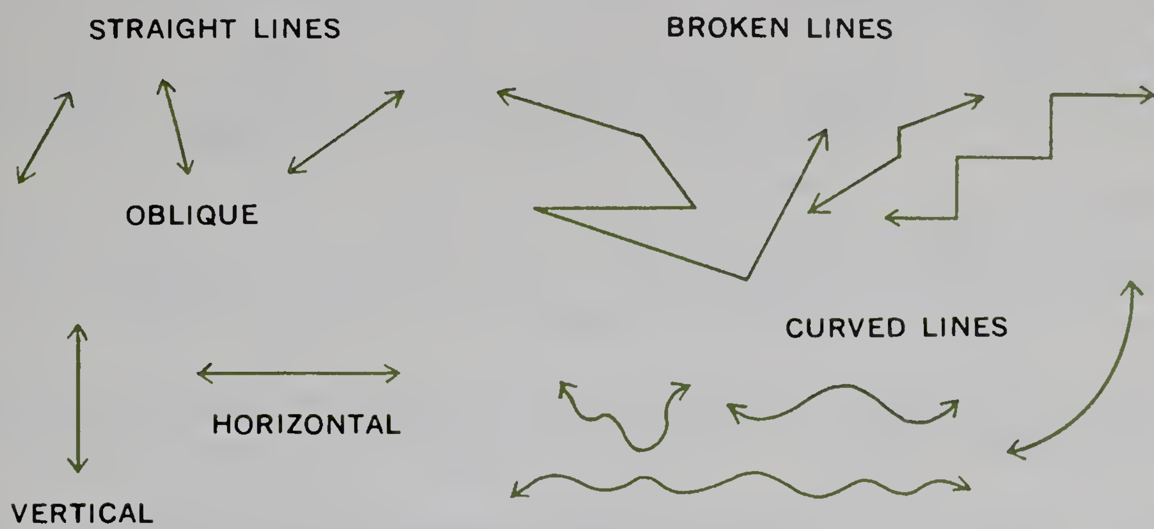
e. With point F as center, and radius equal in measure to FA, mark an arc intersecting $\overline{MO}$. Label the point of intersection G.

f. Now set your compass to a radius equal in measure to AG. This measure represents the length of each of the sides of the regular pentagon. Use it to determine points E, D, C, and B. If your construction is carefully done, the arcs should be equal in length.

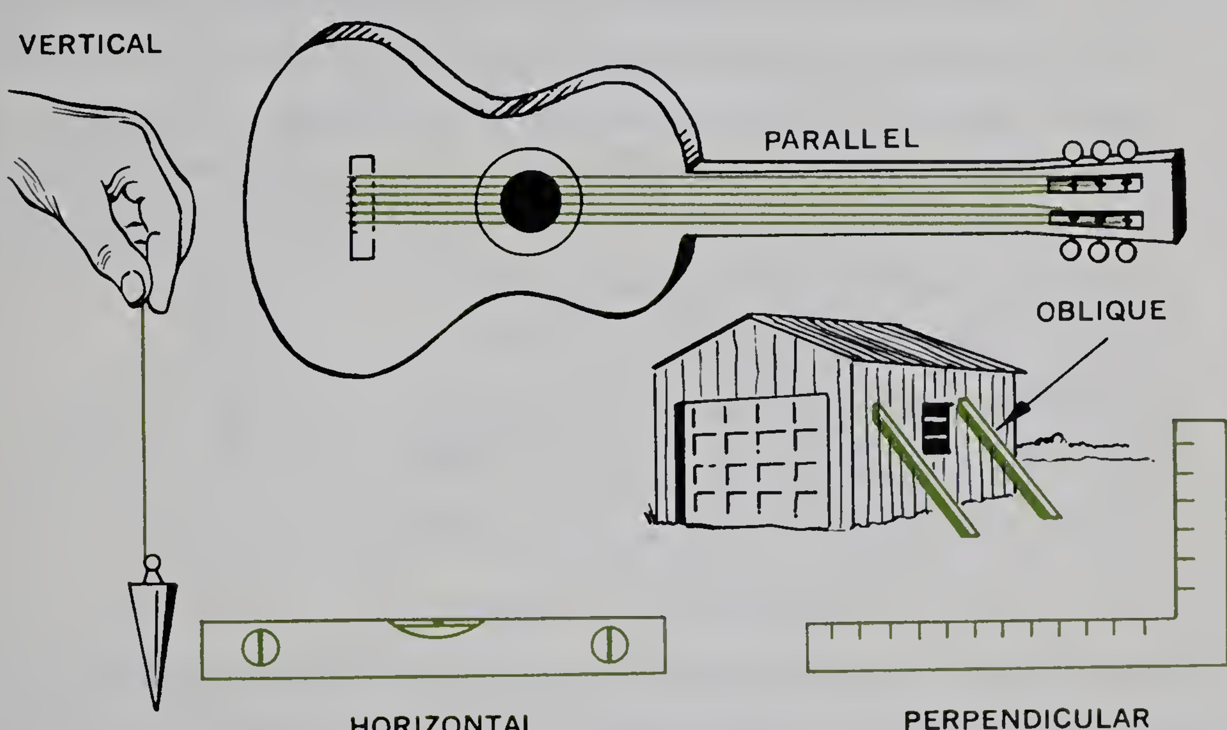
g. Draw segments connecting A and E, E and D, and so forth. Construct a regular pentagon in a circle with radius measuring 2 inches.

2. By connecting alternate points on the circumference of the circle (as A and C, C and E, and so forth), you can draw a 5-point star. Construct a 5-point star in a circle with a radius measuring $2\frac{1}{2}$ inches. How many axes of symmetry does the star have? 5

In geometry we deal with many kinds of lines. Below you see models of straight lines, broken lines, and curved lines. These are the basic kinds of lines that you will be most concerned with in your study.



1. The stretched string of a guitar suggests a straight line. Give five other examples of models that suggest straight lines. Why is it not possible to find a picture or example of a line itself? *A line has no width.*



2. Straight lines can be classified according to position. A string with a weight attached assumes a vertical position. The figure above is a model of a vertical line. Name other examples that suggest vertical lines. *Answers will vary.*

3. A carpenter uses a level to be certain that his work is horizontal. A match stick floating in a pan of water is in a horizontal position. Name four other examples that suggest horizontal lines. *Answers will vary.*

4. As you know, perpendicular lines intersect to determine right angles. A carpenter's square is designed to construct right angles. Name other models of perpendicular lines. *Answers will vary.*

5. The wires used to support a tree, or a ladder leaning against the side of a house suggest oblique lines. Name three other models of oblique lines. Answers will vary.

6. With a simple sketch, show that two oblique lines can be perpendicular to one another.

7. Curved lines are common in nature and in designs. A piece of rope held loosely at both ends is a model of a curved line. Give two other examples. Answers will vary.

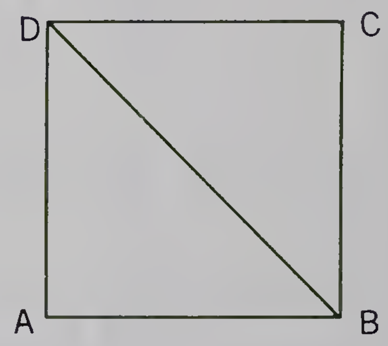
8. A broken line is suggested by the outline of a house. Give three other examples. Answers will vary.

9. In the same plane, straight lines that have no points in common are called *parallel lines*. Can two parallel lines intersect? no

10. Tightly stretched telephone lines suggest parallel lines. Give two other examples. Answers will vary.

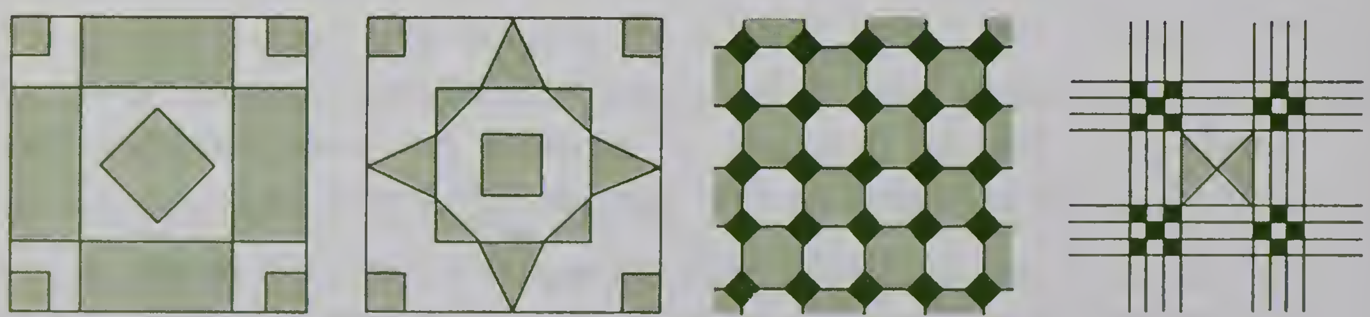
11. The regular polygon pictured below is a model of a square. A *square* is a quadrilateral in which the opposite sides are parallel, all sides have the same measure, and the angles determined by the sides are right angles. In the square, $\overline{BD}$ is called a *diagonal*, as the endpoints B and D are non-adjacent vertices of the figure. Refer to the drawing of the square to answer the following questions.

   a. Name two horizontal segments. $\overline{AB}$, $\overline{DC}$
   b. Name two vertical segments. $\overline{DA}$, $\overline{CB}$
   c. Name two segments that are parallel. $\overline{AB}$ and $\overline{DC}$ or $\overline{DA}$ and $\overline{CB}$
   d. Name one segment that is oblique. $\overline{BD}$
   e. Name an example of a broken line. $\overleftrightarrow{DCB}$
   f. Name another diagonal in the square. $\overline{AC}$



12. In the illustrations on the previous page, you see a plumb bob. This is used a great deal in construction. Use a dictionary or other source of reference to find out how a plumb or plumb line is used. See front.

13. Construct the following designs.



DESIGNS BASED ON SQUARES

Bisecting Segments

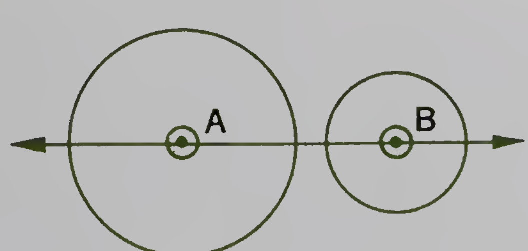
What you have learned about the axis of symmetry and corresponding points is useful in discovering how to make many important constructions. The following exercises, for example, will develop the background for bisecting a segment.

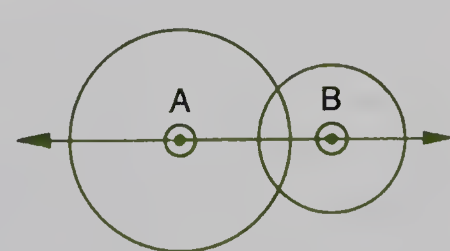
1. Draw a circle. Put a small O around its center and label the point A. We usually name a circle by its center. How would you draw an axis of symmetry of circle A? *a line passing through A*

2. How many axes of symmetry could you draw for circle A?
an infinite number

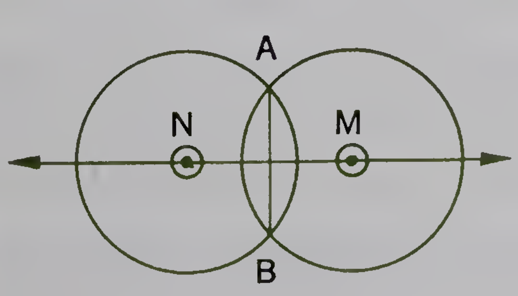
3. If you have a circle and want to draw an axis of symmetry, through what one point must the axis always pass? *the center* The axis of symmetry also passes through two points on the circle. If we consider these two points as endpoints of a segment, this segment will be a diameter.

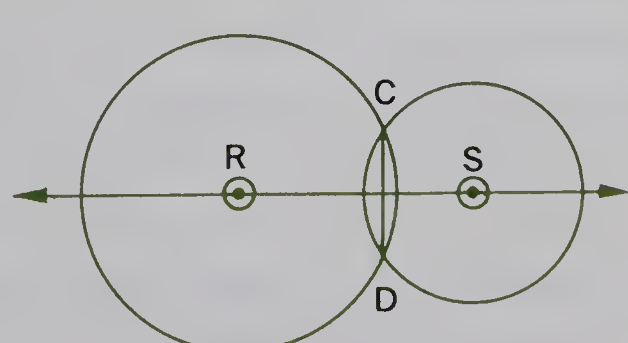
4. Any line which passes through the center of each of two circles divides both circles symmetrically. Use your compass and make drawings like those shown below. Label the centers of the circles A and B respectively. Draw the axis of symmetry for each pair of circles as indicated.





5. Use your compass to determine if the segments $\overline{AB}$ and $\overline{CD}$ that join the intersections of the two circles shown below have been bisected by the axis of symmetry. The axis of symmetry for each of the segments, $\overline{AB}$ and $\overline{CD}$, must pass through what points? We call any point that separates a segment into two equal measures, the *midpoint* of the segment. *yes; N and M, R and S*





6. In Figure 1, A is one of what pair of corresponding points? *A and B*

7. How can you draw the axis of symmetry to bisect $\overline{AB}$ in Figure 1?
It passes through the centers of both circles.

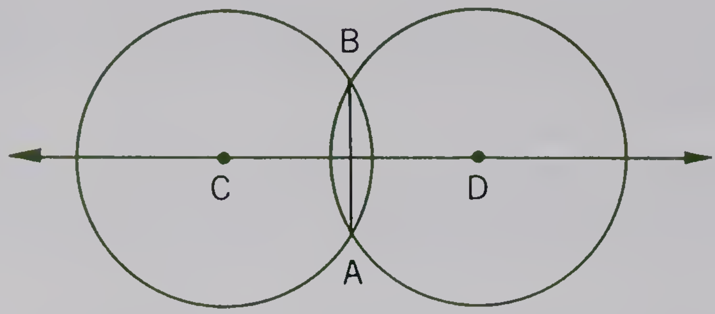


Figure 1

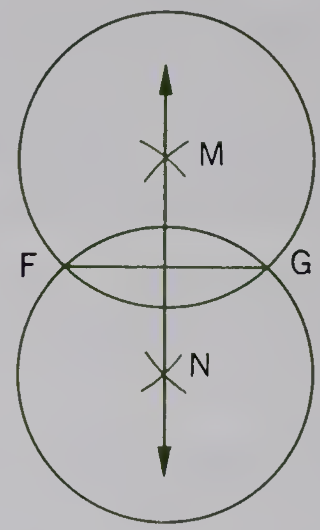


Figure 2

8. To construct the perpendicular bisector of $\overline{FG}$ in Figure 2 you must find the centers of two circles that pass through the endpoints of $\overline{FG}$. Make the construction following the steps below.

a. Set your compass at a measure greater in length than half the measure of $\overline{FG}$. Strike arcs above and below $\overline{FG}$ using as center first F, and then G. Label the points where the arcs intersect M and N.

b. Draw the line through M and N.

c. Thus, $\overleftrightarrow{MN}$ is the perpendicular bisector of $\overline{FG}$. To construct a perpendicular bisector, it is not necessary to draw the circles that pass through the endpoints of the segment that is to be bisected. Instead, we simply find the centers of the two circles that *would* pass through the endpoints if they were drawn. Compare this with Exercise 9, page 10.

9. Draw three segments with the following measures. Construct their perpendicular bisectors.

$AB = 2\frac{9}{16}''$ $\qquad CD = 3\frac{1}{16}''$ $\qquad EF = 4\frac{1}{8}''$

Note: When we refer to the measure of a segment, we do not have a bar over the letters naming the segment. Thus, $AB = 2\frac{9}{16}''$ is read, "the measure of segment AB is $2\frac{9}{16}$ inches."

10. To determine whether you bisected $\overline{FG}$ accurately in Exercise 8, set the point of your compass at F and adjust its radius to the length of one-half FG (as FP in Figure 2). Then with P as center, see if PF equals PG. Use this method to test your constructions in Exercise 9.

11. Construct the perpendicular bisectors of segments $3\frac{3}{4}''$ and $2\frac{3}{8}''$. Test your constructions.

It is often necessary to construct the perpendicular bisector of a segment that is a side of a triangle. First, let us examine the steps for constructing a scalene triangle. Remember that each of the sides of a *scalene triangle* has a different measure.

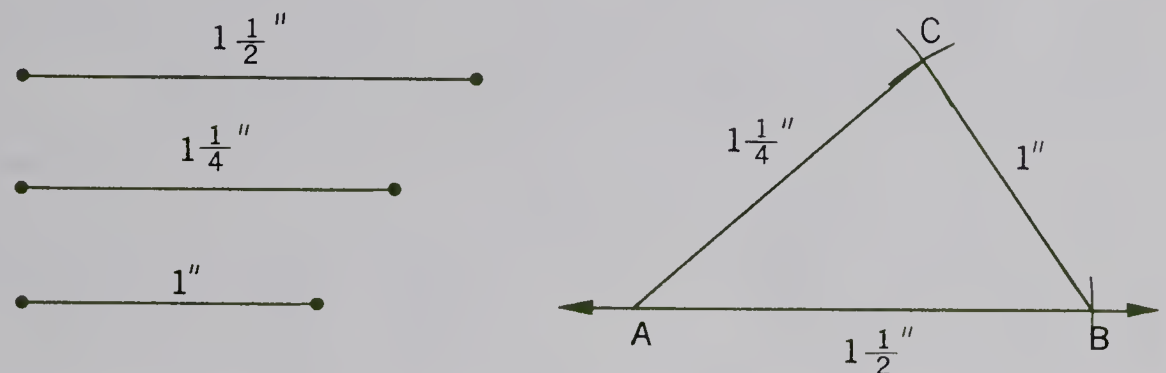
1. Construct a triangle following these directions.

a. Draw a line somewhat longer than $\overline{AB}$ (See the illustration below.) Mark $\overline{AB}$ to the given length with the compass.

b. Set the compass equal in measure to AC. With A as center, strike an arc above $\overline{AB}$.

c. Set the compass equal in measure to BC. With B as center, strike an arc above $\overline{AB}$. Where the two arcs intersect is vertex C.

d. Draw segments $\overline{BC}$ and $\overline{AC}$. Check the measures of each segment and side.



2. Draw a scalene triangle, one of whose angles is obtuse. Remember, the measure of an *obtuse angle* is greater than that of a right angle. Construct the perpendicular bisector of each side of the triangle. Do these perpendicular bisectors appear to meet? If so, do they meet outside the triangle or inside? *yes ; outside*

3. In your construction for Exercise 2, is the point of intersection of the three perpendicular bisectors at an equal distance from the three vertices of the triangle? Use your compass to check. *yes*

4. Notice that the distance from this point to a vertex could be used as the measure for the radius of a circle that would contain the three vertices of the triangle. To show this principle, use the point of intersection in your construction as the center of a circle and use as the radius the distance from this point to one of the vertices of the triangle.

5. Construct a triangle with sides of $2''$, $3''$, and $3''$. Repeat the procedures discussed in Exercises 2 and 3.

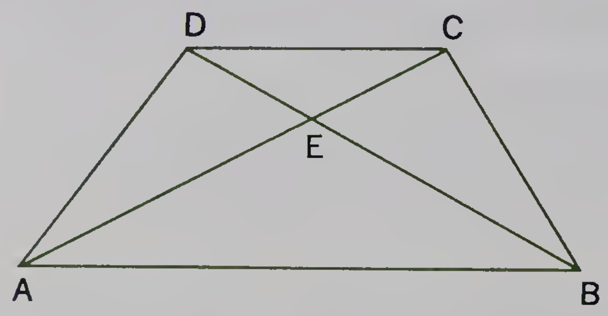
1. The *intersection* of two sets is the set consisting of only those elements which belong to both sets. In the quadrilateral $ABCD$ below, segments AD and DC intersect at D. Using the language of sets we write: $\overline{AD} \cap \overline{DC} = \{D\}$. The intersection of $\overline{AD}$ and $\overline{DC}$ is the point D. What points are at the intersections of these sets?

a. $\overline{AB} \cap \overline{DB} = \square$ B

b. $\overline{AC} \cap \overline{BD} = \square$ E

c. $\overline{BC} \cap \overline{DC} = \square$ C

d. $\overline{AE} \cap \overline{EB} = \square$ E



2. The *union* of two sets includes all elements of both sets. The union of $\overline{PQ}$ and $\overline{QR}$ determines $\overline{PR}$. Using the language of sets we can write $\overline{PQ} \cup \overline{QR} = \overline{PR}$. The symbol $\cup$ means "the union of." What elements are included in the union of these sets?

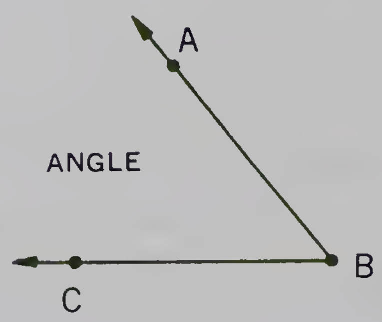
a. $\overline{PR} \cup \overline{RT} = \square$ $\overline{PT}$

b. $\overline{TS} \cup \overline{SQ} = \square$ $\overline{QT}$

c. $\overline{QR} \cup \overline{RS} = \square$ $\overline{QS}$

d. $\overline{PT} \cup \overline{RS} = \square$ $\overline{PT}$

3. We define an angle as the union of two non-collinear rays with a common endpoint. The common point is called the *vertex* and the rays are called the *sides* of the angle. What is the common endpoint in the angle below? B



4. The quadrilateral, a four-sided polygon, of Exercise 1 can be thought of as the union of four line segments. Explain what is meant by $\overline{AD} \cup \overline{DC} \cup \overline{CB} \cup \overline{BA}$. guadrilateral ABCD

5. Refer to the figure for Exercise 1 and answer the following:

a. $\overrightarrow{BA} \cup \overrightarrow{BC} = \square$ $\angle ABC$

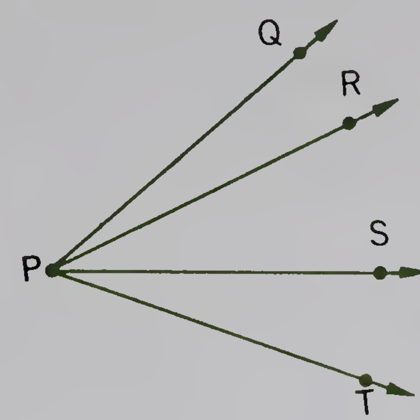
b. $\overline{DE} \cup \overline{EB} = \square$ $\overline{DB}$

c. $\overline{AE} \cup \overline{EC} = \square$ $\overline{AC}$

d. $(\overline{DE} \cup \overline{EC}) \cup \overline{DC} = \square$ $\angle DEC$

6. We say that the union of ray $\overrightarrow{BA}$ and ray $\overrightarrow{BC}$ equals angle ABC. Using the language of sets, we write $\overrightarrow{BA} \cup \overrightarrow{BC} = \angle ABC$. Refer to the diagram below and answer the following:

a. $\angle QPR$
b. $\angle SPT$
c. $\angle RPS$
d. $\angle QPT$
e. $\overrightarrow{PT} \cup \overrightarrow{PR}$
f. $\overrightarrow{PQ} \cup \overrightarrow{PS}$
g. $\overrightarrow{PR} \cup \overrightarrow{PT}$
h. $\overrightarrow{PQ} \cup \overrightarrow{PT}$
i. $\overrightarrow{PS} \cup \overrightarrow{PT}$

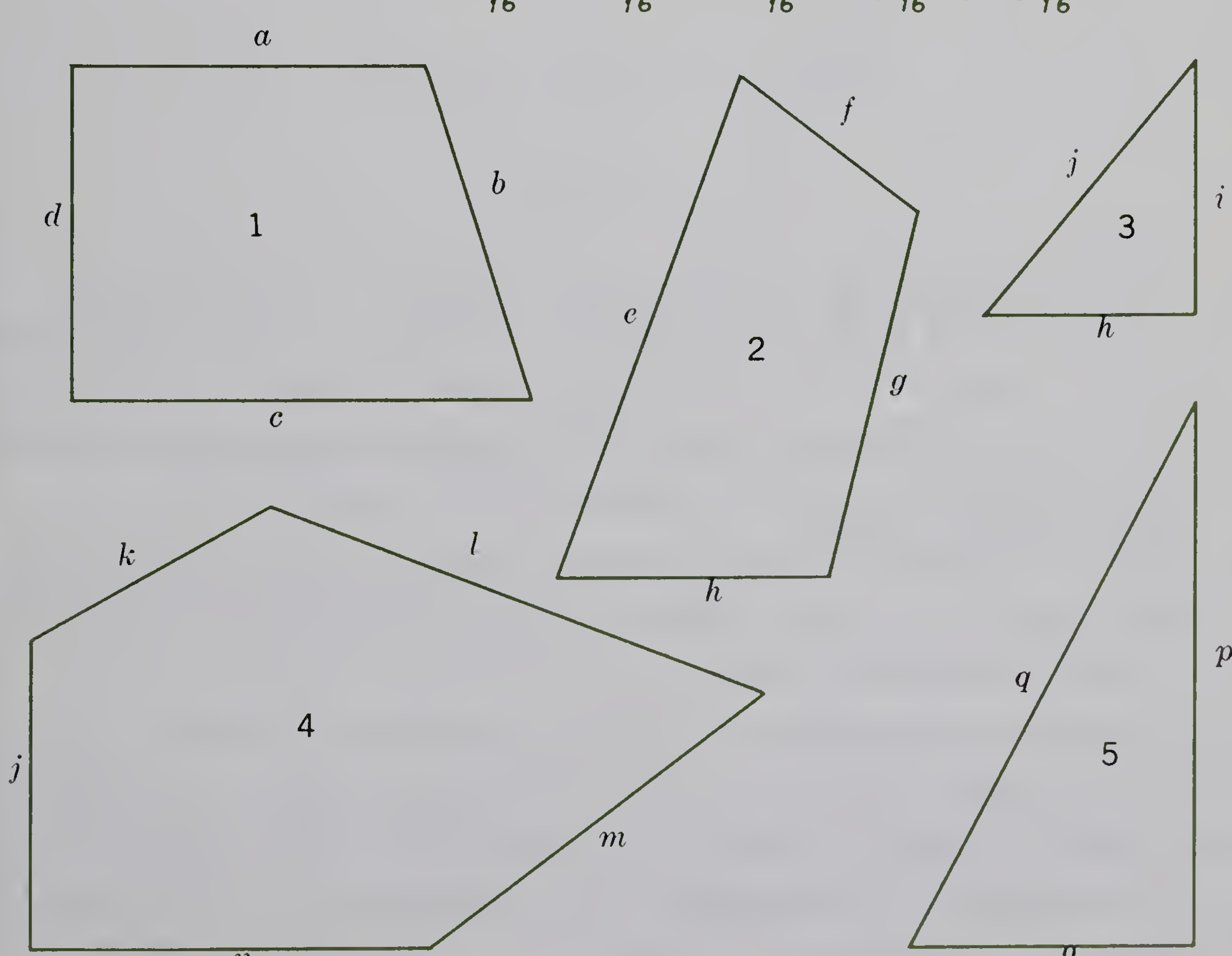


a. $\overrightarrow{PQ} \cup \overrightarrow{PR} = \square$ **d.** $\overrightarrow{PQ} \cup \overrightarrow{PT} = \square$ **g.** $\angle RPT = \square \cup \square$

b. $\overrightarrow{PS} \cup \overrightarrow{PT} = \square$ **e.** $\angle TPR = \square \cup \square$ **h.** $\angle QPT = \square \cup \square$

c. $\overrightarrow{PR} \cup \overrightarrow{PS} = \square$ **f.** $\angle QPS = \square \cup \square$ **i.** $\angle SPT = \square \cup \square$

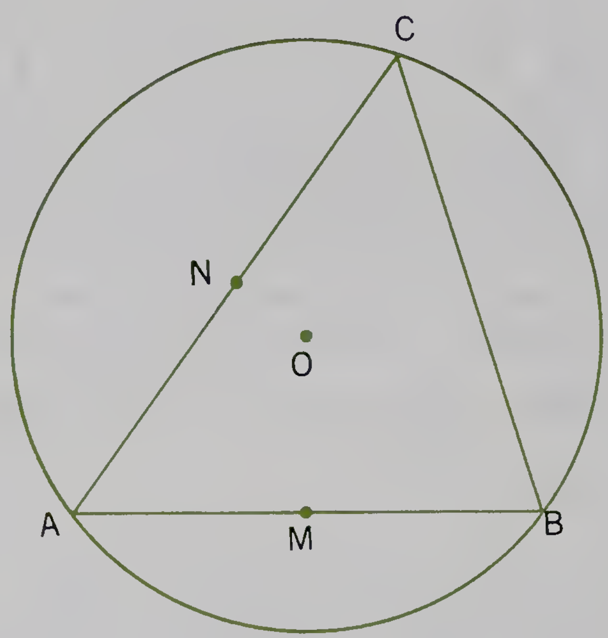
7. One of the skills that you should have is the ability to measure the lengths of segments to the nearest $\frac{1}{16}''$. Each of the following figures is formed by the union of line segments. Measure each segment and find the perimeter of each figure. Record your results in an organized fashion. Make all measurements to the nearest sixteenth of an inch. 1. $5\frac{5}{16}''$ 2. $4\frac{15}{16}''$ 3. $2\frac{13}{16}''$ 4. $6\frac{14}{16}''$ 5. $5\frac{2}{16}''$



The Inscribed Triangle

You may have noticed that many designs such as those in linoleums, carpets, and rugs are based upon a triangle drawn inside a circle. A triangle drawn inside a circle so that each vertex of the triangle is on the circle is said to be *inscribed* in the circle. The circle is said to be *circumscribed* about the triangle. (See Exercise 4 on page 19.)

1. The vertices of an inscribed triangle lie on the circle. In the figure notice that vertices A, B, and C lie on the circle O. What do you know about $\overline{OA}$, $\overline{OB}$, and $\overline{OC}$? Explain how you know this.
   equal in measure; radii of the same circle
2. Compare the distance from the center of circle O to vertex A and to vertex B and C. Give a reason for your answer. same as ex. 1

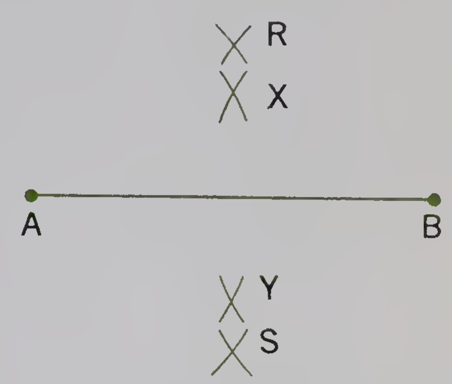


3. Consider the triangle ABC in circle O to see if we can discover how to circumscribe a circle around it. From Exercise 2 you learned that the center of the circle must be equally distant from points A and B. Is the midpoint M of $\overline{AB}$ equally distant from A and B? yes
   Locate four other points equally distant from A and B.
   any points on a line MO
4. Draw a segment on your paper and label the endpoints A and B. Mark four points above and four points below the line that look to be equidistant from A and B.
5. With your compass set at a radius longer than half $\overline{AB}$ and using A as center, strike an arc above $\overline{AB}$. Using the same radius and B as the center, strike an arc that intersects the first arc. Label the point of intersection X. Check with your compass to see if X is the same distance from A that it is from B.

6. Follow the same procedure as you did in Exercise 5, but this time strike your arcs below $\overline{AB}$. Label the point of intersection Y. Show that Y is equally distant from A and B.

7. With your compass set at a radius longer than the radius used in Exercises 5 and 6 and with A and B as center, again strike arcs that intersect above and below the $\overline{AB}$. Label the points of intersection R and S. Show that points R and S are equally distant from A and B.

8. Your construction work should now appear as in the figure at the right. Draw a line through points R, X, Y, and S. Use your compass to check several points on this segment. Do they appear to be equally distant from A and B? Do the original points you drew as part of Exercise 5 lie on the line? *yes; yes*



9. Does $\overline{RS}$ appear to be perpendicular to $\overline{AB}$? Does it bisect $\overline{AB}$? Check with your compass to see. *yes; yes*

10. From your investigations so far, would you say that all points on the perpendicular bisector of a segment are equally distant from the ends of the segment and lie on its perpendicular bisector? Refer back to page 9 for the discussion of a perpendicular bisector. *yes*

11. Now to return to triangle ABC of Exercise 3. The center of the circle O that will circumscribe the triangle is equally distant from A and B. What is its relation to the perpendicular bisector of $\overline{AB}$?
It is on the perpendicular bisector of $\overline{AB}$.

12. The center of the circumscribed circle O is also equally distant from points B and C. What is its relation to the perpendicular bisector of $\overline{BC}$? *It is on the perpendicular bisector of $\overline{BC}$.*

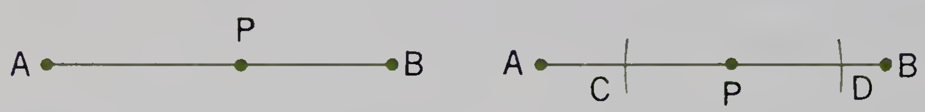
13. Since the center of the circumscribed circle is on the perpendicular bisector of $\overline{AB}$ and also on the perpendicular bisector of $\overline{BC}$, can you locate exactly the center of the circumscribed circle? (See Exercises 2 and 3 on page 19.) *Yes, the center is at the intersection of the two perpendicular bisectors.*

14. List the steps for circumscribing a circle around a triangle. Now draw a triangle like triangle ABC and circumscribe a circle around it.
See front.

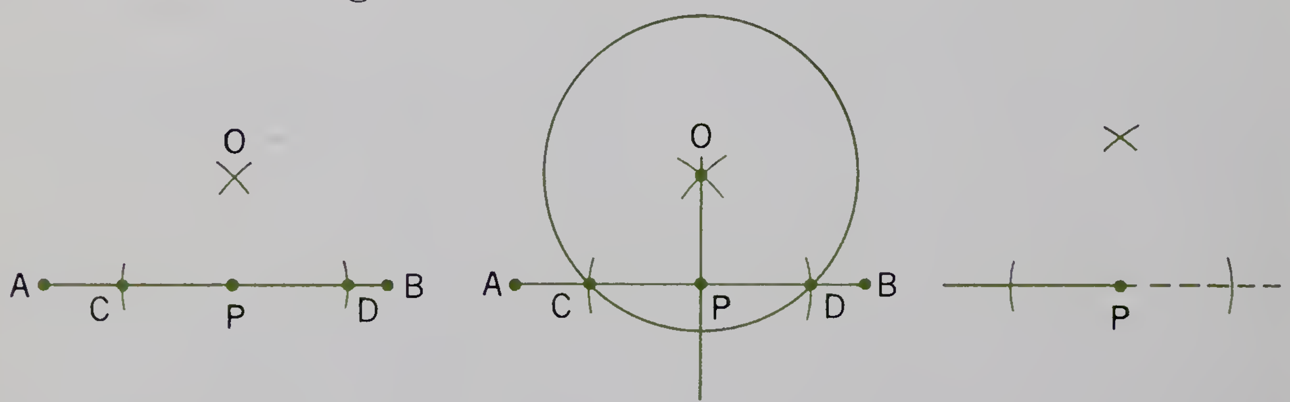
15. Draw a triangle containing an obtuse angle and circumscribe a circle around it. Did you find anything unusual? Draw a triangle containing a right angle and circumscribe a circle around it. Compare this construction to the other one. *See front.*

Often in drawing plans and designs it is necessary to draw a perpendicular to a given line so that it passes through a certain point.

1. How many different lines can be drawn through one point on a line? How many of the lines that you draw on your paper would be perpendicular to the given line? *an infinite number ; one*
2. Mark two points A and B on a sheet of paper. How many straight lines can you draw that contain these two points? *one* You may have heard the expression, "Two points determine a line." Explain in your own words just what this means. *One and only one line can be drawn through two points.*
3. In the figures below, $\overline{AB}$ is a segment. A perpendicular is to be constructed at P. Place the point of your compass at P, and mark off points C and D such that P is the midpoint of $\overline{CD}$. Why would the axis of symmetry of $\overline{CD}$ go through P? Would it be perpendicular to $\overline{AB}$? *See front.*



4. The figure below on the left has O as a point equidistant from C and D. Using your compass, how would you find this point? How do you know O is on the axis of symmetry? How do you know $\overline{OP}$ is perpendicular to $\overline{CD}$? If you wish to erect a perpendicular at P, the end of a segment, the figure below on the right shows extension of the line through P. *See front.*

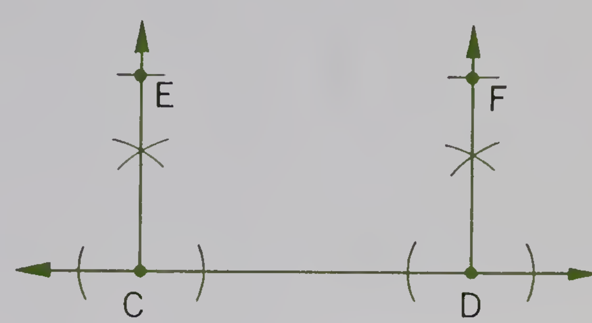


5. Draw $\overleftrightarrow{MN}$ about 4 inches long, and mark a point P somewhere on $\overline{MN}$. Using your compass and ruler, draw a line perpendicular to $\overline{MN}$, which passes through point P. The steps for this construction follow.
   a. Set your compass at a convenient measure, and using the given point P as a center, strike arcs cutting $\overleftrightarrow{MN}$ on either side of P at points C and D.

b. Using points C and D, and a radius somewhat greater than PC, strike arcs that intersect above the line. Label the point of intersection O.

c. Draw a segment connecting points O and P. $\overline{OP}$ will be perpendicular to $\overleftrightarrow{MN}$.

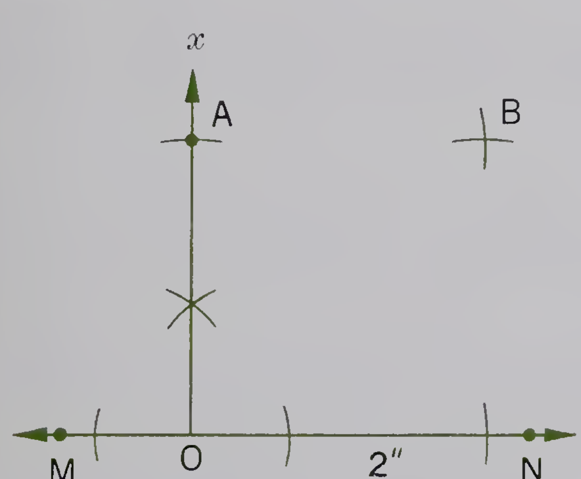
6. On a line about 5″ long, mark two points C and D about $3\frac{1}{2}$″ apart and erect perpendiculars through these points. Set your compass at a radius of $2\frac{1}{2}$″ and using C and D as centers, mark arcs at E and F on these perpendiculars. Draw a segment connecting E and F on your paper. $\overline{EF}$ is *parallel* to $\overline{CD}$, because at all points $\overline{EF}$ is equally distant from $\overline{CD}$. The geometric figure determined by connecting points C, D, F, and E is a *parallelogram* because its opposite sides are parallel. $CDFE$ is also a rectangle because its angles are right angles.

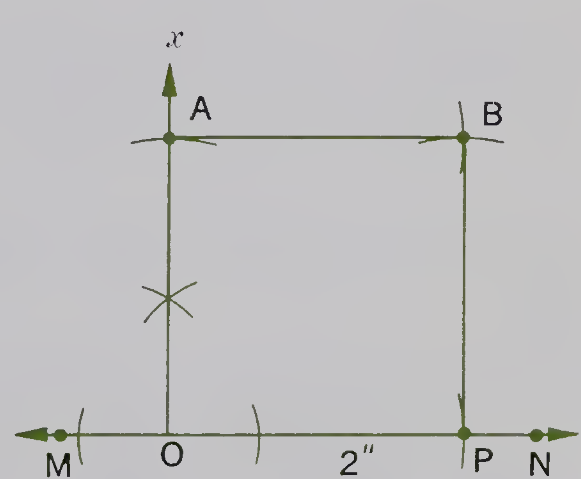


7. Using the same steps, construct rectangles having these dimensions:

a. Length 4″, width $2\frac{1}{2}$″ **b.** Length $3\frac{1}{2}$″, width 3″

8. You can use the same steps to draw a square by making the width equal to the length. The figure below shows you an easier way.





Study the figure and follow the procedures to construct a square with each side 2 inches long.

9. Construct rectangles having these dimensions.

a. Length 4″ width 3″ **c.** Length 2″ width $1\frac{1}{2}$″

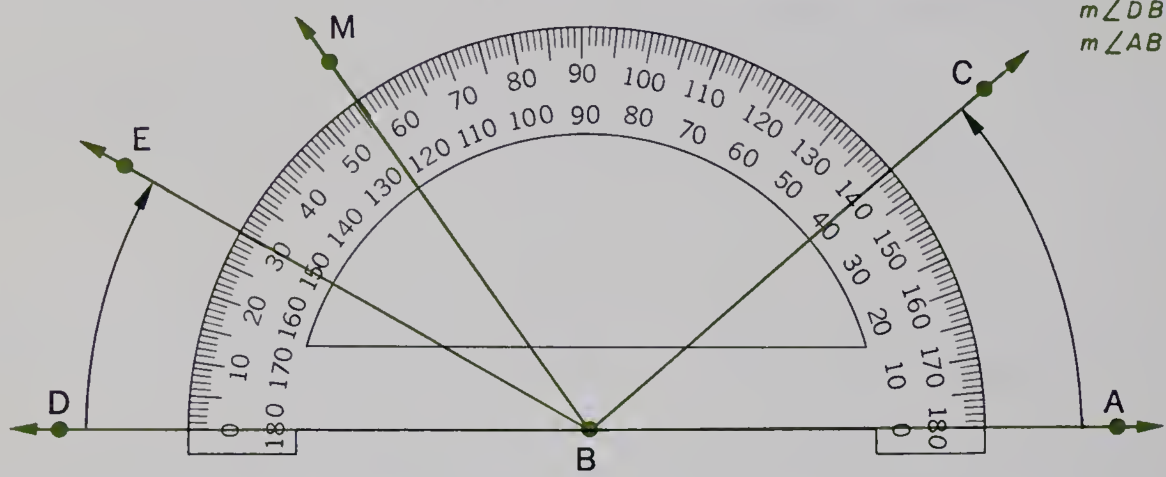
b. Length 3″ width $1\frac{1}{4}$″ **d.** Length 3″ width $\frac{7}{8}$″

10. Draw and measure the diagonal of each rectangle in Exercise 9. If you have made your constructions carefully, the diagonals should measure approximately:

a. 5″ **b.** $3\frac{1}{4}$″ **c.** $2\frac{1}{2}$″ **d.** $3\frac{1}{8}$″

Associated with a given angle is a number, its measure. A unit that we use for finding measure is the *degree,* °. We will use a protractor to find the measure of a given angle. Let's review how to use a protractor.

To measure an angle placed like $\angle ABC$, we use the numerals on the *inner* part of the protractor. Note that the vertex of the angle is placed at the vertex of the protractor and that $\overrightarrow{BA}$ passes through the 0 (or 180) mark on the protractor. Read the measure of the angle by noting where BC would intercept the protractor scale. In using the inner scale, we are measuring an angle in a *counterclockwise* direction. The outer scale is used to measure angles in a *clockwise* direction, such as $\angle DBE$. What are the measures of the angles below? $m\angle EBD = 30°$
$m\angle DBM = 55°$
$m\angle ABC = 40°$



1. Using your protractor, draw an angle that measures 60°. If you divided it into four angles of equal measure, how many degrees would there be in each angle? 15°

2. Use your protractor to divide the angle into four angles of equal measure. Measure each angle carefully. Are they all the same size?

3. Use your protractor to draw the angles listed below, and divide each into the number of angles of equal measure as specified.
   a. 45° (3 equal parts) 15°
   b. 160° (4 equal parts) 40°
   c. 90° (3 equal parts) 30°
   d. 120° (5 equal parts) 24°
   e. 25° (2 equal parts) 12.5°
   f. 100° (4 equal parts) 25°

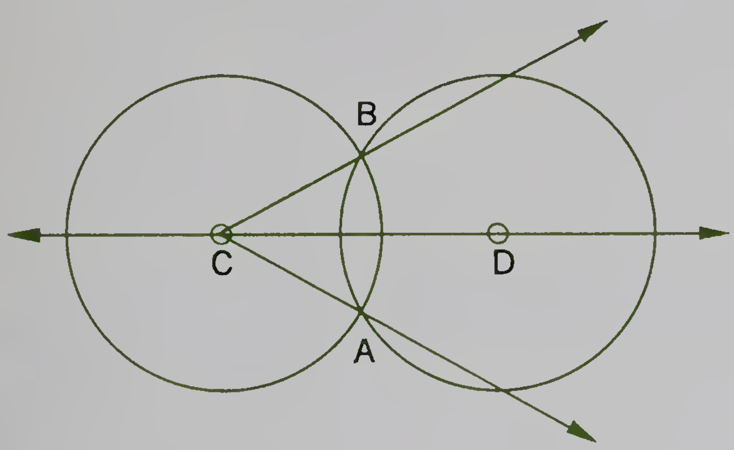
4. Using your protractor, measure the angles of the regular polygons studied in this chapter. What is the sum of the angle measures of a triangle, square, pentagon, hexagon, and octagon? Can you see a pattern that relates the number of sides, n, to the sum s? To help you see the pattern, you might write (3, 180°), (4, 360°), etc. where the first element of the ordered pair represents sides and the second the sum of the angles. triangle, 180°; square, 360°; pentagon, 540°; hexagon, 720°; octagon, 1080°; $2(n-2)$ right angles, where n is the number of sides

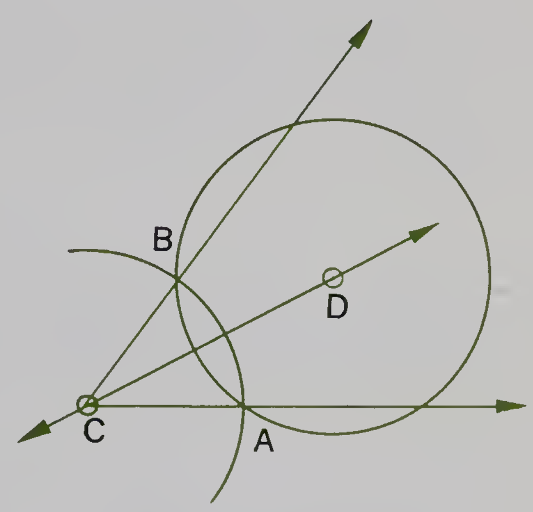
1. You can bisect an angle more accurately by using the principles of symmetry than you can with a protractor. Use your compass to make drawings like those shown below. Draw a small circle around the center of each circle. Draw the axis of symmetry for each figure.

2. Using your protractor, measure to see if the angle in each of the drawings has been divided in half by an axis of symmetry.

3. To draw an axis of symmetry for each of the angles, you must locate what point besides the vertex? *a point on the axis of symmetry*

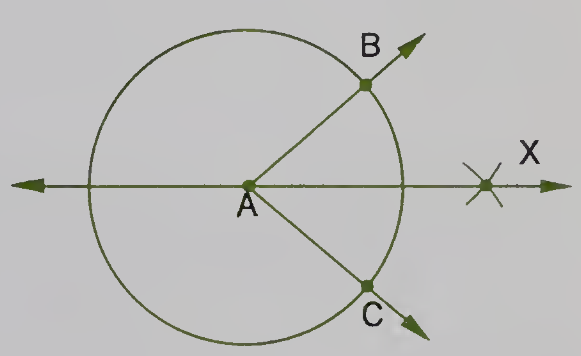
4. In the figures below, A is one of what pair of corresponding points? What two points does the axis of symmetry pass through?
A and B ; C and D





5. $\overleftrightarrow{CD}$ is the axis of symmetry of angle BCA in the figures above. Check with your protractor to see if $\overleftrightarrow{CD}$ bisects the angle.

6. To construct a line which bisects an angle, you must find the centers of two circles. One center is the vertex of the angle and the other is on the angle's axis of symmetry. If you have trouble bisecting an angle, study the figure at the right.

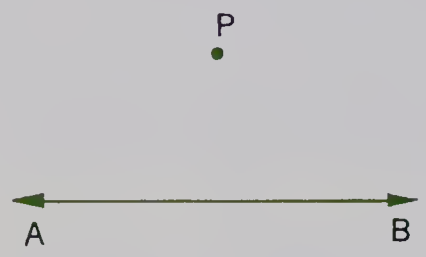


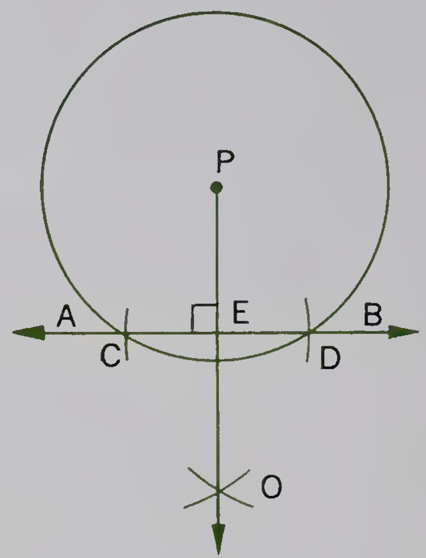
   a. With your compass set at any convenient measure, use the vertex of the angle as a center and draw a circle which intersects the sides of the angle.
   b. Using as centers the points where the circle intersects the sides of the angles, B and C, mark arcs that intersect inside the angle, away from the vertex. Label this intersection point X.
   c. Draw the line through the vertex of the angle and the point X. This line is the angle bisector.

1. Draw $\overleftrightarrow{AB}$. Mark a point P above the line. How many lines can you draw through P that will intersect $\overleftrightarrow{AB}$? *infinite number*

2. How many lines through P will be perpendicular to $\overleftrightarrow{AB}$? *one*

3. If P is the center of a circle that intersects $\overline{AB}$ at points C and D, construct the perpendicular bisector of $\overline{CD}$. Extend $\overline{OE}$ to point P.



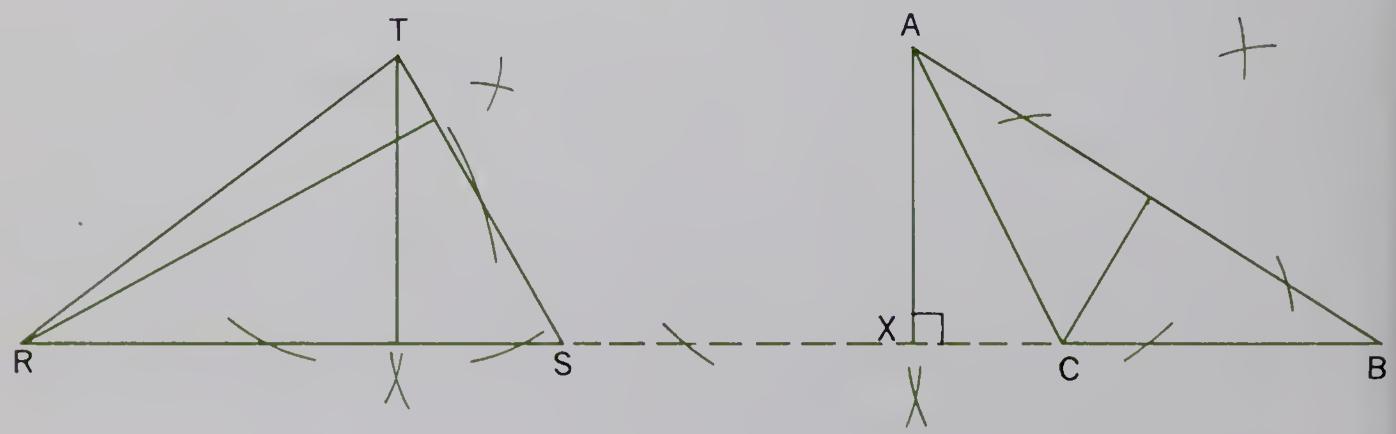


4. How do you know that $\overline{OP}$ is perpendicular to $\overline{AB}$? *C and D are corresponding points; $\overline{OP}$ is part of the axis of symmetry.*

5. Study the diagrams and list the steps for constructing a perpendicular to a segment from a point not on the segment. Practice the steps with your ruler and compass until you can make the construction without referring to the figure or the steps you listed. *See front.*

6. The *altitude* of a triangle is the perpendicular segment drawn from a vertex to the opposite side or the opposite side extended. A triangle has how many vertices? how many altitudes? The opposite side is the *base* of the triangle for that altitude. *3 ; 3*

7. Draw a scalene triangle and construct the altitude from each vertex of the triangle. If the triangle you draw contains an obtuse angle, you will have to extend two of the sides in order to construct two of the altitudes. See the figure on the right below, where $\overline{BC}$ is extended.



8. Construct a scalene triangle with sides of the following lengths: AB = 3 inches; AC = 4 inches; BC = 5 inches. Construct the altitude from each vertex. Do the altitudes intersect? yes

9. Do any of the altitudes that you constructed in Exercise 8 divide the triangle symmetrically? no Explain how you can tell. No altitude bisects the base.

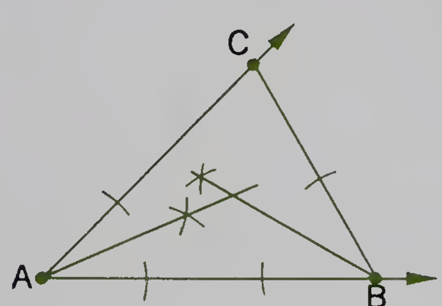
10. Draw an isosceles triangle (one with two sides of equal length) with sides measuring as follows: AB = 2 inches; AC = 2 inches; BC = $2\frac{1}{2}$ inches.

11. Construct the altitudes of the isosceles triangle. Do any of them divide the triangle symmetrically? If so, specify which. yes; the one to the $2\frac{1}{2}$-inch base

12. Construct an equilateral triangle with sides each 2 inches in length. Construct the altitudes. Do any of the altitudes divide the triangle symmetrically? If so, name them. all three

13. Use your protractor to draw three angles which have the following measures. Then use your compass to construct the bisectors.
a. 70° **b.** 40° **c.** 120°
Check your results with the protractor.

14. Draw a triangle and bisect each of its angles. Use your protractor to measure the angles of the triangle and to check the angles formed by the bisectors.



Note: Since an angle is formed by rays and the triangle is formed by segments (sides), the triangle does not contain any angles. Using a vertex as an endpoint, however, rays can be drawn through the opposite vertices. Thus, we say that the sides of a triangle determine three angles, which we will call the angles of the triangle.

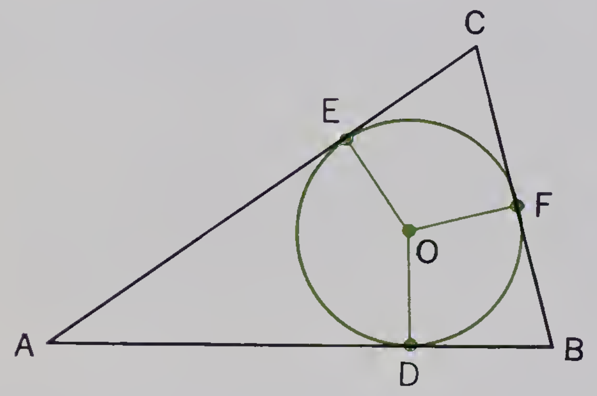
15. Draw an isosceles triangle and bisect each angle. Check to see if any of the bisectors are also altitudes. See front.

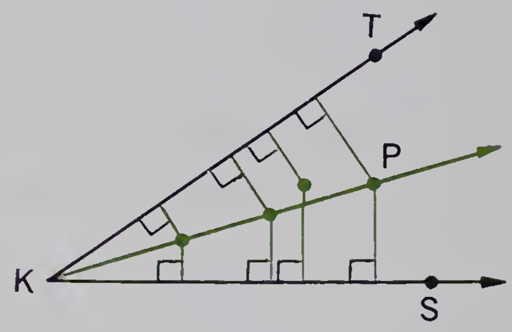
16. Draw a line t and place two points C and D above the line. Construct perpendiculars from these points to the line t. Are they parallel? yes

17. Draw a line m. Mark a point T on the line. Construct a perpendicular at point T. Mark a point K on your paper and construct a perpendicular to line m. Are the perpendiculars parallel? Where could you mark K so that the perpendiculars would *not* be parallel? yes; on the perpendicular from T

A triangle is said to be circumscribed about a circle when the circle lies inside the triangle and each side of the triangle intersects the circle at one, and only one, point. We say that each side of the triangle is *tangent* to the circle.

1. The circle in the figure below at the left touches the three sides of the triangle at D, E, and F. What can you say about the measures OD, OE, and OF? Remember the distance from a point to a line is the length of the perpendicular from the point to the line. *They are equal.*





2. Examine $\angle TKS$ above. To locate a set of points equally distant from the sides of the angle, the points must stay midway between the sides and would therefore divide the angle into two angles of equal measure. What line divides an angle into two equal angles? *bisector; axis of symmetry*
3. Draw an angle measuring 45° on your paper, and construct the bisector of the angle. Choose a point R on the bisector, and construct a perpendicular from R to each of the rays that form the sides of the angle. Check with your compass to see if these perpendiculars are equal in measure.
4. Check other points on the angle bisector to see if they are at an equal distance from the sides of the angle. Remember to measure the perpendicular distance.
5. Let's return to triangle ABC and the inscribed circle of Exercise 1. The center of the inscribed circle is one of the points of the set that makes up the bisector of $\angle BAC$, and is also one of the set of the points that makes up the bisector of $\angle ABC$. How do you determine the point that is the center of the inscribed circle? *where the bisectors intersect*
6. What distance do you select for the measure of the radius of the inscribed circle? Draw a triangle RST and inscribe a circle in it. *the perpendicular distance from the center of the circle to one side*
7. Draw an equilateral triangle and inscribe a circle in it. Also circumscribe a circle around it. Did you find anything unusual? *Yes, the circles have the same center.*

A. *Add:*

1.	3.	5.	7.	9.
54	66	769	7.86	.75
19	80	301	.87	4.16
62	31	560	5.40	20.10
76	20	276	2.16	.09
89	44	908	.58	3.19
300	241	2814	16.87	28.29

2.	4.	6.	8.	10.
205	32	155	700	9
36	307	427	912	206
17	8	3	20	5
109	29	15	6	300
516	506	304	17	4
9	7	66	813	19
892	889	970	2468	543

B. *Subtract:*

1.	3.	5.	7.
84	65	50	54
28	39	26	34
56	26	24	20

2.	4.	6.	8.
$4.78	$2.50	$21.49	$36.16
1.80	2.07	11.23	28.16
$ 2.98	$ .43	$10.26	$ 8.00

C. *Multiply:*

1.	2.	3.	4.
184	234	$35.32	$80.04
73	34	607	750
13,432	7,956	$ 21,439.24	$60,030.00

5.	6.	7.	8.
215	678	9007	8990
77	79	208	790
16,555	53,562	1,873,456	7,102,100

D. *Divide:*

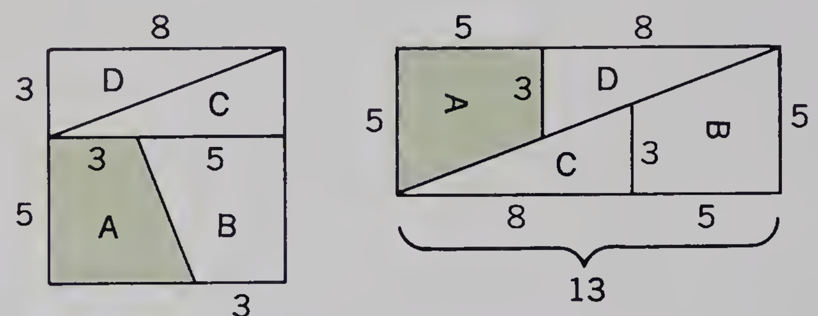
1. 19,402 ÷ 178 *109* **3.** 5645 ÷ 47 *120 $\frac{5}{47}$* **5.** $386.54 ÷ 251 *$ 1.54*

2. 2223 ÷ 57 *39* **4.** $10.92 ÷ 26 *$ .42* **6.** 15,533 ÷ 317 *49*

If you need more practice, turn to the Practice Exercises on page 468 and following. If not, you may work in the Experts' Corner on the next page.

Detective Work

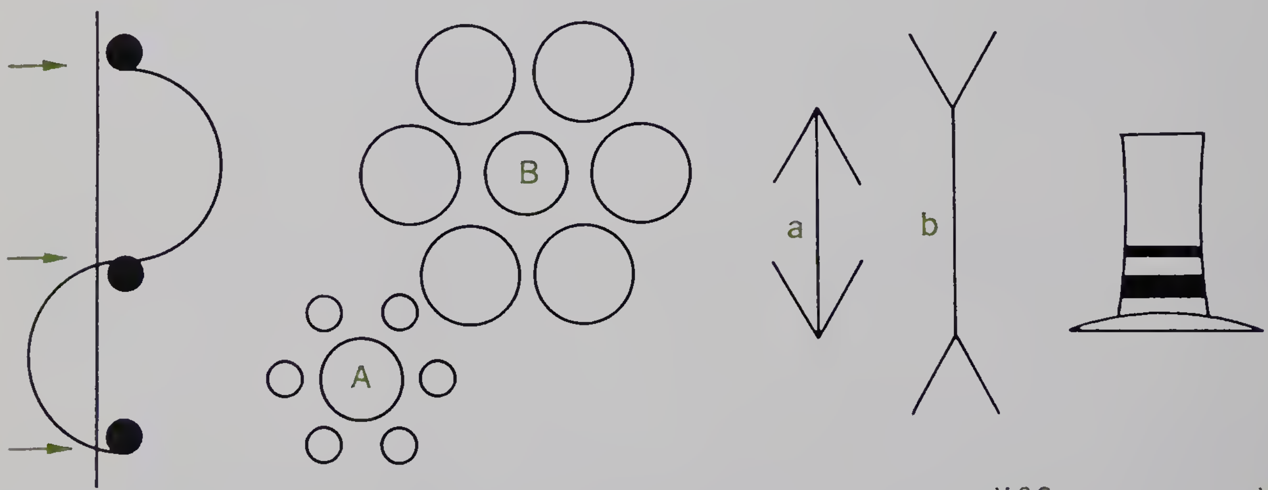
1. Construct a square eight units on a side. Let one unit be equal to $\frac{1}{4}$ of an inch. (If you have forgotten how to draw a square see page 25.) Divide and cut the square into two trapezoids and two triangles as shown. Remember? A trapezoid is a closed four-sided geometric figure, with two sides parallel. Now rearrange the pieces into a rectangle as shown. In the figures below, the numerals represent the number of units. What is the area of the square? What is the area of the rectangle? How do you account for the difference in area? Make a careful drawing of the square on cardboard. Cut it out and see if it can form the rectangle. *See front.*



Square

Rectangle

2. Can you believe what you see? Test yourself on the following questions.



a. Are the colored arrows the same distance apart? *yes* Measure. Is the curve on the right of the line as long as the curve on the left? *yes*

b. Which of the two circles labeled A and B is the larger? Measure the diameters and check yourself. *They are the same size.*

c. Which segment is longer, a or b? Does your ruler agree with you?

d. Which is greater, the width or the height of the hat? How do you account for your answer? *The width and height are the same, but the cross markings make the hat appear taller.*

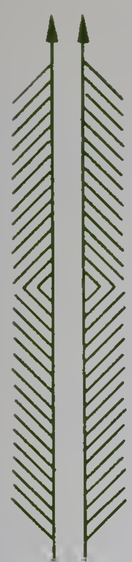


Figure 1

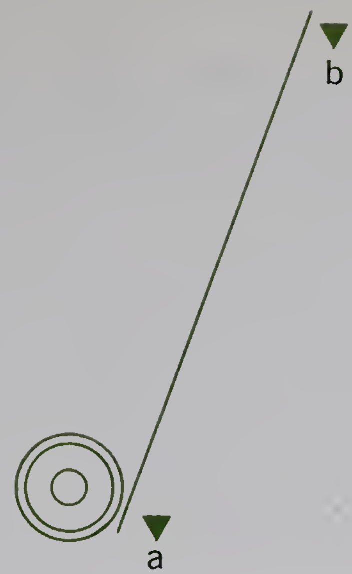


Figure 2

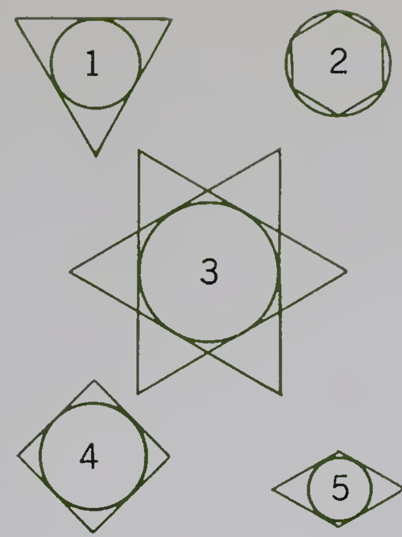


Figure 3

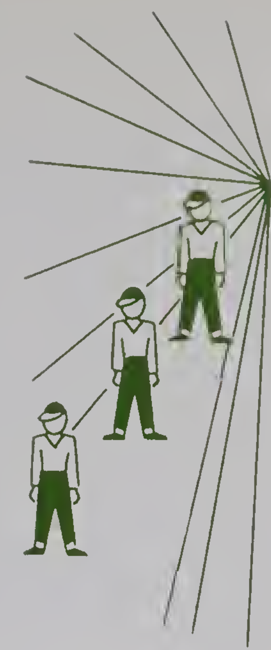


Figure 4

3. **a.** In Figure 1 are the two lines straight or curved? Check with a ruler. *straight*

b. From *a* to *b* is how many turns of the wheel in Figure 2? The circumference of the wheel is half that of a dime. To check, roll a dime along a line twice as long as the given line. $1\frac{1}{2}$

c. In Figure 3, which two circles are the same size? *1, 4*

d. Which soldier is the tallest in Figure 4? Have you learned to depend on your ruler more than on your guess? *same*

4. Geometric Magic Square. Study the illustration below.

a. How many circles are in the first row? *1*

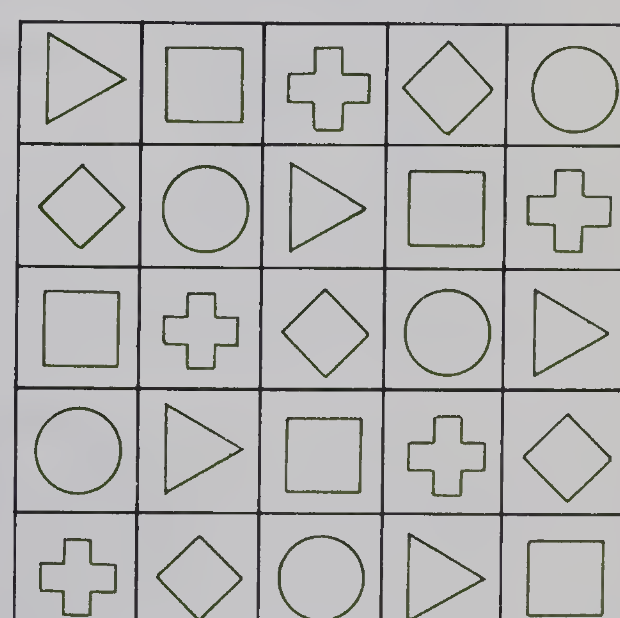
b. How many squares are in the first column? *1*

c. How many triangles are in one of the diagonals? *1*

d. Find any row, column, or diagonal in which a figure is repeated. *none*

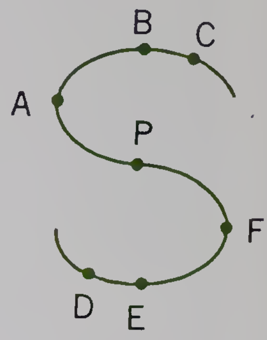
e. Why is it called a geometric magic square? *no repetitions*

f. Make a similar magic square with these symbols: + − × ÷ = *See front.*



5. Optical illusions are very interesting because they deceive the eye. What appears to be is not actually true. You may be interested in trying to make some optical illusions of your own. For example, if you draw a square and then draw a small square inside and connect the vertices, you may obtain the effect of looking down a hallway or looking at the top of a geometric solid.

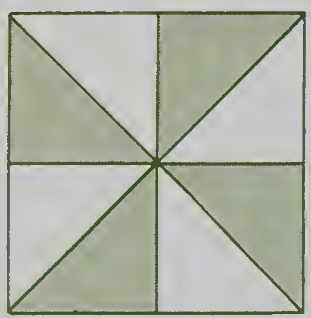
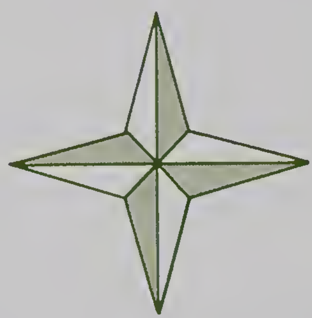
Not all symmetric figures have line symmetry. Some are symmetric in still other ways. For example, they may be symmetric about a point. In addition to line symmetry and point symmetry, solid figures may be symmetric with respect to a plane.



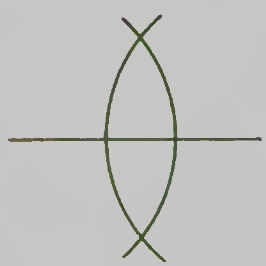
1. In the figure at the right, several points have been named with letters. Notice that point F is in the same relative position in one part of the figure as point A is in the other part. They are called *corresponding points.* Which point corresponds to point C? D

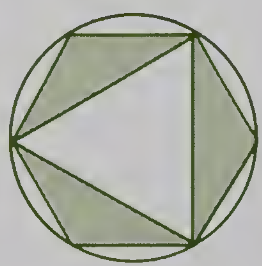
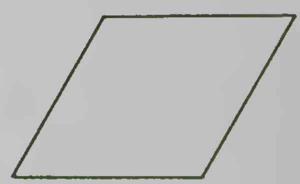
2. Copy carefully the figure above on your paper. Take a ruler and draw a segment from A to F. Draw segments between each of the other pairs of corresponding points. If you have been very accurate, all the segments should cross each other at the same point. This point is called the *center of symmetry.*

3. If a figure has a center of symmetry, it is said to possess *point symmetry.* Copy each of the designs below, and draw a small circle around the center of symmetry for each.


4. Copy each diagram below that has point symmetry, and indicate its center of symmetry. first ; fourth



5. Copy each diagram that has line symmetry. all four

6. When a figure in a plane has both line symmetry and point symmetry, it is said to have *complete symmetry.* Which of the figures of Exercise 4 have complete symmetry? first, fourth

7. Find some other designs that have *only* point symmetry, two that have *only* line symmetry, and two that have complete symmetry.

See front.

Symmetry has interesting and important applications other than those we have touched on in this chapter. In architecture it is one way of securing balance in a design. In Greek architecture symmetry was used extensively for this purpose, as, for example, in the Parthenon. Many buildings are noted for their symmetric design. The Governor's Palace in Williamsburg, Virginia, the Taj Mahal in India, and the Capitol in Washington, D. C. are a few examples.

Often it is necessary to secure balance without symmetry. This is true, for example, in photography. Even though the tree on the right is the center of interest, it should have been balanced by some details on the left. Which scene is more pleasing? The same principle is used in house design and in landscaping. Many magazines have articles stressing the importance of house planning and landscape harmony. See if you can find such articles for display in class.

In mathematics we speak of symmetry not only as it relates to geometric figures, but also in connection with certain *logical relationships*. This is illustrated in the following sentences.

1. Harry goes to the same school as Tom. If this statement is true, does Tom go to the same school as Harry? *yes*

2. Jane rides to school with Kathy. Does it follow that Kathy also rides to school with Jane? *yes*

3. If Babe Ruth was on the same team as Lou Gehrig, do you expect that Lou Gehrig was on the same team as Babe Ruth? *yes*

4. If your answers to the previous questions are "Yes," then we say that the statements have symmetry. We can also say that the phrase "goes to the same school as" is a relation. A relation that has the property of symmetry is said to be *symmetric*. Use the relation in another sentence and check to see if it has the property of symmetry.

5. Does the relation "rides to school with" have the property of symmetry? *yes*

6. What phrase in Exercise 3 expresses a relation? Try the relation in another sentence to see if the relation has the property of symmetry. *was on the same team with*

7. Mr. Clay is the father of George Clay. If this is a true statement, does it follow that George Clay is the father of Mr. Clay? *no*

8. A relation such as the one in Exercise 7 is not symmetric. Why? *Relation is in one direction.*

9. Try the relation "is the father of" in another sentence. Is this relation symmetric? *no*

10. Helen is older than Nancy. Is the relation "is older than" symmetric? *no*

11. Consider the following statements. The phrase in italic type is a relation. Determine whether each relation has the property of symmetry.
   a. Janet *is the mother of* Phyllis. *no*
   b. James *is the brother of* Charles. *yes*
   c. Tim *is a better player than* Jack. *no*
   d. Horace *lives in the same block as* Paul. *yes*

12. We frequently use relation expressions in mathematics. Perhaps the most common is illustrated in this mathematical sentence:

$$6 + 3 + 5 = 14$$

Does the expression "is equal to" have the property of symmetry? *yes*

13. In this sentence: $6 + 3 + 5 > 10$.
Does the relation "is greater than" have the property of symmetry? *no*

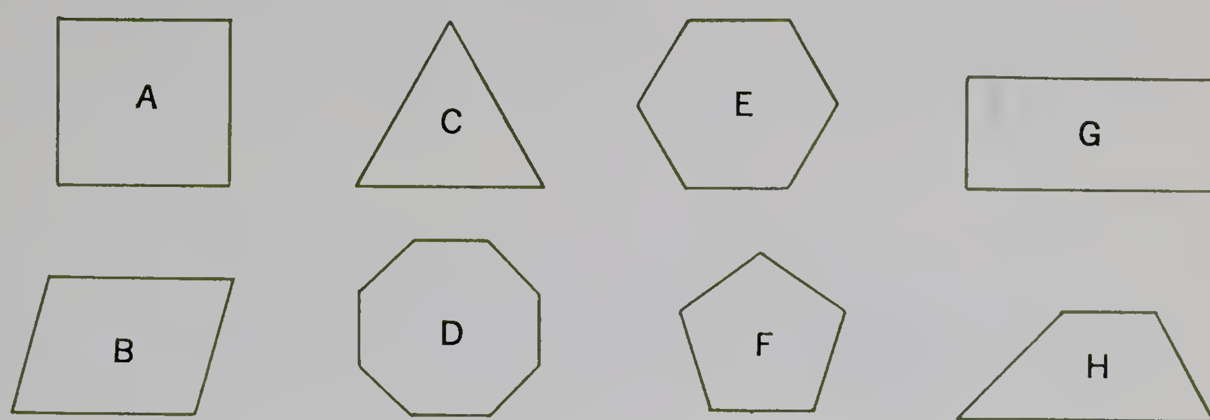
14. Try "is a factor of" to see if the relation is symmetric. *no*

15. Here are some mathematical symbols along with their English translations. State which have the property of symmetry. *a, b*
   a. $=$ is equal to
   b. $\neq$ is not equal to
   c. $\nless$ is not less than
   d. $\subset$ is a subset of
   e. $>$ is greater than
   f. $<$ is less than
   g. $\ngtr$ is not greater than

16. Examine the following sentences, and state whether or not each has the property of symmetry.
   a. $6 + 3 = 3 + 6$ *yes*
   b. $12 - 8 = 4$ *yes*
   c. $14 > 36$ *no*
   d. $13 \times 3 \neq 17 \times 2$ *yes*
   e. $15 \times 3 > 8 \times 5$ *no*
   f. $18 \div 9 < 6 + 7$ *no*
   g. $15 \ngtr 60$ *no*
   h. $\{4\} \subset \{2,4,6,8\}$ *no*
   i. $3 + 2 + 5 \neq 11$ *yes*
   j. $\{1,3,5\} \subset \{1,3,5,7\}$ *no*

Part One

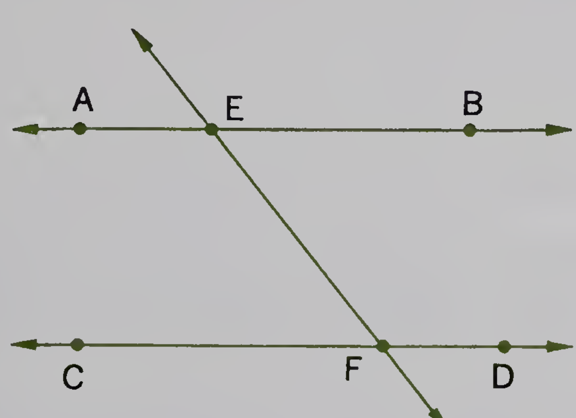
1. Write the following words on a sheet of paper. After each word, write the letter or letters to show to which figure, or figures illustrated below that the word describes.

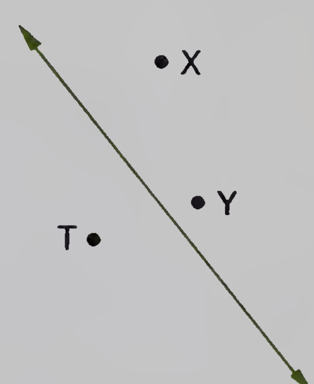


a. square *A*
b. regular hexagon *E*
c. equilateral triangle *C*
d. regular octagon *D*
e. regular pentagon *F*
f. parallelogram *A, B, G*

2. In the figure below on the left, if $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel, describe the following:

a. $\overleftrightarrow{EF} \cap \overleftrightarrow{AB}$ *E*
b. $\overleftrightarrow{EF} \cap \overleftrightarrow{CD}$ *F*
c. $\overleftrightarrow{AB} \cap \overleftrightarrow{CD}$ $\{\ \}$

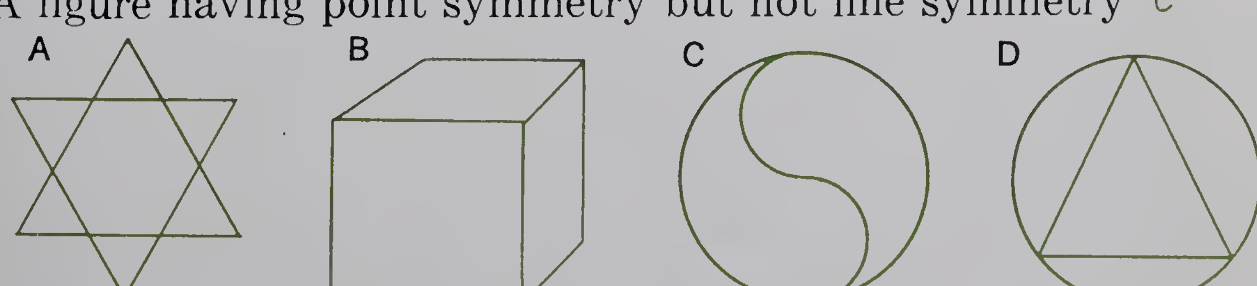




3. Study the diagram above at the right and answer the following:

a. Is $\overline{XY} \cap m$ empty? Why? *yes; no intersection*
b. Is $\overline{XT} \cap m$ empty? Why? *no; intersects*
c. Is $\overrightarrow{XY} \cap m$ empty? Why? *no; intersects*
d. Is $\overrightarrow{YX} \cap m$ empty? Why? *yes; no intersection*

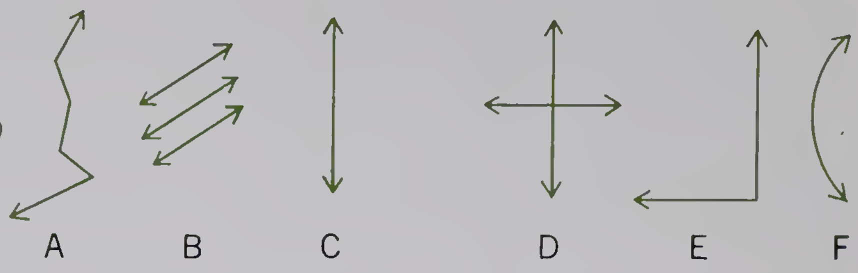
4. Write the following phrases on a sheet of paper. After each phrase write the letter of the figure that the phrase describes.
A figure having both line and point symmetry *A*
A figure having point symmetry but not line symmetry *C*



Part Two

A. Copy the words and after each word write the letter to indicate the figure that it names.

- **a.** straight line *C*
- **b.** right angle *E*
- **c.** broken line *A*
- **d.** perpendicular lines *D*
- **e.** parallel lines *B*
- **f.** curved line *F*



B. Constructions.

1. Construct an equilateral triangle each side of which is $2\frac{1}{2}''$ long.

- **a.** What is the perimeter of this triangle? $7\frac{1}{2}''$
- **b.** Construct the altitude of this triangle and find its length to the nearest $\frac{1}{8}''$. $2\frac{1}{8}''$

2. Construct a regular hexagon given a circle whose radius measures $1\frac{1}{2}''$. By drawing the diagonals of the hexagon that are diameters of the circle, divide the hexagon into equilateral triangles. How many equilateral triangles are there? Construct an altitude of one of the triangles and find its length to the nearest $\frac{1}{8}''$. 6; $1\frac{2}{8}''$

Part Three

Draw a picture of $\overline{ON}$. Mark a point M on the line between N and O.

1. Is point M an element of the set of points in $\overline{ON}$? Is point O an element of the set of points $\overline{MN}$? Refer to your figure and answer the following: *yes; no*

- **a.** $\overline{NM} \cup \overline{MO} = \square$ $\overline{NO}$ **c.** $\overline{ON} \cap \overline{OM} = \square$ $\overline{MO}$ **e.** $\overline{MN} \cap \overline{MO} = \square$ $\overline{NO}$
- **b.** $\overline{NO} \cup \overline{MO} = \square$ $\overline{NO}$ **d.** $\overline{OM} \cup \overline{MO} = \square$ $\overline{MO}$ **f.** $\overline{ON} \cap \overline{OM} = \square$ $\overline{NO}$

2. Mark a point on your paper and label it A. Draw a ray that has A as an endpoint. Choose a point on this ray and label it B. Draw another ray that has A as an endpoint. Choose a point on this ray and label it C. Draw a line through BC.

- **a.** $\overline{AB} \cap \overline{BC} = \square$ *B* **d.** $\overleftarrow{BC} \cup \overleftarrow{AC} = \square$ *∠ACB* **g.** $\angle BAC = \square \cup \square$
- **b.** $\overline{CA} \cap \overline{AB} = \square$ *A* **e.** $\overline{BC} \cap \overline{AC} = \square$ *C* **h.** $(\overline{AC} \cup \overline{CB}) \cup \overline{AB} = \square$
- **c.** $\overleftarrow{CA} \cup \overrightarrow{AB} = \square$ *∠CAB* **f.** $B = \square \cap \square$ $\overline{AB} \cap \overline{BC}$

g. $\overrightarrow{BA} \cup \overrightarrow{AC}$
h. *triangle ABC*

3. Draw lines $\overleftrightarrow{XY}$ and $\overleftrightarrow{RS}$ that intersect at T.

- **a.** $\overleftarrow{XT} \cap \overrightarrow{TS} = \square$ *T* **c.** $\overleftrightarrow{XY} = \square \cup \square$ **e.** $\overleftarrow{YT} \cap \overrightarrow{TR} = \square$ *T*
- **b.** $\overleftarrow{RT} \cup \overrightarrow{TS} = \square$ $\overleftrightarrow{RS}$ **d.** $\overleftrightarrow{XY} \cap \overleftrightarrow{RS} = \square$ *T* **f.** $\overleftrightarrow{XY} \cap \overrightarrow{TS} = \square$ *T*

c. $\overleftarrow{XT} \cup \overrightarrow{TY}$

CHAPTER TWO

MATHEMATICAL SENTENCES

WORDS TO WATCH FOR

addend	*extremes*	*proportion*
closed sentence	*factor*	*rate*
closure	*fractional number*	*repeating decimal*
coefficient	*inverse operation*	*scientific notation*
conditional sentence	*multiplicative inverse*	*subscript*
equation	*open sentence*	*universe*
equivalent	*proper subsets*	*variable*

Mathematical sentences express ideas about numbers. Since there are many kinds of number ideas, it follows that there are many kinds of mathematical sentences. For instance, a mathematical sentence may express equality or inequality; it may be true, conditionally true, or false.

Mathematical sentences have a key role in problem solving. They present the mathematical relationships clearly and without the distracting details that have no bearing on the solution. The mathematician, in fact, usually feels that once the sentence correctly states the problem, the most difficult part of his problem solving is completed.

It is important to recognize that a sentence set up for one problem may serve as a pattern for many different problems of the same type. You have become acquainted with this fact in working with formulas. The interest formula is useful for many calculations, including those dealing with loans, government bonds, and savings accounts. The distance formula is useful whether travel is by bicycle, car, or space capsule. In this chapter, you will have an opportunity to increase your skill in organizing your work and in solving mathematical sentences.

Mathematical sentences are powerful devices to explore new mathematical ideas and relationships and to express them in a form that can be recalled easily. Let's look at an example. We already know that it is not permissible to use zero as a divisor or as a denominator of a fraction. Let's review how we explain this fact. Selecting a number other than zero to use as a dividend, say 35, and dividing this number by zero, we have the sentence:

35	$\div$	0	$=$	n
product		*known factor*		*unknown factor*

The sentence above asks what the quotient will be if the division is performed. We can write an equivalent sentence:

35	$=$	n	$\times$	0
product		*unknown factor*		*known factor*

This sentence tells us that the quotient will be a number that, when multiplied by zero, will give 35 as a product. Since the product of any number and zero is zero, getting 35 as a product is impossible and thus dividing by zero is meaningless.

We could make our original sentence more general by using a variable, such as x, instead of 35 and define x as a member of some specified set of numbers. This would yield a sentence $x \div 0 = n$ and by the same line of reasoning as above we could then establish that division by zero is an undefined operation.

In this chapter we will explore the nature of mathematical sentences, and we will use them in turn to explore number systems and operations and to solve applied problems.

To help you in problem-solving, we have placed displays at appropriate intervals to remind you of the steps for solving applied problems or mathematical problems. We hope that you will use them in your work and increase your ability to use them successfully.

STEPS FOR SOLVING APPLIED PROBLEMS

1. Understand the problem.
2. Note what the problem asks for.
3. Look for hidden questions.
4. Estimate a reasonable answer.
5. Set up and solve the conditional sentence(s).
6. Check your answer.

The language of sets is useful in exploring mathematical sentences and number relationships. We saw in the previous chapter that the word "set" is used to denote such ideas as group, collection, or class. A set is a well-defined assortment of things. By well-defined we mean that the things which belong to the set can be readily determined.

Braces are used to indicate a set and { } is read: "the set of." For example, {even numbers less than ten} is read as "the set of even numbers less than ten." The same set can be represented by listing the elements {0, 2, 4, 6, 8}.

List the members of each set described below.

1. {even numbers greater than 1 and less than 11}. {2, 4, 6, 8, 10}

2. {natural numbers that are factors of 6}. A *factor* is a divisor of a given number. {1, 2, 3, 6}

3. A capital letter is used to name a set. If we describe F as {factors of 10}, then F = {1, 2, 5, 10} is a listing of the same set. Describe A = {natural numbers less than 8} by listing the elements.
A = {1, 2, 3, 4, 5, 6, 7}

4. Describe B = {natural numbers less than 26 which are multiples of 5} by listing the elements. B = {5, 10, 15, 20, 25}

5. In the above exercises, your choice was limited to the natural numbers, which are used in counting. The set from which other sets are selected is called the *universe U*. You are well acquainted with two universes: the set of natural numbers N and the set of whole numbers W.

$$N = \{1, 2, 3, 4, \ldots\}$$
$$W = \{0, 1, 2, 3, \ldots\}$$

Using U = {1, 2, 3, 4, 5, 6, 7, 8, 9} describe the following sets by listing the elements.

a. R = {factors of 18} {1, 2, 3, 6, 9}
b. Z = {numbers less than 5} {1, 2, 3, 4}
c. M = {numbers less than 9 and greater than 4} {5, 6, 7, 8}

6. If the members of a given set are also members of another set, the given set is said to be a *subset* of the other set. The members of sets R, Z, and M of Exercise 5 are also members of U. Therefore, we can say sets R, Z, and M are subsets of U. In fact, since U contains at least one element (member) which is not an element of R, Z, and M, we call R, Z, and M *proper subsets* of U. A set is a subset of itself, but this is *not* considered a proper subset by the definition. Describe two other proper subsets of U.
P = {factors of 8} Q = {1, 3, 5, 7, 9}

(circled)

7. Which of the following are proper subsets of the universe U?

a. $X = \{1, 2\}$ c. $B = \{9, 10\}$ e. $R = \{1, 1.1, 2\}$

b. $A = \{5, 4, 0\}$ d. $Y = \{1, 9\}$ f. $K = \{7, 8, 9.1\}$

8. To indicate that a set is a proper subset, we use the symbol $\subset$. This symbol is read as: "is a proper subset of." Therefore, to indicate that R, Z, and M are proper subsets of U, we would write $R \subset U$, $Z \subset U$ and $M \subset U$. Use this notation to indicate those sets in Exercise 7 that are proper subsets of U. $X \subset U$; $Y \subset U$

9. Using the set of natural numbers, N, describe each of the sets below.

Note: The order for listing the elements of a set is not important. Thus, if $A = \{1, 2, 3\}$ and $B = \{2, 1, 3\}$, we say that $A = B$.

a. $L = \{$numbers less than 4$\}$ $L = \{1, 2, 3\}$

b. $P = \{$odd numbers greater than 30 and less than 40$\}$

c. $M = \{$factors of 16$\}$ $M = \{1, 2, 4, 8, 16\}$ $P = \{31, 33, 35, 37, 39\}$

d. $E = \{$even factors of 36$\}$ $E = \{2, 4, 6, 12, 18, 36\}$

10. If $U = \{$natural numbers$\}$, describe the set A where $A = \{$numbers greater than 6 and also less than 4$\}$. Does set A contain any elements? $A = \{\ \}$; no

11. Can you find the members of the following sets? no

a. $T = \{$numbers greater than 5 that will divide 4 exactly$\}$

b. $N = \{$odd numbers that contain two as a factor$\}$

c. $Q = \{$members in your class that are 92 years of age$\}$

12. Each of the sets in Exercise 11 is called the *empty set*. The symbols ϕ or $\{\ \}$ are used to denote the empty set. We always consider the empty set as a proper subset of any given set. An empty set has how many members? none

$\{\ \}, \{7\}, \{11\}, \{14\}, \{7, 11\}, \{7, 14\}, \{11, 14\}, \{7, 11, 14\}$

13. If $A = \{7, 11, 14\}$, list the eight possible subsets of A. Which subset of A is not a proper subset? Can we say $\phi \subset A$? $\{7, 11, 14\}$; yes

14. The number of elements in a set is called its *cardinal number*. What is the cardinal number of set A? 3

15. What is the cardinal number of the empty set? 0

16. If $A = \{6, 4, 2\}$ and $B = \{7, 5, 3\}$, how many members are common to A and B? Note that the set of elements common to A and B is an empty set. 0

17. If $M = \{0, 1, 3, 5\}$ and $N = \{0, 2, 4, 6\}$, describe set P which is the set of members common to M and N. Is it an empty set? 0 ; no

18. What is the cardinal number of set P? Is it an empty set? 1 ; no

19. What is the difference between the sets $X = \phi$ and $Y = \{0\}$?

The cardinal number of set X is 0, but the cardinal number of set Y is 1.

You are familiar with the symbols for the operations of arithmetic +, −, ×, and ÷. Another useful group of symbols can be used as verbs in mathematical sentences. While you have used some of them before, it is useful to summarize them here.

Positive		*Negative*	
$=$	is equal to	$\neq$	is not equal to
$>$	is greater than	$\ngtr$	is not greater than
$<$	is less than	$\nless$	is not less than
$\geq$	is greater than or equal to	$\ngeq$	is not greater than or equal to
$\leq$	is less than or equal to	$\nleq$	is not less than or equal to

What symbols on the positive side of the chart have equivalent meanings in the symbols listed on the negative side of the chart? List the forms that are equivalent using the symbols from the chart.

Using one of the symbols listed above rewrite each of the following sentences.

1. 8 is greater than 3 $8>3$

2. 4 + 2 is equal to 7 − 1 $4+2=7-1$

3. 15 − 9 is less than 4 + 3 $15-9<4+3$

4. 7 + 2 is not equal to or less than 9 − 5 $7+2 \nleq 9-5$

5. 5 + 3 is greater than 15 − 9 $5+3>15-9$

6. 8 + 5 is less than 7 + 7 $8+5<7+7$

7. 17 − 2 is equal to 5 + 10 $17-2=5+10$

8. 14 + 5 is not equal to or greater than 6 + 11 $14-5 \ngeq 6+11$

9. 5 + 9 is less than 20 − 1 $5+9<20-1$

10. 20 − 7 is not equal to or less than 11 + 1 $20-7 \nleq 11+1$

A sentence that expresses a true statement is a *true sentence.* If a sentence expresses a false statement, it is a *false sentence.* Rewrite each of these statements in words, replacing the symbols for the verbs, and state whether it is true or false.

11. $(4+6) < (3+7)$ F

12. $(8-2) \ngtr (10-3)$ T

13. $(27-13) \neq (9+5)$ F

14. $(16+3) = 38-19$ T

15. $(6+5) \leq (28-17)$ T

16. $(3+8) \geq (5+4)$ T

17. $(56-48) \leq (14+4)$ T

18. $(7+9) \neq (25-11)$ T

19. $(15+5) \ngtr (29-15)$ F

20. $(2+15) = (36-18)$ F

21. $(8+7) < (20-3)$ T

22. $(29-5) < (25+4)$ T

23. $(5 \times 6) \leq 31$ T

24. $(20 \div 4) \geq 6$ F

25. $(7+6) > (2 \times 6)$ T

26. $(18+6) = (4 \times 6)$ T

A mathematical sentence that expresses an idea unconditionally about mathematical relationships, whether true or false, is a *closed sentence.* If the idea is *conditional,* it is an *open sentence.* In such sentences a letter, called a variable, is used to represent the number or numbers that will fulfill the stated conditions. The set of numbers that satisfy the conditions is called the *solution set* of the open sentence. To *solve* means to find the solution set.

In later topics you will study some systematic procedures for finding the solution set. The conditions in these open sentences, however, are simple enough so that you can identify the solution set.

Solve: $U = \{\text{whole numbers}\}$.

$8 \div a = 2$ 4 $\frac{1}{4}x = 20$ 80 $t + 9 = 15$ 6

When the verb in a conditional sentence is "=" the solution set may be one number. When the symbol is $<$ or $>$ or a related symbol, more than one number may satisfy the conditions.

EXAMPLE

Solve: $n + 15 \geq 21$ where $U = \{\text{whole numbers}\}$.

It is evident that the sentence is true if n is 6. The sentence is also true if n is any number greater than 6. Thus, the solution set is not one number, but a *set* of numbers, that may be written in either of two ways: {all numbers greater than 5}, or $\{6, 7, 8, \ldots\}$.

Solve: $U = \{\text{whole numbers}\}$. See front.

1. $(n + 3) \leq 8$

2. $(18 - n) \ngtr 5$

3. $(12 \div n) \geq 3$

4. $(n + 6) < 9$

5. $(8 + n) \leq 15$

6. $(5 + n) \nless 14$

7. $(18 \div n) \nless 6$

8. $(24 \div n) \nleq 6$

9. $(n - 5) \geq 5$

10. $(x + 5) = 9$

11. $(n - 3) > 2$

12. $(a + 5) \neq 10$

13. $(x - 3) < 4$

14. $3y = 15$

15. $(12 \div 3) \neq n$

16. $18 = 3n$

17. $x = (5 + 18)$

18. $(a + 6) \ngtr 10$

19. $(y + 20) \leq 31$

20. $(x + 13) \geq 25$

21. $4b > 12$

22. $(y + 19) \neq 30$

23. $(21 + n) \nless 25$

24. $(x + 10) \nless 30$

25. $(a + 16) \leq 20$

26. $(n - 6) < 7$

27. $(x \div 5) = 4$

28. $6n = 24$

29. $13 = (n + 6)$

30. $(x \div 4) = 2$

31. $(36 \div x) \nless 5$

32. $(n - 7) < 1$

33. $(n + 1) < 1$

Addend-Addend-Sum Relationship

An *equation* is a mathematical sentence containing an equals sign (=). The symbols to the right of the equals sign name the same number as do the symbols to the left. If the equation contains a variable, the equation is a conditional sentence. We shall be dealing primarily with solving equations that contain variables. Which of the exercises on page 44 are equations? *10, 14, 16, 17, 27, 28, 29, 30*

Many equations may be solved readily if you identify the variable as one part of an arithmetic operation. We can then use an equivalent form of the relationship to isolate the variable. Let's look at an equation and examine the equivalent forms.

$$\begin{array}{ccccc} 19 & + & 14 & = & 33 \\ \text{addend}_1 & + & \text{addend}_2 & = & \text{sum} \end{array}$$

Note: The small numerals, *subscripts*, denote first and second addends.

The equivalent forms of the equation may be described by

$$\begin{array}{ccccc} 19 & = & 33 & - & 14 \\ \text{addend}_1 & = & \text{sum} & - & \text{addend}_2 \end{array} \quad \text{or} \quad \begin{array}{ccccc} 14 & = & 33 & - & 19 \\ \text{addend}_2 & = & \text{sum} & - & \text{addend}_1 \end{array}$$

The addend-addend-sum relationship can be expressed in the following equivalent forms, where a_1 and a_2 represent the addends and s the sum.

$$a_1 + a_2 = s \qquad \text{or} \qquad a_1 = s - a_2 \qquad \text{or} \qquad a_2 = s - a_1$$

The left hand equation may be read "a sub-one plus a sub-two equals s." Translated into English, it means, "The first addend added to the second addend equals the sum." Read each of the other formulas, and write its English translation. *See front.*

Suppose one of the addends were missing, say 14, in the addition sentence above, and we let x represent that missing addend.

$$19 + x = 33 \qquad \text{is in the form} \qquad a_1 + a_2 = s$$

To solve this equation, we can write its equivalent forms

$$\begin{array}{ccccc} 19 & = & 33 & - & x \\ a_1 & = & s & - & a_2 \end{array} \quad \text{or} \quad \begin{array}{ccccc} x & = & 33 & - & 19 \\ a_2 & = & s & - & a_1 \end{array}$$

The equation on the right quickly yields $x = 14$. Substituting 14 for x in the original equation $19 + x = 33$, we have $19 + 14 = 33$ and $33 = 33$. Therefore, $\{14\}$ is the solution set of the equation.

Isolating the variable becomes easier with practice. The exercises on the following page will provide practice in understanding this relationship as well as increasing the speed with which you can determine the equivalent form that will quickly lead to the solution set.

EXAMPLE

Equation	*Relationship*	
$x - 9 = 16$	$s - a_1 = a_2$	*Check:* $x - 9 = 16$
$x - 16 = 9$	$s - a_2 = a_1$	$25 - 9 = 16$
$x = 9 + 16 = 25$	$s = a_1 + a_2$	$16 = 16$

The solution set is $\{25\}$.

(Ordinarily, we do not use braces to indicate the single-element solution to an equation. You may omit them in your answers.)

For each of the following equations, write the two equivalent forms, showing the addend-addend-sum relationship for each form. Then using the form that has the variable alone on one side, solve the equation. Check your answer by substituting the value into the original equation.

1. $16 + x = 35$ *19*
2. $45 - b = 35$ *10*
3. $12 + 28 = n$ *40*
4. $27 - a = 9$ *18*
5. $n - 14 = 16$ *30*
6. $34 - 15 = y$ *19*
7. $x = 25 + 19$ *44*
8. $21 = 40 - n$ *19*
9. $a + 13 = 30$ *17*
10. $x + 15 = 21$ *6*
11. $15 - y = 6$ *9*
12. $n - 13 = 25$ *38*
13. $25 + x = 41$ *16*
14. $y = 14 + 17$ *31*
15. $19 - n = 9$ *10*
16. $x - 12 = 28$ *40*
17. $13 + n = 20$ *7*
18. $n = 31 - 15$ *16*
19. $80 - x = 53$ *27*
20. $y - 31 = 19$ *50*
21. $18 + 13 = n$ *31*
22. $45 + y = 61$ *16*
23. $28 - a = 19$ *9*
24. $25 - 16 = x$ *9*
25. $51 = 18 + n$ *33*
26. $32 + a = 60$ *28*
27. $b - 22 = 13$ *35*
28. $16 + x = 40$ *24*
29. $x - 50 = 24$ *74*
30. $18 = x + 6$ *12*

Factor-Factor-Product Relationship

In many equations, the variable replaces a factor or product from a multiplication operation.

$$13 \times 15 = 195$$
$$\text{factor}_1 \times \text{factor}_2 = \text{product}$$

The sentence could be written in its equivalent forms.

$$13 = \frac{195}{15} = (195 \div 15) \qquad 15 = \frac{195}{13} = (195 \div 13)$$

$$\text{factor}_1 = \frac{\text{product}}{\text{factor}_2} \qquad \text{factor}_2 = \frac{\text{product}}{\text{factor}_1}$$

The factor-factor-product relationship can be expressed in the following equivalent forms, where f_1 and f_2 represent the factors and p the product:

$$f_1 \times f_2 = p \qquad \text{or} \qquad f_1 = p \div f_2 \qquad \text{or} \qquad f_2 = p \div f_1$$

Note that we have used subscript notation, as in the addend-addend-sum formulas. The first formula is read, "f sub one times f sub two equals p." The English translation is: "The first factor times the second factor gives the product." Read each of the other formulas and write the English translation. *See front.*

Suppose one of the factors were missing in the multiplication sentence above and we let n represent that missing factor:

$$13 \times n = 195$$
$$f_1 \times f_2 = p$$

To solve the equation, we can write the equivalent forms

$$13 = \frac{195}{n} \qquad \text{or} \qquad n = \frac{195}{13}$$

$$f_1 = \frac{p}{f_2} = (p \div f_2) \qquad\qquad f_2 = \frac{p}{f_1} = (p \div f_1)$$

The equation on the right quickly yields $n = 15$. Substituting 15 in the original equation, we have $13 \times 15 = 195$ and $195 = 195$. Therefore, $\{15\}$ is the solution set of the equation.

For each of the following equations, write the two equivalent forms showing the factor-factor-product relationship for each form. Then using the form that has the variable alone on one side, solve the equation. Check your answer by substituting the value into the original equation.

1. $7y = 63$ *9*

2. $14 \times 7 = k$ *98*

3. $28 \div n = 4$ *7*

4. $75 = 5z$ *15*

5. $18 \div 6 = x$ *3*

6. $y \div 6 = 5$ *30*

7. $4 = 32 \div n$ *8*

8. $9x = 72$ *8*

9. $6x = 54$ *9*

10. $36 \div n = 9$ *4*

11. $y \div 16 = 4$ *64*

12. $a = 8 \times 15$ *120*

13. $n = 54 \div 6$ *9*

14. $15y = 60$ *4*

15. $k \div 16 = 8$ *128*

16. $56 \div n = 8$ *7*

17. $7y = 14$ *2*

18. $63 \div x = 7$ *9*

19. $72 \div x = 6$ *12*

20. $63 \div b = 9$ *7*

21. $a \div 5 = 15$ *75*

22. $12n = 96$ *8*

23. $72 \div x = 12$ *6*

24. $16z = 80$ *5*

25. $48 \div 1 = y$ *48*

26. $3n = 39$ *13*

27. $b \div 7 = 9$ *63*

28. $k \div 9 = 17$ *153*

29. $48 \div y = 8$ *6*

30. $36 \div x = 4$ *9*

Using a variable to represent the unknown number, write an equation to express the idea in each sentence. Then solve the equation. Be sure to check your answer according to the original sentence. The equations for the first four exercises are given. *(Any letter may be used for the variable in each case.)*

1. If a certain number is added to 9, the sum is 25. $n + 9 = 25$ *16*

2. When a certain number is subtracted from 50, the result is 32. $50 - y = 32$ *18*

3. Five times a certain number is 235. $5r = 235$ *47*

4. If a certain number is divided by 8, the quotient is 13. $\frac{t}{8} = 13$ or $(t \div 8) = 13$ *104*

5. Fourteen more than a certain number is 49. *n+14=49; 35*

6. If a certain number is decreased by 27, the result is 66. *n−27=66; 93*

7. If a certain number is divided by 14, the result is 17. *n÷14=17; 238*

8. If 112 is divided by a certain number, the result is 16. *112÷n=16; 7*

9. If a certain number is added to 52, the sum is 81. *n+52=81; 29*

10. When a certain number is subtracted from 93, the result is 67. *93−n=67; 26*

11. Eight times a certain number is 136. *8n=136; 17*

12. If a certain number is divided by 13, the result is 117. *n÷13=117; 1521*

13. If 192 is divided by a certain number, the quotient is 24. *192÷n=24; 8*

14. If a certain number is increased by 37, the result is 111. *n+37=111; 74*

15. If 92 is increased by a certain number, the result is 200. *92+n=200; 108*

16. If 19 is multiplied by a certain number, the result is 209. *19n=209; 11*

17. If a certain number is decreased by 45, the result is 160. *n−45=160; 205*

18. When a certain number is added to 193, the sum is 350. *n+193=350; 157*

19. When a certain number is subtracted from 803, the difference is 350. *803−n=350; 453*

20. If 333 is divided by a certain number, the quotient is 37. *333÷n=37; 9*

21. The quotient when a certain number is divided by 9 is 27. *n÷9=27; 243*

22. The difference when a certain number is subtracted from 84 is 49. *84−n=49; 35*

23. When 456 is divided by a certain number, the quotient is 12. *456÷n=12; 38*

24. If a certain number is subtracted from 127, the difference is 75. *127−n=75; 52*

25. When a certain number is divided by 132, the quotient is 19. *n÷132=19; 2508*

EQUATIONS IN PROBLEM SOLVING

The equation serves a very useful purpose in problem solving. By writing the equation, you can identify the operations to be performed, and at the same time you remove the irrelevant elements that may confuse the problem situation. You can then concentrate solely on solving the equation.

EXAMPLE

A truck averaged 48 miles per hour on a 384 mile trip. How long did the trip take?

First define the variable.

Let H represent the number of hours. Then: $d = rt$ tells us

$384 = 48H$	$p = f_1 \times f_2$
$384 \div 48 = H$	$f_2 = p \div f_1$

$H = \square$ 8 (Complete the solution.)

Frequently a problem calls for more than one conditional equation.

EXAMPLE

On a 640-mile trip Jim's car averaged 16 miles on a gallon of gasoline. The gasoline cost 34.9¢ per gallon. What was the cost of gasoline for the trip?

a. Let x represent the number of gallons used. Then: $640 \div 16 = x$. Thus, $x = 40$
He used 40 gallons of gasoline.

b. Let y represent the total cost of the trip. Then: the relationship $c = np$ tells us $y = 40 \times .349$. Do you remember what c, n, p represent?
$y = \square$ (Complete the solution.) The gasoline cost $\square$ for the trip. $13.96

You will notice that all dollar signs and other labels are omitted to keep the equation as simple as possible. Only numerals, variables, and operational symbols are included in the equation.

Many problems encountered in everyday activities arise out of the *applications* of mathematics. The solution of such problems requires a systematic approach similar to the one used in solving mathematical problems as described in Chapter One. This is because you must first identify the mathematical elements in the problem situation, using only those that are relevant, and decide what computations are needed.

STEPS FOR SOLVING APPLIED PROBLEMS

1. Understand the problem.	**2.** Note what the problem asks for.	**3.** Look for hidden questions.
6. Check your answer.	**5.** Set up and solve the conditional sentence(s).	**4.** Estimate a reasonable answer.

Solve each of the following problems. First specify the variable, then set up an equation and solve it. Check your answer.

1. Henry needed to earn \$400 during the summer. He earned \$85 in June and \$155 in July. How much did he need to earn in August?
$85 + 155 + x = 400$; \$160

2. A school library contained 5776 volumes at the beginning of the year. During the year, 1250 volumes were added and 136 volumes were discarded. How many volumes were in the library at the close of the year? $5776 + 1250 - 136 = x$; 6890

3. By how much does the sum of 46 and 27 exceed their difference?
$(46+27) - (46-27) = x$; 54

4. In the morning the manager of a filling station had 2283 gallons of gasoline put into an empty storage tank. During the day he sold 975 gallons from the tank. If the tank has a 5000 gallon capacity, how many gallons are needed to fill the tank?
$5000 - 2283 + 975 = x$; 3692

5. Mike drove over to Springville, a distance of 265 miles. He returned by a longer route. At the start of his trip the odometer reading was 56073.6. At the end of the trip, the odometer read 56634.6. How much longer was the return route? $56634.6 - 56073.6 = x$
$x - 265 = y$; 31

6. Henry's car averages 16 miles on a gallon of gasoline. How far will it travel on \$6.78 worth of gasoline at 33.9¢ a gallon?
$6.78 \div 0.339 = x$; $16x = y$; 320

7. Tom drove to a town 140 miles away in three hours. On his return trip he took four hours. What was his average speed for the entire trip? $280 \div 7 = x$; 40 m.p.h.

8. A scout troop gathered a bushel of elderberries. The troop divided the berries equally among eight families. How many quarts of berries did each family get? $32 \div 8 = x$; 4 quarts

9. The quotient of two numbers is 13. The smaller of the two numbers is 6. What is the larger number? $x \div 6 = 13$; 78

10. One-half the sum of 18 and 30 is divided by twice their difference. What is the quotient? $\frac{18+30}{2} \div 2(30-18) = x$; 1

11. A grocer bought 16 bags of potatoes. Each bag weighed 98 pounds. He packed them in smaller bags, each weighing 32 pounds. How many smaller bags did he fill? $(16)(98)=x$; $x \div 32 = y$; 49

12. Jim sold 23 papers in the afternoon and 28 in the evening. If the price of each paper was 8¢, what were his receipts? $23+28=x$; $8x=y$; \$4.08

13. A contractor estimated the price of a 60-foot wall at \$2 a foot, plus \$195 labor costs. What was the total estimate? $(60)(2)+195=x$; \$315

14. Jon worked at a filling station last summer for \$1.85 an hour. He worked 38 hours per week for 12 weeks. How much did he earn? $(38)(12)(1.85)=x$; \$843.60

15. The Jones family has purchased a home for \$2800 down and monthly payments of \$80 for a period of 15 years. By the time payments are completed, how much will they have paid for the house? $(12)(15)(80)=x$; $x+2800=y$; \$17,200

16. The school band raised \$1965 for new uniforms. If they purchased 38 uniforms at \$45 each, how much of what they raised was left over? $1965-(38)(45)=x$; \$255

17. The product of two numbers is 924. The smaller number is 28. What is the larger number? $28x=924$; 33

18. The product of 18 and 24 is how many times their difference? $(18)(24)=x$; $x \div (24-18)=y$; 72

19. If a car traveling 48 miles an hour goes 16 miles on a gallon of gasoline, how many gallons will it use in 4 hours? $(4)(48) \div 16 = x$; 12

20. Eric earned \$22 a week for 8 weeks last summer. He saved all but \$24.75 of his earnings. How much did he save? $(8)(22)=x$; $x-24.75=y$; \$151.25

21. A school bus travels a daily route of 63 miles. If the bus averages 9 miles per gallon, how much gasoline will it use for a school week of 5 days? $(5)(63)=x$; $x \div 9 = y$; 35

22. A car used 40 gallons of gasoline on a trip of 600 miles. At 35.9¢ per gallon, what did the gasoline cost per mile? $(40)(35.9)=x$; $x \div 600 = y$; 2.4¢

23. Last year Mr. Jones' orchard produced 2400 bushels of apples which he sold at \$2.50 per bushel. The costs of labor and other expenses amounted to \$4500. How much did he have left after these costs had been paid? $(2400)(2.50)=x$; $x-4500=y$; \$1500

24. Martin works part-time in a restaurant for \$1.65 an hour. Last Saturday he worked from 1:00 P.M. to 5:00 P.M. and from 7:00 P.M. to 9:00 P.M. How much did he earn on Saturday? $(6)(1.65)=x$; \$9.90

25. Bob wants to buy a bicycle that will cost \$48. He now has \$18. He plans to earn the rest by working two hours per day after school five days a week, at \$1.50 an hour. How many weeks will it take him to earn the rest of the cost of the bicycle? $48-18=x$; $x \div 15 = y$; 2

A. *Add:*

1.	2.	3.	4.	5.	6.
63	75	764	$7.08	$ .68	$62.74
22	53	505	.63	8.42	.87
54	70	624	8.20	40.20	18.62
47	44	310	5.12	.07	.06
96	50	402	.37	6.26	.48
282	*292*	*2605*	*$21.40*	*$55.63*	*$82.77*

7.	8.	9.	10.	11.
303	717	$300.20	$ 1.19	$100.00
5	17	.45	301.20	.43
92	209	1.06	.16	.07
117	6	20.51	.07	6.00
6	4	.07	150.61	.03
43	703	500.00	.13	250.00
566	*1656*	*$822.29*	*$453.36*	*$356.53*

B. *Subtract:*

1.	2.	3.	4.	5.	6.
97	76	96	57	596	378
34	43	31	37	340	105
63	*33*	*65*	*20*	*256*	*273*

7.	8.	9.	10.	11.
$500.00	$250.60	$185.16	$320.70	$309.08
87.50	3.19	169.00	31.05	185.54
$412.50	*$247.41*	*$ 16.16*	*$289.65*	*$123.54*

C. *Multiply:*

1.	2.	3.	4.	5.
314	750	402	506	607
203	618	320	495	458
63,742	*463,500*	*128,640*	*250,470*	*278,006*

D. *Divide:* Find quotients to the nearest cent:

$ .65	*$ .96*	*$ .89*
1. 71)$46.15	**2.** 38)$36.48	**3.** 56)$49.84
$ .09	*$ 2.31*	*$ 8.45*
4. 703)$65.85	**5.** 861)$1987.35	**6.** 739)$6243.50

If you need more practice, turn to page 471. If not, you may work in the Experts' Corner on the following page.

Tests For Divisibility

It is very useful to know without actually carrying out the division whether one number can be divided by another, with no remainder. There are a number of easy tests. One that you doubtless are familiar with is this.

1. Even numbers are divisible by 2. Odd numbers are not divisible by 2. Which of the following are divisible by 2? a, b, d, e, h

a. 144	**c.** 419	**e.** 258	**g.** 345
b. 10	**d.** 346	**f.** 627	**h.** 402

2. *A natural number is divisible by 3 if the sum of the numbers named by its digits is a multiple of 3.*

EXAMPLES

1. Is 1251 divisible by 3? $1 + 2 + 5 + 1 = 9$
   Since 9 is a multiple of 3, 1251 is divisible by 3
2. Is 4596 divisible by 3?
   $4 + 5 + 9 + 6 = 24$ (Add the digits again) $2 + 4 = 6$
   Since 6 is a multiple of 3, 4596 is divisible by 3

Test each of the following to see if it is divisible by 3. Then divide by 3 to see if you are right. a, d, e, h

a. 264	**c.** 421	**e.** 111	**g.** 922
b. 515	**d.** 582	**f.** 613	**h.** 711

3. *A natural number is divisible by 9 if and only if the sum of the numbers named by its digits is a multiple of 9.* If the sum is a two-digit numeral, and you add them again, the sum will be 9 if the number is divisible by 9.

EXAMPLE

Is 5661 divisible by 9?
$5 + 6 + 6 + 1 = 18$ (Add the digits again) $8 + 1 = 9$
Then 5661 is divisible by 9. What is the quotient?

Test each of the following to see if it is divisible by 9. Then divide the number by 9 to see if you are right. a, c, d, h

a. 432	**c.** 333	**e.** 7417	**g.** 32,721
b. 116	**d.** 513	**f.** 6342	**h.** 59,814

4. *A natural number is divisible by 4 if and only if the last two digits name a number divisible by 4.* This also includes multiples of 100—that is, numbers divisible by 100.

EXAMPLE

Is 624 divisible by 4? The last two digits name the number 24. Since this is divisible by 4, 624 is divisible by 4.

Test each of the following to see if it is divisible by 4. Then divide the number by 4 to see if you are right. a, e, f, h

a. 936	**d.** 214	**g.** 30,114
b. 826	**e.** 1720	**h.** 68,256
c. 154	**f.** 1500	**i.** 99,870

5. *Only those natural numbers whose numerals end in 5 or 0 are divisible by 5.*

Test each of the following to see if it is divisible by 5. Then divide it by 5 to see if you are right. a, c, d, g, h

a. 135	**d.** 720	**g.** 325
b. 273	**e.** 216	**h.** 645
c. 400	**f.** 419	**i.** 551

By combining two or more of the above tests you can find tests of divisibility for other numbers.

Even numbers that are divisible by 3 are divisible by 6.
Numbers divisible both by 3 and by 4 are divisible by 12.
Numbers divisible both by 3 and by 5 are divisible by 15.
Even numbers divisible by 9 are divisible by 18.

6. a. Which of the numbers named in Exercises 1 and 2 above are divisible by 6? (ex. 1) a, e, h (ex. 2) a, d

a, h **b.** Which of the numbers named in Exercise 3 are divisible by 18?

a, f, h **c.** Which of the numbers named in Exercise 4 are divisible by 12?

a, d, h **d.** Which of the numbers named in Exercise 5 are divisible by 15?

e. Write a rule for testing a number for divisibility by 20. See front.

f. Write five numerals that name numbers divisible by 20. See front.

g. Write a rule for testing a number for divisibility by 36. See front.

h. Write five numerals that name numbers divisible by 36. See front.

i. Write a rule for divisibility by 45. See front.

j. Write five numerals that name numbers divisible by 45.
1035; 5040; 7110; 810 (answer not unique)

7. Investigate other references to see if you can find other tests for divisibility. The rules for 7 and 11 are interesting.

CLOSURE

The set of whole numbers $W = \{0, 1, 2, 3, 4, 5 \ldots\}$ may be represented as points on a number line like this:

1. Each number associated with a point on this line is greater than any to its left. How much greater is each number than the one associated with a point immediately to its left? *1*

2. How much less is each number than the one associated with a point immediately to its right? *1*

3. Using the symbol $<$ or $>$, describe the relation between:
   a. 6, 3   **b.** 2, 9   **c.** 7, 12   **d.** 8, 13   **e.** 15, 11
   6>3   *2<9*   *7<12*   *8<13*   *15>11*

4. If you add any two of the elements in the set of whole numbers, is the sum a member of the set of whole numbers? Before you answer, find the sums: $14 + 19$, $15 + 9$, and $18 + 25$. *yes*

5. In the relationship $s = a_1 + a_2$ and $U = W$ is it reasonable to suppose that s must be a whole number? *yes*

6. Is the product of two whole numbers also a member of the set of whole numbers? Before you answer, find the products: 7×9, 11×16, and 14×17. *yes*

7. In the relationship $p = f_1 \times f_2$ and $U = W$ you are combining f_1 sets, each having f_2 elements, into one larger set, p. If your answer to Exercise 5 above was "yes," then you must also agree that p must be a whole number. Why? *It is the same statement. Multiplication is repeated addition.*

8. Exercises 4 through 7 illustrate a property of numbers called *closure*. If addition, subtraction, multiplication, or division is performed on any set of numbers, and the result is always a member of the set, the set is said to be closed under that operation. Is the set of whole numbers closed under addition? Is it closed under multiplication?
   yes   *yes*

Before we continue with the topic of closure, it will be necessary to add some further information about sets. If we can count the number of elements of a set, we say it is a *finite set* and it has a cardinal number. That is, the number of elements in the set can be expressed as a whole number. This was mentioned on page 42. If a set has an unlimited number of elements, we say the set is an *infinite set*. Familiar infinite sets are $N = \{1, 2, 3, 4, \ldots\}$ or $W = \{0, 1, 2, 3, 4, \ldots\}$. The three dots after 4 means that the elements continue in the same pattern without end.

EXAMPLES

If $U = N$ describe the following sets.

$E = \{$all natural numbers containing a factor of 2$\}$	$\{2, 4, 6, \ldots\}$
$O = \{$all odd numbers$\}$	$\{1, 3, 5, \ldots\}$
$T = \{$all multiples of 10$\}$	$\{10, 20, 30, \ldots\}$
$V = \{$all multiples of 5$\}$	$\{5, 10, 15, \ldots\}$

Is the set containing the two elements $\{0, 1\}$ closed under the operation of multiplication? The answer is *Yes* because all possible products using these elements will be 0 or 1. Is this set closed under the operation of addition? The answer is *No* because $1 + 1 = 2$ and 2 is not an element of the set $\{0, 1\}$. A finite set is easier to check than an infinite set.

If $T = \{7, 14, 21, \ldots\}$, is this set closed under addition? *YES.* If the set T is not closed, we need to show an example where the sum of two members of the set is not a multiple of 7. If you test many examples, it will seem reasonable that this set is closed under addition. This set T is also closed under multiplication.

9. Which of these sets are closed under addition? *b, c, d*

a. $\{1, 3, 5, 7, 9, \ldots\}$
b. $\{3, 6, 9, 12, \ldots\}$
c. $\{2, 4, 6, 8, 10, \ldots\}$
d. $\{5, 10, 15, 20, \ldots\}$

10. Which of the sets in Exercise 9 are closed under multiplication? *all*

11. Is the set of whole numbers closed under subtraction? Before you answer, find the differences: $(17 - 9)$, $(5 - 9)$, and $(13 - 25)$. *no*

12. If you can find a *counterexample*, one that disproves the idea being tested, then you must conclude that the set is *not* closed under subtraction. Using the symbol $<$ or $>$, complete this statement: The set of whole numbers is *not* closed in the relationship $a_1 = s - a_2$ when $a_2 \square s$. $>$

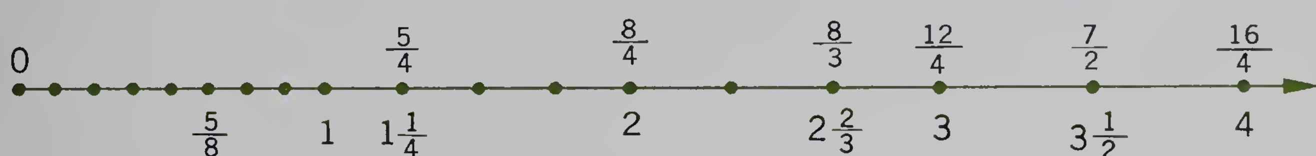
13. Is the set of whole numbers closed under division? Before you answer, find the quotients: $(24 \div 6)$, $(72 \div 5)$, and $(38 \div 7)$. *no*

14. Since you can find a counter example, you must conclude that the set of whole numbers is *not* closed under division.
The set of whole numbers is *not* closed in the relationship $f_1 = p \div f_2$ unless f_2 is a factor of p.
Are the natural numbers closed under addition? subtraction? multiplication? division?

In the next section we shall learn about a set of numbers that is closed under division. Later on, we shall learn about a set of numbers that is closed under subtraction. *yes ; no ; yes ; no*

In the relationship $f_1 = p \div f_2$ which may also be stated $f_1 = \frac{p}{f_2}$, where p is an element of W and f_2 is an element of N, we have the set of fractional numbers of arithmetic. A fractional number of arithmetic is associated with a point on the number line. If f_2 is *not* a factor of p, then the point will be on the space between two whole numbers. In general, the set of fractional numbers of arithmetic F_a are expressed as $\frac{a}{b}$ where a is a whole number and b is a natural number. Why is $b \neq 0$? If you do not remember, see page 40.

Any fraction $\frac{a}{b}$ can be associated with a point on the number line. The number below the bar, the denominator, tells how many parts the segment "between" two whole numbers is to be divided into. The number above the bar, the numerator, tells how many of these parts to count from the left, in order to locate the point on the number line.



1. Locate each of the following on a number line.

a. $\frac{7}{8}$ **c.** $\frac{7}{4}$ **e.** $\frac{4}{8}$
b. $\frac{3}{4}$ **d.** $\frac{6}{8}$ **f.** $\frac{3}{2}$

2. If the numerator is greater than the denominator (this is an *improper* fraction), the number named can be more readily located on the number line if it is expressed as a *mixed numeral*. To express an improper fraction as a mixed numeral, divide the numerator by the denominator, and express the remainder as a fraction. For example, $\frac{15}{4} = 3\frac{3}{4}$ and $\frac{19}{2} = 9\frac{1}{2}$. Express each of the following as mixed numerals.

a. $\frac{15}{4}$ $3\frac{3}{4}$ **c.** $\frac{18}{7}$ $2\frac{4}{7}$ **e.** $\frac{9}{4}$ $2\frac{1}{4}$
b. $\frac{12}{5}$ $2\frac{2}{5}$ **d.** $\frac{3}{2}$ $1\frac{1}{2}$ **f.** $\frac{97}{8}$ $12\frac{1}{8}$

3. If the numerator is exactly divisible by the denominator, the fraction can be expressed as a whole number.

a. $\frac{8}{4} = \square$ 2 **d.** $3 = \frac{\square}{4}$ $\frac{12}{4}$ **g.** $\frac{12}{6} = \square$ 2

b. $5 = \frac{\square}{2}$ $\frac{10}{2}$ **e.** $\frac{9}{3} = \square$ 3 **h.** $1 = \frac{\square}{10}$ $\frac{10}{10}$

c. $\frac{16}{8} = \square$ 2 **f.** $2 = \frac{\square}{8}$ $\frac{16}{8}$ **i.** $\frac{27}{9} = \square$ 3

4. As you have seen, any fractional number associated with a point on the number line has many names.

$$\frac{3}{8} = \frac{\square\ 6}{16} = \frac{\square\ 9}{24} = \frac{\square\ 12}{32}$$

Equivalent fractions name the same number. Both numerator and denominator of $\frac{3}{8}$ were multiplied by what number to express $\frac{3}{8}$ with a denominator of 16? to express $\frac{3}{8}$ with a denominator of 24? to express $\frac{3}{8}$ with a denominator of 32? *2 ; 3 ; 4*

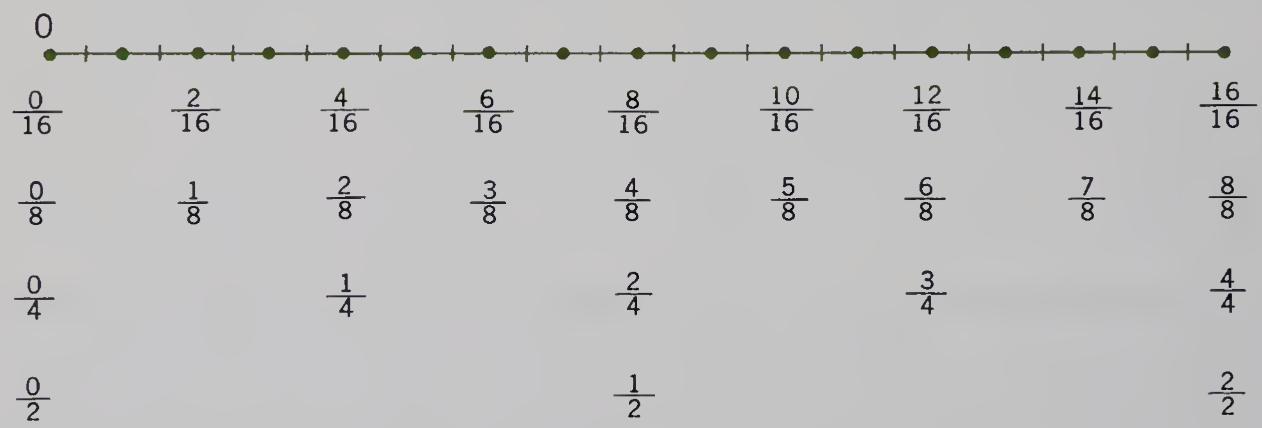
5. If any number is multiplied by 1, the product is the original number. Since $\frac{n}{n} = 1$, then $\frac{a \times n}{b \times n} = \frac{an}{bn} = \frac{a}{b}$.

Use this property to solve the following.

a. $\frac{3}{4} = \frac{x}{32}$ *24* c. $\frac{5}{16} = \frac{x}{48}$ *15* e. $\frac{2}{3} = \frac{10}{x}$ *15*

b. $\frac{5}{9} = \frac{x}{45}$ *25* d. $\frac{3}{8} = \frac{12}{x}$ *32* f. $\frac{6}{x} = \frac{2}{5}$ *15*

6. Can you find a number between $\frac{1}{4}$ and $\frac{5}{16}$? That is, a number n, such that $n > \frac{1}{4}$ and $n < \frac{5}{16}$, or simply, $\frac{5}{16} > n > \frac{1}{4}$. *yes*



FRACTIONAL EQUIVALENTS

A simple method is to find the average of the given numbers.

$$\frac{1}{2}\left(\frac{1}{4} + \frac{5}{16}\right) = \frac{1}{2}\left(\frac{4}{16} + \frac{5}{16}\right) = \frac{1}{2} \times \frac{9}{16} = \frac{9}{32}$$

Find a fractional number between each of the following pairs.

7/32 a. $\frac{3}{16}$ and $\frac{1}{4}$ *7/64* b. $\frac{3}{32}$ and $\frac{1}{8}$ *1/4* c. $\frac{1}{5}$ and $\frac{3}{10}$ *5/64* d. $\frac{1}{16}$ and $\frac{3}{32}$

7. Look at your answer for Exercise 6d. Find a number between that number and $\frac{3}{32}$. *11/128*

8. Find a number between your answer for Exercise 7 and $\frac{3}{32}$. *23/256*

9. Can that process be continued indefinitely? How many numbers are between $\frac{1}{16}$ and $\frac{3}{32}$? Since we can always find a fractional number between two given fractional numbers of arithmetic, we say that the set of fractional numbers of arithmetic is a *dense set*.
yes ; infinite

A. *Solve:*

1. $\frac{3}{16} = \frac{n}{48}$ 9
2. $\frac{5}{9} = \frac{15}{n}$ 27
3. $\frac{5}{8} = \frac{15}{n}$ 24
4. $\frac{4}{7} = \frac{n}{21}$ 12
5. $\frac{2}{3} = \frac{n}{15}$ 10
6. $\frac{7}{12} = \frac{28}{n}$ 48

B. Find the least common multiple (L.C.M.) of the denominators.

1. $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}$ 8
2. $\frac{2}{5}, \frac{7}{15}, \frac{5}{6}$ 30
3. $\frac{4}{9}, \frac{5}{6}, \frac{7}{12}$ 36
4. $\frac{1}{2}, \frac{3}{4}, \frac{1}{6}$ 12
5. $\frac{3}{4}, \frac{3}{8}, \frac{5}{16}$ 16
6. $\frac{1}{6}, \frac{1}{9}, \frac{1}{2}$ 18
7. $\frac{3}{16}, \frac{5}{8}, \frac{3}{4}, \frac{1}{2}$ 16
8. $\frac{4}{7}, \frac{1}{2}, \frac{9}{14}$ 14
9. $\frac{5}{6}, \frac{2}{3}, \frac{3}{8}$ 24
10. $\frac{4}{15}, \frac{5}{6}, \frac{4}{5}$ 30

C. *Add:*

1. $\begin{array}{r} \frac{1}{8} \\ \frac{7}{8} \\ \frac{3}{8} \\ \hline 1\frac{3}{8} \end{array}$

2. $\begin{array}{r} \frac{3}{16} \\ \frac{5}{16} \\ \frac{1}{16} \\ \frac{7}{16} \\ \hline 1 \end{array}$

3. $\begin{array}{r} \frac{3}{8} \\ 2\frac{3}{4} \\ 7\frac{5}{8} \\ \frac{1}{2} \\ \hline 11\frac{1}{4} \end{array}$

4. $\begin{array}{r} 2\frac{1}{3} \\ \frac{7}{12} \\ 3\frac{5}{6} \\ 6\frac{1}{2} \\ \hline 13\frac{1}{4} \end{array}$

5. $\begin{array}{r} \frac{3}{4} \\ 1\frac{2}{3} \\ 3\frac{1}{6} \\ 5\frac{5}{8} \\ \hline 11\frac{5}{24} \end{array}$

D. *Subtract:*

1. $\begin{array}{r} \frac{5}{6} \\ \frac{1}{3} \\ \hline \frac{1}{2} \end{array}$

2. $\begin{array}{r} 1\frac{3}{4} \\ \frac{5}{8} \\ \hline 1\frac{1}{8} \end{array}$

3. $\begin{array}{r} 5\frac{1}{2} \\ 3\frac{1}{4} \\ \hline 2\frac{1}{4} \end{array}$

4. $\begin{array}{r} 6\frac{7}{12} \\ 3\frac{1}{2} \\ \hline 3\frac{1}{12} \end{array}$

5. $\begin{array}{r} 2\frac{1}{8} \\ 1\frac{3}{4} \\ \hline \frac{3}{8} \end{array}$

E. *Solve:*

1. $x - 1\frac{3}{4} = 5\frac{1}{2}$ $7\frac{1}{4}$
2. $2\frac{3}{8} - x = 1\frac{1}{4}$ $1\frac{1}{8}$
3. $15\frac{3}{4} + x = 25\frac{1}{3}$ $9\frac{7}{12}$
4. $x - 4\frac{1}{3} = 2\frac{3}{4}$ $7\frac{1}{12}$
5. $x + 15\frac{1}{2} = 30$ $14\frac{1}{2}$
6. $x + \frac{2}{3} = \frac{3}{4}$ $\frac{1}{12}$
7. $2\frac{5}{12} = x + 1\frac{1}{2}$ $\frac{11}{12}$
8. $17\frac{1}{9} = x - 2\frac{1}{2}$ $19\frac{11}{18}$
9. $38\frac{1}{2} - 19\frac{3}{4} = x$ $18\frac{3}{4}$
10. $13\frac{3}{4} + x = 25\frac{1}{3}$ $11\frac{7}{12}$

If you need more practice, turn to the Practice Exercises on page 473 and following. If not, you may work in the Experts' Corner on the following page.

Some Properties of Zero

Zero as a number has several interesting and important properties. Its invention and use as a digit made possible the development of our positional system of numeration. Zero is the only member of the set of whole numbers that is not a natural number. Some of the special properties of zero are illustrated in the following exercises.

1. *Add:*

   a. $0 + 9$ 9 c. $1\frac{7}{8} + 0$ $1\frac{7}{8}$ e. $0 + 0$ 0

   b. $3\frac{1}{2} + 0$ $3\frac{1}{2}$ d. $0 + 13.5$ 13.5 f. $7.18 + 0$ 7.18

2. If a is any fractional number, $a + 0 = ?$ Write a statement regarding the sum of zero and any fractional number of arithmetic. See front.

3. If a is any fractional number of arithmetic and $a + n = a$, then n is called the *identity element* for addition, or the *additive identity*. What is the identity element for addition? 0

4. *Subtract:*

   a. $18 - 0$ 18 c. $5\frac{3}{7} - 0$ $5\frac{3}{7}$ e. $0 - 0$ 0

   b. $4\frac{1}{3} - 0$ $4\frac{1}{3}$ d. $19.7 - 0$ 19.7 f. $83.33 - 0$ 83.33

5. If a is any fractional number of arithmetic, $a - 0 = ?$ a Write a statement regarding the result of subtracting zero from any fractional number of arithmetic. See front.

6. If a is any fractional number of arithmetic, and $a - n = a$, then n is called the *identity element* for subtraction, or the *subtractive identity*. What is the identity element for subtraction? 0

7. Zero is the number associated with the number of elements in the empty set: How many elements are contained in the intersection of two parallel lines r and s? $r \cap s = ?$ 0 ; $\emptyset$

8. Suppose $A = \{0, 2, 4\}$ and $B = \{0, 1, 3\}$. Complete the statement: $A \cap B = ?$ $\{0\}$

9. How many elements do sets A and B have in common? Is their intersection the empty set? 1 ; no

10. Explain the difference between $\{0\}$ and $\{\ \}$. Review page 42. Does $\{0\} \cup \{1, 2, 3, \ldots\} = W$? yes Their cardinal numbers are 0 and 1, respectivly.

11. An important step in the development of the positional decimal system of numeration was the introduction of zero as a place holder. What is the value of each of the numbers named by the digits in the numeral 2050? What do the zeros tell you? See front.

12. Write 8, then place a zero after it. What number have you now named? How does the zero change the value of the number named by the digit 8? If we wrote 80 and placed a zero after this numeral, how does the extra zero change the number named by 80?
80 , multiplied by 10 ; multiplied by 10

13. Write an explanation of how one can multiply mentally:

a. 4×100 **b.** 4×1000 **c.** $4 \times 10{,}000$
Place the given number of zeros after 4.

14. Write an explanation of how one can divide mentally:

a. 0.7 by 10 **b.** 0.7 by 100 **c.** 0.7 by 1000
Place the given number of zeros between 7 and the decimal point.

15. *Subtract:*

a. $13 - 13$ *0* **c.** $6\frac{1}{2} - 6\frac{1}{2}$ *0* **e.** $0 - 0$ *0*
b. $4.5 - 4.5$ *0* **d.** $8.13 - 8.13$ *0* **f.** $0.03 - 0.03$ *0*

16. What is the result when you subtract any fractional number of arithmetic from itself? *0*

17. *Multiply:*

a. 8×0 *0* **c.** 0×5.3 *0* **e.** 0×0 *0*
b. 9.7×0 *0* **d.** $4\frac{1}{7} \times 0$ *0* **f.** $0 \times 4\frac{2}{3}$ *0*

18. Write a statement regarding the product of any fractional number of arithmetic and zero. *The product of any fractional number and zero is zero.*

19. If we divide zero by any fractional number, as $0 \div \frac{1}{3}$, we are to find n so that: $0 \div \frac{1}{3} = n$ or $0 = \frac{1}{3}n$. Then n must be zero. Why? *If the product of two factors is zero, then one of its factors must be zero.*

20. *Divide:*

a. $0 \div 15$ *0* **c.** $0 \div 3.7$ *0* **e.** $0 \div 0$ *undefined*
b. $0 \div 3\frac{1}{2}$ *0* **d.** $0 \div 0.83$ *0* **f.** $0 \div 0.05$ *0*

21. Write a statement regarding division of zero by any non-zero fractional number of arithmetic. *If zero is divided by any fractional non-zero number, the quotient is zero.*

22. The question of finding the quotient of a non-zero fractional number divided by zero has been considered earlier in the chapter. If $\frac{2}{7} \div 0 = n$, then $\frac{2}{7} = n \times 0$. Why is this an impossible condition? Similarly, why is $0 \div 0$ meaningless? *If one of two factors is zero, the product is zero.*

23. Using a variable, say y, show that the division by zero in the set of fractional numbers of arithmetic is undefined. *If $y \div 0 = n$, then $n \cdot 0 = y$. If $y \neq 0$, this is impossible (see ex. 18). If $y = 0$, n is not unique.*

24. The variable n is an element of {fractional numbers of arithmetic}.

a. $n + 0 = \square$ *n* **c.** $n - n = \square$ *0* **e.** $0 \div n = \square$ *0*
b. $n - 0 = \square$ *n* **d.** $n \times 0 = \square$ *0* **f.** $0 + 0 = \square$ *0*

25. *Solve:*

a. $1\frac{3}{4} - 1\frac{3}{4} = x$ *0* **d.** $0 + 7.057 = x$ *7.057* **g.** $0 \div 18\frac{1}{3} = x$ *0*
b. $3\frac{5}{16} + 0 = x$ *$3\frac{5}{16}$* **e.** $13.05 - 13.05 = x$ *0* **h.** $0 \times 17\frac{5}{8} = x$ *0*
c. $11.07 \times 0 = x$ *0* **f.** $78.9 + 0 = x$ *78.9* **i.** $13.7 - 0 = x$ *13.7*

When numbers are compared by division, ratios express the comparison. A ratio states what part one number is of another, or how many times one number is as great as another. When the ratio is expressed in the form $\frac{a}{b}$, the denominator is the number to which the numerator is compared.

EXAMPLE

A football team has won six out of nine games. What part of its games has it won? If the ratio is to be expressed as $\frac{a}{b}$, then the b must be 9. Why? $\frac{6}{9}$; *the number to which 6 is compared*

The ratio is $\frac{6}{9}$, which may be written in simpler form as $\frac{2}{3}$. Remember, a fraction is in *simplest form* if the numerator and denominator do not have a common factor other than 1.

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1. A basketball team has won 15 games and lost 10. What is the ratio of the number of games won to the number of games played? $\frac{3}{5}$

2. A city lot measures 120 feet in length and 80 feet in width. What is the ratio of width to length? the ratio of length to width? $\frac{2}{3}$; $\frac{3}{2}$

3. The height of the city water tower is 40 feet. The measure of a diameter is 18 feet. What is the ratio of the measure of a diameter to the height? the ratio of height to the measure of a diameter? $\frac{9}{20}$; $\frac{20}{9}$

4. Jim earned \$250 last summer, while Harry was earning \$200. Jim's earnings were how many times as great as Harry's earnings? Harry's earnings were what fraction of the amount earned by Jim? $\frac{5}{4}$; $\frac{4}{5}$

5. Mr. Jensen earns \$7000 a year. His family expenditure for food is \$1750. What part of his income is spent for food? $\frac{1}{4}$

6. At a meeting of the Camera Club, six members were absent. There are 32 members in the club. What part of the membership was absent? was present? $\frac{3}{16}$; $\frac{13}{16}$

7. On a test of 24 questions Elaine had three wrong. What part of the test did she have correct? wrong? $\frac{7}{8}$; $\frac{1}{8}$

8. Cindy bought a book for \$2.80 that was regularly priced at \$3.50. What part of the regular price was the reduction? $\frac{1}{5}$

9. What is the ratio of a centimeter to a meter? $\frac{1}{100}$ to a millimeter? $\frac{10}{1}$

A statement of equality between two ratios is a *proportion*. This equation, for example, is a proportion:

$$\begin{matrix} \textit{extreme} \rightarrow \\ \textit{mean} \rightarrow \end{matrix} \frac{5}{8} = \frac{15}{24} \begin{matrix} \leftarrow \textit{mean} \\ \leftarrow \textit{extreme} \end{matrix}$$

You should remember the names because of the proportion property:

$$\text{If } \frac{e_1}{m_1} = \frac{m_2}{e_2}, \text{ then } m_1 m_2 = e_1 e_2$$

The product of the means is equal to the product of the extremes.

If one of the terms of a proportion is represented by a variable, the property of proportions can be applied to yield an equation.

EXAMPLE

$$\frac{5}{13} = \frac{25}{e}$$

The product of the extremes is $5e$. The product of the means is 325. Since these products are equal:

$325 = 5e$ $\qquad$ $p = f_1 f_2$ $\qquad$ *Check:* $\frac{5}{13} = \frac{25}{65}$

$65 = e$ $\qquad$ $f_2 = p \div f_1$ $\qquad$ $325 = 325$

Solve: Check your answer.

1. $\frac{3}{16} = \frac{x}{48}$ 9
2. $\frac{4}{19} = \frac{n}{57}$ 12
3. $\frac{4}{5} = \frac{16}{x}$ 20
4. $\frac{5}{y} = \frac{25}{30}$ 6
5. $\frac{3}{11} = \frac{15}{n}$ 55
6. $\frac{6}{x} = \frac{12}{34}$ 17
7. $\frac{9}{13} = \frac{y}{39}$ 27
8. $\frac{8}{a} = \frac{32}{20}$ 5
9. $\frac{15}{44} = \frac{x}{11}$ 3.75
10. $\frac{18}{n} = \frac{3}{8}$ 48
11. $\frac{y}{38} = \frac{2}{19}$ 4
12. $\frac{9}{4} = \frac{5}{x}$ $2\frac{2}{9}$
13. $\frac{x}{13} = \frac{21}{39}$ 7
14. $\frac{6}{7} = \frac{n}{35}$ 30
15. $\frac{7}{z} = \frac{4}{5}$ 8.75
16. $\frac{8}{5} = \frac{n}{17}$ 27.2
17. $\frac{19}{n} = \frac{76}{16}$ 4
18. $\frac{28}{29} = \frac{7}{x}$ 7.25
19. $\frac{y}{100} = \frac{5}{8}$ 62.5
20. $\frac{n}{100} = \frac{8}{15}$ $53\frac{1}{3}$
21. $\frac{x}{100} = \frac{3}{16}$ 18.75

The product of two fractional numbers is a fractional number such that the numerator of the new fraction is the product of the numerators and the denominator of the new fraction is the product of the denominators.

EXAMPLE

Solve: $$n = \frac{3}{4} \times \frac{8}{9}$$

$$\frac{3}{4} \times \frac{8}{9} = \frac{3 \times 8}{4 \times 9} = \frac{24}{36}$$

Then $$n = \frac{2}{3}$$

If we can identify factors common to numerator and denominator and "remove" them before multiplying, we can work with smaller numbers. Above we see that $\dfrac{\cancel{3} \times (\cancel{2} \times \cancel{2} \times 2)}{(\cancel{2} \times \cancel{2}) \times (\cancel{3} \times 3)} = \dfrac{2}{3}$

Mixed numerals should be expressed as fractions before multiplying.

EXAMPLE

Solve: $n = 1\frac{5}{8} \times 2\frac{2}{3}$

$n = \frac{13}{8} \times \frac{8}{3}$ Renaming $1\frac{5}{8}$ and $2\frac{2}{3}$

$n = \frac{13}{\cancel{8}_1} \times \frac{\cancel{8}^1}{3}$ Dividing the numerator and denominator by the common factor 8

$n = \frac{13 \times 1}{1 \times 3} = \frac{13}{3}$ or $4\frac{1}{3}$ The product is $4\frac{1}{3}$

Multiply: Express answers in simplest form.

1. $\frac{2}{3} \times \frac{15}{16}$ *$\frac{5}{8}$*

2. $\frac{3}{8} \times \frac{4}{9}$ *$\frac{1}{6}$*

3. $\frac{5}{8} \times \frac{2}{3}$ *$\frac{5}{12}$*

4. $\frac{3}{14} \times \frac{7}{8}$ *$\frac{3}{16}$*

5. $\frac{4}{5} \times \frac{1}{2}$ *$\frac{2}{5}$*

6. $\frac{7}{8} \times \frac{4}{21}$ *$\frac{1}{6}$*

7. $\frac{4}{15} \times \frac{5}{8}$ *$\frac{1}{6}$*

8. $\frac{21}{25} \times \frac{5}{7}$ *$\frac{3}{5}$*

9. $18\frac{3}{5} \times \frac{5}{31}$ *3*

10. $16\frac{2}{3} \times \frac{4}{15}$ *$4\frac{4}{9}$*

11. $14\frac{2}{3} \times 1\frac{7}{11}$ *24*

12. $12\frac{1}{2} \times \frac{24}{25}$ *12*

13. $4\frac{2}{5} \times \frac{5}{11}$ *2*

14. $14\frac{2}{7} \times 5\frac{3}{5}$ *80*

15. $6\frac{1}{4} \times 2\frac{4}{5}$ *$17\frac{1}{2}$*

16. $12\frac{4}{5} \times 7\frac{5}{8}$ *$97\frac{3}{5}$*

17. $\frac{2}{3} \times 3\frac{1}{3}$ *$2\frac{2}{9}$*

18. $\frac{8}{9} \times 7\frac{7}{8}$ *7*

19. $2\frac{7}{16} \times 13\frac{2}{3}$ *$33\frac{5}{16}$*

20. $2\frac{5}{8} \times 9\frac{1}{7}$ *24*

21. $\frac{7}{12} \times 3\frac{1}{3}$ *$1\frac{17}{18}$*

If the product of two numbers is 1, each factor is the *multiplicative inverse* of the other.

Solve:	$n \times 15 = 1$	$f_1 \times f_2 = p$
Then	$n = 1 \div 15$	$f_1 = p \div f_2$
And	$n = \dfrac{1}{15}$	$f_1 = \dfrac{p}{f_2}$

We can see that the multiplicative inverse of this natural number has 1 for its numerator and the given number as its denominator. Is this generally true? Let the natural numbers m and n be multiplicative inverses.

Then	$mn = 1$	$f_1 \cdot f_2 = p$
or	$m = 1 \div n$	$f_1 = p \div f_2$
Therefore	$m = \dfrac{1}{n}$	$f_1 = \dfrac{p}{f_2}$
Also	$n = 1 \div m$	$f_2 = p \div f_1$
and	$n = \dfrac{1}{m}$	$f_2 = \dfrac{p}{f_1}$

1. Find the multiplicative inverse of each of the following:

a. 9 $\frac{1}{9}$	**c.** 47 $\frac{1}{47}$	**e.** 16 $\frac{1}{16}$	**g.** 36 $\frac{1}{36}$	**i.** $\frac{1}{45}$ 45
b. 25 $\frac{1}{25}$	**d.** 19 $\frac{1}{19}$	**f.** 13 $\frac{1}{13}$	**h.** $\frac{1}{7}$ 7	**j.** $\frac{1}{6}$ 6

What is the multiplicative inverse of a fraction whose numerator is not equal to one? *a fraction whose numerator is the denominator of the original fraction and whose denominator is the numerator of the original fraction*

EXAMPLE

What is the multiplicative inverse of $\frac{5}{8}$?

$$\text{If } n \times \frac{5}{8} = 1, \text{ then } n = 1 \div \frac{5}{8} \text{ or } n = \frac{1}{\frac{5}{8}}$$

We will multiply numerator and denominator by $\frac{8}{5}$ so that the denominator will be 1:

$$\frac{1}{\frac{5}{8}} \times \frac{\frac{8}{5}}{\frac{8}{5}} \qquad \text{or} \qquad \frac{1 \times \frac{8}{5}}{\frac{5}{8} \times \frac{8}{5}} = \frac{\frac{8}{5}}{1} = \frac{8}{5}$$

Thus you see that the multiplicative inverse of a given fraction is the fraction with numerator and denominator interchanged. This is reasonable, since the product of the fractions is 1.

2. Find the multiplicative inverse of the following. Change mixed numerals to fractions before finding the multiplicative inverse.

a. $\frac{4}{9}$ *$\frac{9}{4}$*	**c.** $\frac{3}{8}$ *$\frac{8}{3}$*	**e.** $\frac{9}{15}$ *$\frac{15}{9}$*	**g.** $1\frac{3}{5}$ *$\frac{5}{8}$*	**i.** $3\frac{1}{3}$ *$\frac{3}{10}$*
b. $\frac{2}{5}$ *$\frac{5}{2}$*	**d.** $\frac{4}{7}$ *$\frac{7}{4}$*	**f.** $5\frac{2}{5}$ *$\frac{5}{27}$*	**h.** $2\frac{1}{7}$ *$\frac{7}{15}$*	**j.** $6\frac{2}{3}$ *$\frac{3}{20}$*

3. Since multiplication is the inverse operation of division, it seems reasonable that division can be performed by an equivalent multiplication operation by using the multiplicative inverse of the divisor. That is, $a \div b$ or $\frac{a}{b}$ $(b \neq 0)$ is equivalent to $a \cdot \frac{1}{b}$.

EXAMPLES

$$35 \div 5 = 35 \cdot \frac{1}{5} = 7 \qquad 91 \div 13 = 91 \times \frac{1}{13} = 7$$

$$54 \div 27 = 54 \cdot \frac{1}{27} = 2 \qquad 65 \div 15 = 65 \cdot \frac{1}{15} = \frac{\cancel{5} \cdot 13}{\cancel{5} \cdot 3} = 4\frac{1}{3}$$

Solve using the multiplicative inverse.

a. $48 \div 6 = m$ *8*	**d.** $49 \div 7 = c$ *7*	**g.** $36 \div 4 = s$ *9*
b. $18 \div 2 = n$ *9*	**e.** $24 \div 3 = d$ *8*	**h.** $16 \div 8 = b$ *2*
c. $27 \div 3 = a$ *9*	**f.** $12 \div 4 = r$ *3*	**i.** $21 \div 7 = t$ *3*

4. You can use the multiplicative inverse to divide fractional numbers other than counting numbers.

$$\frac{9}{16} \div \frac{3}{4} = y \qquad n = \frac{11}{4} \div \frac{22}{5}$$

$$\frac{9 \cdot 4}{16 \cdot 3} = \frac{3}{4} = y \qquad n = \frac{11}{4} \cdot \frac{5}{22} = \frac{5}{8}$$

Divide: Mixed numerals should be written as fractions.

a. $\frac{15}{16} \div \frac{5}{8}$ *$1\frac{1}{2}$*	**f.** $\frac{3}{7} \div \frac{13}{14}$ *$\frac{6}{13}$*	**k.** $17\frac{1}{2} \div \frac{7}{8}$ *20*
b. $\frac{5}{9} \div \frac{2}{3}$ *$\frac{5}{6}$*	**g.** $3 \div \frac{3}{4}$ *4*	**l.** $5\frac{5}{7} \div 2\frac{3}{5}$ *$2\frac{18}{91}$*
c. $\frac{7}{8} \div \frac{3}{16}$ *$4\frac{2}{3}$*	**h.** $20 \div \frac{10}{11}$ *22*	**m.** $7\frac{1}{2} \div 6\frac{1}{4}$ *$1\frac{1}{5}$*
d. $\frac{4}{5} \div \frac{14}{15}$ *$\frac{6}{7}$*	**i.** $10\frac{1}{2} \div \frac{3}{8}$ *28*	**n.** $2\frac{1}{8} \div 1\frac{17}{64}$ *$1\frac{55}{81}$*
e. $\frac{3}{4} \div \frac{2}{3}$ *$1\frac{1}{8}$*	**j.** $\frac{4}{9} \div 8$ *$\frac{1}{18}$*	**o.** $6\frac{2}{5} \div 1\frac{3}{7}$ *$4\frac{12}{25}$*

5. Use the relationship $f_1 f_2 = p$ to solve for the variable in the following exercises.

a. $\frac{3}{4} \div x = \frac{3}{8}$ *2*	**f.** $\frac{5}{6} \div c = \frac{1}{3}$ *$2\frac{1}{2}$*	**k.** $s \div 3 = \frac{1}{3}$ *1*
b. $\frac{4}{7} \div y = 4$ *$\frac{1}{7}$*	**g.** $t \div \frac{1}{16} = 2$ *$\frac{1}{8}$*	**l.** $m \div 5 = \frac{3}{5}$ *3*
c. $\frac{2}{3} \div a = \frac{4}{9}$ *$1\frac{1}{2}$*	**h.** $v \div \frac{2}{3} = \frac{3}{4}$ *$\frac{1}{2}$*	**m.** $n \div \frac{2}{9} = \frac{3}{8}$ *$\frac{1}{12}$*
d. $\frac{9}{16} \div f = \frac{3}{4}$ *$\frac{3}{4}$*	**i.** $\frac{7}{8} \div r = \frac{1}{2}$ *$1\frac{3}{4}$*	**n.** $\frac{7}{8} \div x = \frac{3}{16}$ *$4\frac{2}{3}$*
e. $20 \div m = \frac{10}{11}$ *22*	**j.** $10\frac{1}{2} \div y = \frac{3}{8}$ *28*	**o.** $7\frac{1}{2} \div y = 6\frac{1}{4}$ *$1\frac{1}{5}$*

Suppose you are asked to solve the equation:

$$\frac{3n}{7} = 18$$

Remember, $\frac{3n}{7}$ is a more convenient way of writing $\frac{3}{7}n$. In the equation above, $\frac{3}{7}$ is the *coefficient* of n; that is, $\frac{3}{7}$ is the number by which n has been multiplied. The equation is the relationship $f_1f_2 = p$; 18 is the product while $\frac{3}{7}$ and n are the factors. So:

$\frac{3n}{7} = 18$	$f_1 \times f_2 = p$
$n = 18 \div \frac{3}{7}$	$f_2 = p \div f_1$
$n = 18 \times \frac{7}{3}$	Use of multiplicative inverse
$n = 42$	*Check:* $\frac{3 \times 42}{7} = 18$

The solution set is $\{42\}$.

1. $x = 35 \cdot \frac{6}{5}$
2. $a = 63 \cdot \frac{9}{7}$
3. $n = 16 \cdot \frac{5}{4}$
4. $36 \cdot \frac{3}{4} = y$
5. $24 \cdot \frac{4}{3} = x$
6. $\frac{27}{15} \cdot \frac{5}{9} = n$
7. $n = \frac{8}{3} \cdot \frac{1}{32}$
8. $n = 36 \cdot \frac{5}{6}$
9. $\frac{5}{6} \cdot 18 = x$
10. $n = \frac{9}{10} \cdot \frac{1}{3}$
11. $\frac{9}{5} \cdot 45 = x$
12. $x = 42 \cdot \frac{3}{7}$

Write each equation in an equivalent form having the variable alone on one side. Solve. Check the answer in the original equation.

1. $\frac{5x}{6} = 35$ 42 **5.** $24 = \frac{3x}{4}$ 32 **9.** $18 = x \div \frac{5}{6}$ 15

2. $\frac{7a}{9} = 63$ 81 **6.** $\frac{5}{9} = n \div \frac{27}{15}$ 1 **10.** $3n = \frac{9}{10}$ $\frac{3}{10}$

3. $\frac{4n}{5} = 16$ 20 **7.** $32n = \frac{8}{3}$ $\frac{1}{12}$ **11.** $45 = \frac{5x}{9}$ 81

4. $36 = y \div \frac{3}{4}$ 27 **8.** $\frac{6n}{5} = 36$ 30 **12.** $\frac{7x}{3} = 42$ 18

Translate each of these sentences into equations using x for the variable. Solve. Check your answer in the original sentence.

13. Two-thirds of a certain number is 12. $\frac{2x}{3} = 12$; 18

14. If six times a certain number is divided by 5, the quotient is 48. $\frac{6x}{5} = 48$; 40

15. A certain number is $\frac{7}{8}$ of 112. $x = \frac{7}{8} \cdot 112$; 98

16. Three times a certain number divided by 4 is 15. $\frac{3x}{4} = 15$; 20

17. If a certain number is divided by $\frac{3}{5}$, the quotient is 9. $x \div \frac{3}{5} = 9$; $5\frac{2}{5}$

A. *Multiply:*

1. $\frac{7}{8} \times \frac{4}{5}$ $\frac{7}{10}$
2. $\frac{3}{5} \times \frac{5}{16}$ $\frac{3}{16}$
3. $\frac{7}{8} \times 3\frac{1}{5}$ $2\frac{4}{5}$
4. $5\frac{5}{6} \times \frac{3}{5}$ $3\frac{1}{2}$
5. $3\frac{3}{8} \times 5\frac{1}{3}$ 18
6. $3\frac{3}{7} \times 35$ 120
7. $6\frac{5}{9} \times 18$ 118
8. $2\frac{2}{5} \times 3\frac{4}{7}$ $8\frac{4}{7}$
9. $\frac{5}{9} \times 3\frac{1}{5}$ $1\frac{7}{9}$
10. $2\frac{2}{3} \times \frac{3}{8}$ 1
11. $5\frac{7}{10} \times \frac{5}{19}$ $1\frac{1}{2}$
12. $6\frac{3}{5} \times \frac{5}{11}$ 3

B. *Divide:*

1. $\frac{8}{15} \div \frac{2}{3}$ $\frac{4}{5}$
2. $\frac{5}{8} \div \frac{3}{4}$ $\frac{5}{6}$
3. $3\frac{1}{3} \div \frac{5}{6}$ 4
4. $14\frac{2}{7} \div 16\frac{2}{3}$ $\frac{6}{7}$
5. $7\frac{1}{5} \div 5\frac{1}{7}$ $1\frac{2}{5}$
6. $25 \div 6\frac{2}{3}$ $3\frac{3}{4}$
7. $6 \div \frac{3}{8}$ 16
8. $6\frac{2}{3} \div 15$ $\frac{4}{9}$
9. $18 \div \frac{9}{10}$ 20
10. $2\frac{2}{3} \div 5\frac{1}{3}$ $\frac{1}{2}$
11. $5\frac{7}{10} \div 3\frac{1}{5}$ $1\frac{25}{32}$
12. $7\frac{1}{8} \div 19$ $\frac{3}{8}$

C. *Solve:*

1. $\frac{5x}{8} = 35$ 56
2. $3\frac{1}{3} \div x = 10$ $\frac{1}{3}$
3. $x \div \frac{5}{9} = 72$ 40
4. $x \div \frac{5}{9} = 22\frac{1}{2}$ $12\frac{1}{2}$
5. $\frac{16x}{3} = 64$ 12
6. $x \div 2\frac{3}{4} = 15$ $41\frac{1}{4}$
7. $18\frac{3}{4} \div x = 15$ $1\frac{1}{4}$
8. $x \div \frac{5}{8} = 16$ 10
9. $\frac{75x}{4} = 6\frac{1}{4}$ $\frac{1}{3}$
10. $\frac{5x}{8} = 75$ 120
11. $33 = \frac{3x}{5}$ 55
12. $16 \div x = \frac{1}{2}$ 32
13. $x \div \frac{5}{3} = 12$ 20
14. $\frac{3x}{4} = \frac{15}{20}$ 1
15. $12 \div x = \frac{3}{4}$ 16
16. $15x = 5$ $\frac{1}{3}$

If you need more practice, turn to the Practice Exercises on page 476. If not, you may work in the Experts' Corner on the following page.

The Calendar

Man has been concerned with the length of the year probably since the time our primitive ancestors found it desirable to store food to last from season to season. Many nations developed calendars independently. The earliest Roman calendar had 300 days, divided into ten months. Later two additional months were added. When the government felt the need of a more exact calendar as the basis for tax collection and other fiscal operations, Julius Caesar introduced the year of 365 days, plus a leap year every fourth year.

One difficulty in the calendar has been that the natural units of time–the day and the year–come from two different sources. The day is the length of time it takes for the earth to rotate once on its axis. The year is the time required for the earth to make an orbit around the sun. The number of days required for an orbit is not exact–it takes 365 days, 5 hours, 48 minutes, 46 seconds; or more conveniently, 365.2422 days. The Julian calendar, introduced by Julius Caesar, approximates the year at 365.25 days. Can you explain how?

The Julian year was a little too long, and in 1582 Pope Gregory pointed out that the spring equinox came on March 11 instead of March 21 as it should. By a decree, he ordered that the ten days between March 11 and March 21 be skipped, and that in the future the last year of a century should *not* be a leap year unless the number of the *century* was divisible by 4. Thus 1700, 1800, and 1900 were not leap years, but the year 2000 will be a leap year.

A calendar year is 52 weeks plus one day, except in leap year when it is 52 weeks plus two days. Accordingly, a given date falls on a different day of the week each year. Have you ever noticed that New Year's Day does not fall on the same day of the week in any two successive years? Ordinarily it falls one day later each year; that is, January 1 came on Sunday in 1950, and on Monday in 1951. Each leap year, however, is a day longer than an ordinary year, so a New Year's Day following a leap year comes two days later in the week than the previous one. For this reason, January 1, 1953 was on Thursday. The day of the week on which New Year's Day falls is shown for a number of years in Table 1.

There is, however, a system as to the days on which dates in the same year will fall. If you will look at a calendar, you will see that February 1 always comes three days later in the week than January 1. That is, February 1, 1959 was on Sunday, which is three days later than

Thursday, the day January 1, 1959, fell on. Look at Table 1 and decide on what day of the week was February 1, 1900? 1940? 1950?

Knowing what day of the week January 1 fell on, it is easy to find what day the first of any month fell on. In ordinary years, October 1 is on the same day as January 1. May 1 is one day later; August 1 is two days later; February 1, March 1, and November 1 are three days later. Leap year affects every month following February. You can see this in Table 2, which shows how many week days to count from the day January 1 falls on.

TABLE 1. THE DAY OF THE WEEK ON WHICH JANUARY 1 FALLS

Sun.	*Mon.*	*Tues.*	*Wed.*	*Thurs.*	*Fri.*	*Sat.*
—	1900	1901	1902	1903	1904	—
1905	1906	1907	1908	—	1909	1910
1911	1912	—	1913	1914	1915	1916
—	1917	1918	1919	1920	—	1921
1922	1923	1924	—	1925	1926	1927
1928	—	1929	1930	1931	1932	—
1933	1934	1935	1936	—	1937	1938
1939	1940	—	1941	1942	1943	1944
—	1945	1946	1947	1948	—	1949
1950	1951	1952	—	1953	1954	1955
1956	—	1957	1958	1959	1960	—
1961	1962	1963	1964	—	1965	1966
1967	1968	—	1969	1970	1971	1972
—	1973	1974	1975	1976	—	1977
1978	1979	1980	—	1981	1982	1983
1984	—	1985	1986	1987	1988	—
1989	1990	1991	1992	—	1993	1994
1995	1996	—	1997	1998	1999	2000

1. On what day did New Year's Day fall in 1930? 1948? 1950? 1956? Wed.; Thurs.; Sun.; Sun.

2. In what years between 1930 and 1940 did New Year's Day fall on Sunday? 1933; 1939

3. Christmas and New Year's are just a week apart. Thus Christmas of one year is on the same week day as New Year's Day of the following year. Christmas, 1938, and New Year's Day, 1939, were both on Sunday. On what day of the week did Christmas fall in 1940? in 1945? in 1957? Wed.; Tues.; Wed.

4. In what year will Christmas next fall on Sunday? 1977

TABLE 2. THE DAY ON WHICH THE FIRST OF ANY MONTH FALLS IF JANUARY 1 FALLS ON SUNDAY

Month	*Ordinary Year*	*Leap Year*
January	Sunday	Sunday
February	Wednesday	Wednesday
March	Wednesday	Thursday
April	Saturday	Sunday
May	Monday	Tuesday
June	Thursday	Friday
July	Saturday	Sunday
August	Tuesday	Wednesday
September	Friday	Saturday
October	Sunday	Monday
November	Wednesday	Thursday
December	Friday	Saturday

5. In an ordinary year when January 1 falls on Sunday, on what day will the first of each of the other months fall? as above

6. On what day will they fall in a leap year? as above

7. In a year when February 1 falls on Thursday, on what day will July 1 fall, if it is not a leap year? if it is a leap year? Sun.; Mon.

8. Find what day of the week each of these dates fell on: February 22, 1949; July 4, 1932. Tue.; Sun.

9. Suppose we wish to know on what day of the week Lincoln's Birthday (February 12) fell in 1911. Table 3 is a calendar for general purposes, when you know what day the first of the month fell on. If you are using it for a month beginning on Wednesday, for example, *A* is Wednesday, *B* is Thursday, and so on. Using Table 1 and Table 2 you find that February 1, 1911, fell on Wednesday. Hence, using Table 3, "*A*" becomes Wednesday, 12 falls under *E*. If *A* is Wednesday, what is *E*? Sun.

TABLE 3. MONTHLY CALENDAR

A	*B*	*C*	*D*	*E*	*F*	*G*
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31				

Substitute for *A* the day of the week the first of the month falls on and this Table then will become a Calendar for that month (consider the number of days to fit the month.)

10. The United States entered World War I on April 6, 1917. On what day of the week was it? What day did April 1 fall on? (Table 1, 2) What day does *A* in Table 3 become? *Fri.; Sun.; Sun.*

11. Use Table 3 to find on what day the 15th will fall if the 1st falls on Tuesday; on Friday; on Monday; on Saturday; on Wednesday; on Sunday; on Thursday. *the same day*

12. What was the date of your birth? Use the tables to find what day of the week it was.

13. Select ten important dates since 1900, and use the tables to find what day of the week they fell on.

14. Thanksgiving Day is commonly celebrated the last Thursday in November. On what date did it fall in 1943? *Nov. 25*

15. On what date will Thanksgiving be celebrated next year?

16. General election day, when the President is elected in most states, is the first Tuesday after the first Monday in November in leap years. (If November 1 comes on Tuesday, election day is a week later.) On what day was the voting in 1920? 1924? 1940? 1956? 1960? 1964? 1968? *Nov. 2; Nov. 4; Nov. 5; Nov. 6; Nov. 8; Nov. 3; Nov. 5*

17. Set up ten additional problems similar to those above that you can answer, using the tables.

QUESTIONS FOR RESEARCH AND DISCUSSION

a. Investigate the Babylonian calendar.

b. It is stated above that the Julian calendar approximates a year as 365.25 days. What is the length according to the Gregorian calendar?

c. What is the historical basis for the month? for the week?

d. Many proposals have been made for calendar reforms, aimed at making the same dates fall on the same days each year. Find out some of these and explain their advantages. Why were they not adopted?

e. Looking at Table 1 you may be able to find that it repeats itself, so you only need part of it. How could you use part of it, and still find on what date January 1 fell some time in the past, or will fall in the future, not listed on the table?

f. Do you know the rhyme for remembering the number of days in the months, ("Thirty days hath September, etc.")? If not, you should learn it. It is very useful.

g. Stonehenge mystery cleared up by computer! Report on this research to the class.

The Hindu-Arabic system of numeration is a positional system. By *positional* we mean that the value of a digit in a numeral is the product of the number named by the digit and the value associated with its place in the numeral which is a power of 10. The value associated with any given place in a numeral is ten times that of the place to its right, and $\frac{1}{10}$ that of the place to its left. In other words, the value of each digit increases by a *power* of 10 as we move to the left. Similarly, the value of each digit decreases by a power of 10 as we move to the right. For this reason the system of numeration is called *base 10,* and also the *decimal* system, from *decem,* Latin for ten.

This property can be illustrated by analyzing the values of the numbers named by the digits in any numeral, or in vertical analysis.

EXAMPLE

$$9537 = (9 \times 10^3) + (5 \times 10^2) + (3 \times 10^1) + (7 \times 10^0)$$

$$\begin{aligned} 9 \times 10^3 &= 9 \times 1000 &= 9000 \\ 5 \times 10^2 &= 5 \times 100 &= 500 \\ 3 \times 10^1 &= 3 \times 10 &= 30 \\ 7 \times 10^0 &= 7 \times 1 &= \underline{7} \\ & & 9537 \end{aligned}$$

Notice that as the exponent of 10 decreases, we end up with 10^0. From the above pattern, 10^0 must be considered as equal to 1. In fact we will define any number raised to the zero power as being equal to 1. What is

6^0 *1* $\quad 8^0$ *1* $\quad x^0$ *1* $\quad 2(x^0)$ *2* $\quad (2x)^0$ *1* $\quad 2x^0$? *2*

1. Following the example above, show the value of the number named by each of the digits in these numerals. *See front.*

a. 5836 **b.** 76 **c.** 8359 **d.** 16,257 **e.** 82,359 **f.** 40,005 **g.** 123,509 **h.** 200,312 **i.** 2,038,010

2. The value of the number associated with any digit extends to digits to the right of the units digit. The value associated with the place to the right of the units digit in the numeral 222.2 is what fraction of the value associated with the units place? $\frac{1}{10}$

3. The decimal point is used to locate the separation between the units place and the next place to the right. The value associated with the second place to the right of the decimal place in the numeral 777.77 is what fraction of the value associated with the units place? $\frac{1}{100}$

4. Notice that $\frac{3}{10} = 0.3$; $\frac{7}{100} = 0.07$; $\frac{9}{1000} = 0.009$. A zero is placed in front of the decimal point to call attention to the place of the number and to avoid error. Write these fractions as decimals.

a. $\frac{3}{100}$ *0.03* **b.** $\frac{9}{10}$ *0.9* **c.** $\frac{1}{100}$ *0.01* **d.** $\frac{13}{10,000}$ *0.0013*

5. The value of the number named by each digit in the numeral 875.214 can be stated as follows:

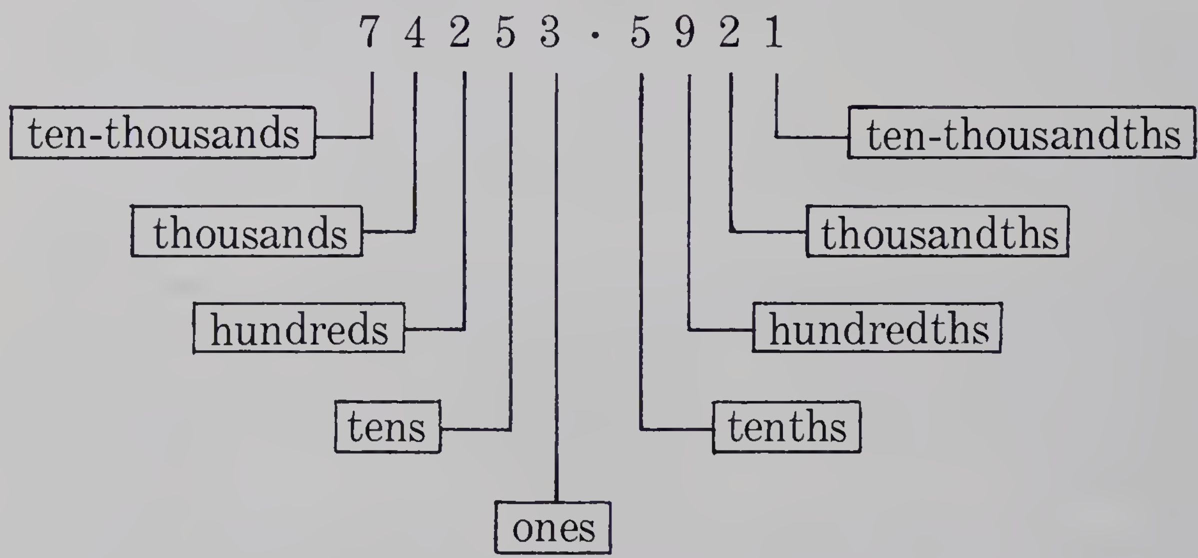
$$\begin{aligned}
8 \times (10 \times 10) &= 8 \times 100 &&= 800 \\
7 \times (10) &= 7 \times 10 &&= 70 \\
5 \times (1) &= 5 \times 1 &&= 5 \\
2 \times \left(\frac{1}{10}\right) &= 2 \times 0.1 &&= 0.2 \\
1 \times \left(\frac{1}{10} \times \frac{1}{10}\right) &= 1 \times 0.01 &&= 0.01 \\
4 \times \left(\frac{1}{10} \times \frac{1}{10} \times \frac{1}{10}\right) &= 4 \times 0.001 &&= 0.004 \\
& && \overline{875.214}
\end{aligned}$$

List the value of the number named by each digit in these numerals.

a. 598.6235 **b.** 10,075.432 **c.** 187,635.025

See front.

6. As you have noticed, each place to the left of the units place has a corresponding fractional place to the right.



You can see a symmetric pattern around the ones place. Write each of the following as numerals, using the decimal point.

a. Four thousand six hundred seventy-seven and eight hundred ninety-two ten-thousandths *4677.0892*

564.0083

b. Five hundred sixty-four and eighty-three ten-thousandths

c. Six thousand five and four hundred three hundred-thousandths

d. Five hundred seventy-five and eighty-five ten-thousandths

e. Eighteen thousand three hundred twelve and three hundredths

c. 6005.00403 d. 575.0085 e. 18,312.03

Frequently you need to get a quick, general idea of a large number. To do this you can *round* the number. Thus, "In 1900 the population of the United States was 75,994,575. By 1960 the population had increased to 179,323,175."

Here the main idea has been blurred by too many digits. To see the relationship more clearly, let us think in millions rather than ones. Is 75,994,575 nearer to 75 million, or to 76 million? You can see that to the nearest million it is 76,000,000. Is 179,323,175 nearer to 179 million, or to 180 million? *179 million*

1. Round 179,323,175 to the nearest million. *179,000,000*

2. Rewrite the sentences in quotes, above, with each number rounded to the nearest million. Which is easier to understand? *See front.*

3. A TV announcer said there were about 100,000 people attending one of the New Year's Day Bowl games. The reported attendance was 101,576. The announcer had rounded the figure to the nearest ten thousand. What would the announced number be rounded to the nearest thousand? *102,000*

4. There are some simple rules that are generally followed in rounding numbers. If rounding to the nearest 10, the ones digit is examined. Numerals containing 5 or greater in the ones place are written to the nearest 10 above. Those with 4 or less are rounded to the nearest 10 below. Thus 64 is rounded *down* to 60, while 185 is rounded *up* to 190. Round to the nearest 10:

a. 73 *70* **b.** 57 *60* **c.** 92.3 *90* **d.** 348 *350* **e.** 532.7 *530* **f.** 267 *270*

5. To round a mixed numeral to the nearest whole number, first examine the fraction. If it is less than $\frac{1}{2}$, the fraction is dropped and the whole number remains unchanged. If the fraction is equal to or greater than $\frac{1}{2}$, the fraction is dropped and the whole number is rounded *up* to the next higher whole number. Round each of these to the nearest whole number:

a. $4\frac{5}{8}$ *5* **c.** $13\frac{4}{9}$ *13* **e.** 13.7 *14* **g.** 21.48 *21* **i.** 175.85 *176*
b. $12\frac{1}{2}$ *13* **d.** $215\frac{13}{16}$ *216* **f.** 118.5 *119* **h.** 32.175 *32* **j.** 219.88 *220*

6. The same principle applies to rounding to the nearest cent. Fractions of a cent less than $\frac{1}{2}$ are not considered and the number of cents does not change. If the fraction is $\frac{1}{2}$¢ or more, the number of cents is increased by 1¢. Round to the nearest cent:

a. \$1.275 *\$1.28* **b.** \$3.833 *\$3.83* **c.** \$12.517 *\$12.52* **d.** \$37.394 *\$37.39*

7. Describe a method to round to the nearest hundred; thousand. *See front.*

An important use of rounded numbers is in finding an approximate answer without having to use pencil and paper. For many problems an exact answer is unnecessary, as in determining how long a trip will take, how much gasoline will be consumed on a trip, or what a trip will cost. A useful way to check a problem is to find an approximate answer using rounded numbers. This tells you if your answer is reasonable. For this purpose use a convenient number a little greater or less than the given number.

In the following problems, mentally determine a reasonable answer using rounded numbers, and make note of this answer, stating whether it is too large or too small. Then solve each problem and see how close your approximation was to the exact answer.

1. What will 29 pounds of nails cost at $7\frac{1}{2}$¢ a pound? (Round 29 to 30, and $7\frac{1}{2}$¢ to 8¢.) *$ 2.18*

2. A truck driver plans to drive at an average rate of 47 miles an hour on a 387-mile trip. How long will the trip take at that rate? (Round 387 to 400, and 47 to 50.) *8 hr. 14 min.*

3. Jim works 12 hours a week taking care of lawns after school. He is paid $1.15 an hour. How much does he earn per week? (Round 12 to 10.) *$ 13.80*

4. Margaret bought 32 yards of ribbon at 19¢ a yard. How much change should she receive from a $10 bill? *$ 3.92*

EXAMPLES

1.		$278 \times 19 = \square$
	Estimate:	$300 \times 20 = 6000$
	Exact:	$278 \times 19 = 5282$

2.	$1935 \div 43 = \square$
	$2000 \div 50 = 40$
	$1935 \div 43 = 45$

Before performing each of these computations, estimate a reasonable answer. Find the exact answer and see how close your estimate was.

5. 487×21 *10,227*	**11.** 517×85 *43,945*	**17.** 2123×54 *114,642*
6. 652×46 *29,992*	**12.** 1017×93 *94,581*	**18.** 4782×36 *172,152*
7. 955×37 *35,335*	**13.** 1122×49 *54,978*	**19.** 6167×48 *296,016*
8. $891 \div 27$ *33*	**14.** $1591 \div 37$ *43*	**20.** $4176 \div 58$ *72*
9. $665 \div 19$ *35*	**15.** $3599 \div 59$ *61*	**21.** $10,864 \div 97$ *112*
10. $6364 \div 86$ *74*	**16.** $2496 \div 48$ *52*	**22.** $6384 \div 76$ *84*

Multiplication of numbers expressed as decimals differs from multiplication of whole numbers only in one respect—we need to locate a decimal point in the product. Using rounded numbers to estimate an answer provides a useful guide for this purpose.

EXAMPLES

1. $3.7 \times 5.2 = \square$

Estimate: $4 \times 5 = 20$

Exact:
$$\begin{array}{r} 3.7 \\ \underline{5.2} \\ 74 \\ \underline{185} \\ 19.24 \end{array}$$

2. $29.3 \times .082 = \square$

$$30 \times \frac{1}{10} = 3$$

$$\begin{array}{r} 29.3 \\ \underline{.082} \\ 586 \\ \underline{2344} \\ 2.4026 \end{array}$$

Use rounded numbers to locate the decimal point in these products.

a. $10.36 \times 6.8 = 70448$ 70.448

b. $1.95 \times 4.3 = 8385$ 8.385

c. $5.75 \times 3.15 = 181125$ 18.1125

d. $0.913 \times 17.2 = 157036$ 15.7036

e. $3.83 \times 0.117 = 44811$ .44811

f. $9.89 \times 4.85 = 479665$ 47.9665

If you examine the factors and product in each exercise, you will note a useful relationship between the number of digits to the right of the decimal point in the factors and product.

The number of digits to the right of the decimal point in the product is equal to the sum of the numbers of digits to the right of the decimal point in the two factors.

Use this relationship to locate the decimal point in each product. Use rounded numbers to estimate your answer.

1. 3.5×14.7 51.45
2. $.075 \times 6.135$ .460125
3. 0.85×1.375 1.16875
4. 0.35×6.5 2.275
5. 16.75×8.05 134.8375
6. 68.5×4.78 327.430
7. 28.9×3.19 92.191
8. 3.5×4.05 14.175
9. 25×13.5 337.5
10. $0.95 \times .0105$ .009975
11. 8.23×5.08 41.8084
12. 26.9×8.07 217.083
13. 50.08×6.67 334.0336
14. $0.09 \times .098$ .00882
15. $23.7 \times .007$ 0.1659
16. 5.09×1.07 5.4463
17. 0.013×29 0.377
18. 8.25×0.37 3.0525
19. 3.47×0.256 0.88832
20. 8.49×3.092 26.25108
21. $8.053 \times .016$ .128848

Write a conditional equation before solving each exercise.

STEPS FOR SOLVING APPLIED PROBLEMS

1. Understand the problem.	2. Note what the problem asks for.	3. Look for hidden questions.
6. Check your answer.	5. Set up and solve the conditional sentence(s).	4. Estimate a reasonable answer.

1. During a sale at the Campus Shop all prices were reduced by 0.15 of the regular price. The regular price of a certain camera is $36. What is its sale price? *$ 30.60*

2. A tire is on sale for 0.85 of its regular price of $28. What is the sale price of the tire? *$ 23.80*

3. On a test of 30 problems Mike had 0.7 of them correct. How many did he have wrong? *9*

4. In 1940, the population of Oak Grove was 22,764. In 1960 the population was 1.75 as great. What was the population in 1960? *39,837*

5. The income of the Jones family is $7500 per year. Of this figure, 0.25 is allowed for food and 0.35 for rent. How much is allowed for food? rent? *$ 1875 ; $ 2625*

6. Jon purchased 16.7 gallons of gasoline at 35.9¢ per gallon. What was the cost of the gasoline? Give your answer to the nearest cent. *$ 6.00*

7. If the Green Sox hope to win 0.65 of the games they play and they play 160 games, how many games do they expect to lose? *56*

8. A truck driver averaged 46.8 miles per hour for 6.75 hours. How far did he travel during that time? *315.9 mi.*

9. The Hendrum High School football team has played 16 games and has won 0.875 of them. How many did the team lose? *2*

10. A mechanic earns $4.40 per hour. His helper earns 0.625 as much. What is their combined earnings per hour? *$ 7.15*

11. A supermarket purchased 1200 bushels of potatoes for $1800. If they were sold for 1.6 of the cost, how much did they sell for per bushel? *$ 2.40*

In dividing numbers expressed as decimals, you have two problems that do not occur with whole numbers:

1. Making the divisor a natural number. We do this by multiplying divisor and dividend by the same power of 10.
2. Locating the decimal point in the quotient.

Let us examine the first of these problems.

EXAMPLE

Divide: 7.35 ÷ 1.5 $\qquad$ 7.35 ÷ 1.5 is equivalent to $\frac{7.35}{1.5}$.

$$\frac{7.35}{1.5} \times \frac{10}{10} = \frac{73.5}{15}$$

Multiplying by $\frac{10}{10}$ we can form an equivalent fraction with a natural number as denominator. The value of the fraction has not been changed since $\frac{10}{10} = 1$ which is the multiplicative identity. The division then becomes 73.5 ÷ 15.

Indicate what power of 10 you would multiply by to make the divisor a whole number.

1. 46 ÷ 1.6 *1*	**3.** 73.5 ÷ 0.315 *3*	**5.** 50.45 ÷ 7.405 *3*
2. 65 ÷ 3.05 *2*	**4.** 14.31 ÷ 3.9 *1*	**6.** 18.62 ÷ .072 *3*

EXAMPLE

Divide: 4.95 ÷ 2.75

```
         1.8
2.75‸)4.95‸0
      2 75
      2 20 0
      2 20 0
```

Estimate: 5 ÷ 2.5 = 2
$10 > q > 1$

To keep in mind what the divisor and dividend were to start with, the new location of the decimal point (after making the divisor a whole number by multiplying by a power of 10) is shown in the dividend and divisor by a *caret* (‸). By what power of 10 were dividend and divisor multiplied? *2*

Notice that the decimal point in the quotient is located above the caret in the dividend. An important procedure, as illustrated in the Example, is to make an estimate using rounded numbers and to state between what powers of 10 the quotient should lie.

If the quotient is to be stated to a specified number of decimal places, the correct procedure is to carry the division one place beyond the specified number and round to the desired number of places.

EXAMPLE

Find: $24.9 \div 3.7$ to the nearest thousandth.
Estimate: $24 \div 4 = 6$ $\quad 10 > q > 1$

$$\begin{array}{r} 6.7297 \\ 3.7_{\wedge}\overline{)24.9_{\wedge}0000} \\ \underline{22\;2} \\ 2\;7\;0 \\ \underline{2\;5\;9} \\ 1\;10 \\ \underline{74} \\ 360 \\ \underline{333} \\ 270 \\ \underline{259} \end{array}$$

Both dividend and divisor were multiplied by what power of 10 to make the divisor a natural number?

To the nearest thousandth the quotient is 6.730.

Explain how the decimal point was located, and then check your answer. Note that you can add as many zeros as necessary after the last non-zero digit following the decimal point in the dividend ($24.9 = 24.90000$).

Divide:

1. $22.4 \div 0.32$ *70.0*
2. $1.68 \div 1.2$ *1.4*
3. $6.86 \div 4.9$ *1.4*
4. $2.24 \div 3.2$ *0.7*
5. $17.52 \div 0.24$ *73.0*
6. $16.65 \div 4.5$ *3.7*
7. $34.56 \div 4.8$ *7.2*
8. $0.64 \div 1.6$ *0.4*
9. $4.2 \div 5.6$ *0.75*
10. $16.32 \div 0.136$ *120.0*
11. $8.4 \div 9.6$ *0.875*
12. $20.24 \div 1.84$ *11.0*

Find the quotients to the nearest hundredth.

13. $52.85 \div 6.21$ *8.51*
14. $19.6 \div 4.1$ *4.78*
15. $34.05 \div 6.8$ *5.01*
16. $62.4 \div 6.4$ *9.75*
17. $0.565 \div 0.13$ *4.35*
18. $12.42 \div 2.1$ *5.91*
19. $36.42 \div 6.07$ *6.00*
20. $6.624 \div 1.6$ *4.14*
21. $5.55 \div 3.7$ *1.50*

Find the quotients to the nearest thousandth.

22. $10.35 \div 5.1$ *2.029*
23. $6.095 \div 5$ *1.219*
24. $83.286 \div 9$ *9.254*
25. $1.936 \div 1.6$ *1.210*
26. $6.7 \div 2.8$ *2.393*
27. $0.3762 \div .3$ *1.254*
28. $4.872 \div 3$ *1.624*
29. $29.8 \div 18$ *1.656*
30. $1.3875 \div .09$ *15.417*

Write the conditional equation for each problem before undertaking its solution. Make an estimate of the answer before each computation.

1. A bus traveled 363.75 miles in 7.5 hours. What was its average speed in miles per hour? *48.5 m.p.h.*

2. A ribbon 42 inches long is to be cut into pieces, each 2.8 inches long. How many pieces will there be? *15*

3. A board 18 feet long is to be cut into pieces for shelves, each 3.5 feet long. If you disregard the width of the saw cut, how many shelves will there be? *5*

4. In Exercise 3 how many inches long is the piece that is left? *6 in.*

5. A plane makes a flight of 3437.5 miles in 6.25 hours. What is the average speed of the plane? *550 m.p.h.*

6. A truck averages 47.5 miles per hour on a long trip. At that rate, how long should the driver plan to take on a trip of 308.75 miles? *6.5 hr.*

7. Last Saturday, Tom earned $11.25 by working 7.5 hours. What was his hourly wage? *$ 1.50*

8. In many wholesale book houses an order of books is weighed instead of being counted. If one book weighs 1.75 pounds, how many books are in a pile that weighs 49 pounds? *28*

9. A pile of sheet metal strips, each 0.375 inches thick, is 24 inches high. How many strips are in the pile? *64*

10. Eric bought a used motorcycle for $240. He paid 0.375 of the price in cash, and the rest in six equal monthly payments. How much was each payment? *$25*

11. When the Jones family started on a 750-mile trip the gasoline tank was full, and the speedometer reading was 15,365.2. When they stopped for lunch the reading was 15,533.2. It took 11.2 gallons of gasoline to refill the tank. If the average number of miles traveled per gallon remains the same, find the cost of gasoline at 35.9¢ per gallon for the entire trip. *$17.95*

12. Mr. Jones spends $900 per year on car expenses. He says that this is 0.12 of his income. How much is his income? *$7500*

13. At the end of the year the bookstore sold all books at a reduction of 0.15 off the regular price. On one of the expensive books the reduction was $1.80. What was the regular price of the book? *$12*

A. Copy each product and locate the decimal point correctly.

1. $0.16 \times 0.3 = 48$ .048
2. $3.2 \times 11 = 352$ 35.2
3. $2.09 \times 1.8 = 2717$ 2.717
4. $0.5 \times 0.9 = 45$ 0.45
5. $3.5 \times 0.35 = 1225$ 1.225
6. $0.37 \times .09 = 333$ .0333
7. $11.0 \times 1.7 = 187$ 18.7
8. $0.7 \times 0.15 = 105$ 0.105
9. $1.3 \times .09 = 117$ 0.117
10. $3.19 \times 2.6 = 8294$ 8.294
11. $31.9 \times 0.26 = 8294$ 8.294
12. $.0319 \times 2.6 = 8294$ .08294

B. Find the products.

1. 1.8×0.7 1.26
2. 5.9×0.8 4.72
3. 3.7×0.17 0.629
4. 9.0×0.16 1.44
5. $.156 \times 9.0$ 1.404
6. 0.32×1.9 0.608
7. 52.9×11.2 592.48
8. 0.687×4.3 2.9541
9. 5.3×0.96 5.088
10. 6.17×3.08 19.0036
11. $52.9 \times .08$ 4.232
12. $9.03 \times .009$ .08127

C. Copy each quotient and locate the decimal point correctly.

1. $32.8 \div 8 = 41$ 4.1
2. $8.58 \div 30 = 286$ 0.286
3. $7.0 \div 35 = 2$ 0.2
4. $3.57 \div 1.7 = 21$ 2.1
5. $3.366 \div .06 = 561$ 56.1
6. $12.0 \div 75.0 = 16$ 0.16
7. $0.64 \div 1.6 = 4$ 0.4
8. $62.5 \div 0.25 = 25$ 250.0
9. $7.29 \div 2.7 = 27$ 2.7
10. $9.6 \div 0.12 = 8$ 80.0
11. $2.88 \div 8.0 = 36$ 0.36
12. $8.1 \div .09 = 9$ 90.0

D. Find the quotients.

1. $4.5 \div 3.0$ 1.5
2. $5.4 \div 0.6$ 9.0
3. $8.1 \div .027$ 300
4. $0.95 \div 0.5$ 1.9
5. $10.8 \div .012$ 900.0
6. $4.8 \div .08$ 60.0
7. $2.121 \div .03$ 70.7
8. $23.76 \div 1.2$ 19.8
9. $56.44 \div 0.16$ 352.75
10. $.0378 \div 4.2$ .009
11. $25.6 \div 0.64$ 40.0
12. $48.0 \div .025$ 1920.0

If you need more practice, use the Practice Exercises on page 480. If not, you may work in the Experts' Corner on the following page.

Scientific Notation

Many studies now being made in space science require the use of very large numbers. Usually these numerals can most conveniently be written as powers of 10. Use of such numerals is called *scientific notation.*

1. The number of molecules in 1.008 grams of hydrogen gas is 606,000,000,000,000,000,000,000. This numeral can more conveniently be written and read as 6.06×10^{23}. In 1 gram of water there are 30,000,000,000,000,000,000,000,000 molecules. Use scientific notation in writing this numeral. 3×10^{25}

2. A scientist has estimated that the total number of atoms in the matter composing the earth is 3×10^{54}. This numeral can be written as 3 followed by how many zeros? 54

3. The brightest star in the sky is named Sirius. It is about 520 trillion miles from the earth. Use scientific notation in writing this numeral. 52×10^{13}

4. It has been calculated that there are 4×10^{16} tons of salt in the ocean. Write this numeral in words. 40 quadrillion

5. Distances in space are so great that astronomers use, as a linear unit, the distance light travels in one year. This is called a light-year. If light travels 186,000 miles in one second, how far does light travel in one minute? in one hour? one day? one year ($365\frac{1}{4}$ days)? To the nearest trillion miles, how many miles is a light-year? Write the numeral in scientific notation. 11,160,000; 669,600,000; 16,070,400,000; 5,869,713,600,000; 6 trillion; 6×10^{12}

6. How many years does it take for the light from Sirius to reach the earth? 87 yr.

7. The most distant planet from the sun is Pluto, which is 3668×10^{6} miles from the sun. How long does it take for light to travel from the sun to Pluto? 55 hr.

8. It takes about $1\frac{1}{3}$ seconds for the light from the moon to reach the earth. About how many miles is it from the moon to the earth? 248,000 mi.

9. The North Star is about 40 light-years from the earth. 40 light-years is how many miles (approximately)? Write your answer in scientific notation. 24×10^{13}

10. The distance from the earth to the nearest star outside our solar system, Alpha Centauri, is 4.3 light-years. Express this distance in miles using scientific notation. 25.8×10^{12}

Equivalent fractions, as you know, name the same number. Each fraction also has an equivalent decimal. Some of these you are familiar with, as $\frac{1}{2} = 0.5$ and $\frac{1}{4} = 0.25$. Any fraction whose denominator is 10 or a power of 10 can readily be written as a decimal. If the denominator of a fraction is not a power of 10, we can readily determine an equivalent fraction when the denominator is a factor of ten or of a power of ten. In $\frac{1}{5}$, 5 is a factor of ten and $\frac{1}{32}$ has 32 as a factor of a power of ten, namely 100,000. $\frac{1}{5} = 0.2$ and $\frac{1}{32} = 0.03125$.

EXAMPLE

Find the decimal equivalent of $\frac{3}{25}$.

Is the denominator 25 a factor of 10? NO. Is it a factor of 100? YES, because we know $100 \div 25 = 4$. To find the fractional equivalent of $\frac{3}{25}$ that has a denominator of 100, we write $\frac{3}{25} = \frac{n}{100}$. Then using the proportion property, we obtain $25n = 300$ which yields $n = 12$ using the relationship $f_2 = \frac{p}{f_1}$. Therefore, $\frac{3}{25} = \frac{12}{100} = 0.12$

1. Find the decimal equivalent.

a. $\frac{7}{25}$ *0.28* **b.** $\frac{1}{4}$ *0.25* **c.** $\frac{2}{5}$ *0.4* **d.** $\frac{7}{20}$ *0.35* **e.** $\frac{21}{50}$ *0.42*

A fraction can be written as an equivalent fraction whose denominator is a power of 10 only if the prime factors of its denominator are 2 or 5 or both. A simple test is to rename the denominator as a product of its prime factors.

EXAMPLE

Express $\frac{5}{8}$ as a decimal.

Since $8 = 2 \times 2 \times 2$, 8 is a factor of a power of 10 which is $10 \times 10 \times 10$ or 1000 and $1000 \div 8 = 125$.

$$\frac{5}{8} = \frac{n}{1000} \text{ or } 8n = 5000 \text{ or } n = 625$$

$$\frac{5}{8} = \frac{625}{1000} = 0.625$$

2. Find the decimal equivalent.

a. $\frac{3}{8}$ 0.375 **c.** $\frac{6}{125}$ .048 **e.** $\frac{5}{16}$ 0.3125

b. $\frac{7}{8}$ 0.875 **d.** $\frac{9}{40}$ 0.225 **f.** $\frac{9}{16}$ 0.5625

Not all fractions have denominators that are factors of powers of 10. In $\frac{1}{3}$, 3 is not a factor of 10 and in $\frac{1}{17}$, 17 is not a factor of a power of 10. We shall apply the definition $\frac{a}{b}$ means $a \div b$, $b \neq 0$ in order to find the equivalent decimal.

EXAMPLE

Find the decimal equivalent of $\frac{1}{3}$.

$$\begin{array}{r} .333 \\ 3\overline{)1.000} \\ \underline{9} \\ 10 \\ \underline{9} \\ 10 \\ \underline{9} \\ 1 \end{array}$$

As indicated above $\frac{1}{3}$ means $1 \div 3$. As we carry out the division, it is clear that we can continue the operation indefinitely and get 3's in the quotient. How can you tell this? Since the repetition begins with a 3 in the quotient, we place a bar over the 3 to show what part of the quotient repeats. Thus, $0.\overline{3}$ is a repeating decimal.

3. Write each fraction as a repeating decimal.

a. $\frac{2}{3}$ $0.\overline{6}$ **c.** $\frac{5}{9}$ $0.\overline{5}$ **e.** $\frac{8}{9}$ $0.\overline{8}$

b. $\frac{2}{9}$ $0.\overline{2}$ **d.** $\frac{7}{9}$ $0.\overline{7}$ **f.** $\frac{2}{30}$ $0.0\overline{6}$

Some decimals begin to repeat with the second digit.

EXAMPLE

Find the decimal equivalent of $\frac{1}{6}$.

$$\begin{array}{r} .166 \\ 6\overline{)1.000} \\ \underline{6} \\ 40 \\ \underline{36} \\ 40 \\ \underline{33} \\ 7 \end{array}$$

You can see that if the operation is continued, the 6 in the quotient will continue to repeat as the remainder will continue to be 4. We place a bar over the 6 to indicate what part of the quotient repeats. We write $\frac{1}{6} = 0.1\overline{6}$.

4. Find the decimal equivalent.

a. $\frac{1}{15}$ $.0\overline{6}$ **c.** $\frac{7}{15}$ $0.4\overline{6}$ **e.** $\frac{1}{45}$ $.0\overline{2}$

b. $\frac{5}{6}$ $0.8\overline{3}$ **d.** $\frac{5}{18}$ $0.2\overline{7}$ **f.** $\frac{7}{30}$ $0.2\overline{3}$

More than one digit may repeat when finding a decimal equivalent.

EXAMPLE

Express $\frac{4}{11}$ as an equivalent decimal.

$$\begin{array}{r} .3636 \\ 11\overline{)4.0000} \\ \underline{3\,3} \\ 70 \\ \underline{66} \\ 40 \\ \underline{33} \\ 7 \end{array}$$

A decimal will repeat when the remainder repeats. How can you tell where the division can stop in this example? Can you notice where the decimal begins to repeat? We write $\frac{4}{11} = 0.\overline{36}$. Notice the bar is used to indicate the repeating digits.

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5. Express each of the following as an equivalent decimal. $0.58\overline{3}$

a. $\frac{7}{9}$ $0.\overline{7}$ **c.** $\frac{9}{11}$ $0.\overline{81}$ **e.** $\frac{1}{13}$ $0.\overline{076923}$ **g.** $\frac{2}{7}$ $0.\overline{285714}$ **i.** $\frac{7}{12}$

b. $\frac{5}{14}$ $0.3\overline{571428}$ **d.** $\frac{23}{24}$ $0.958\overline{3}$ **f.** $\frac{47}{60}$ $0.78\overline{3}$ **h.** $\frac{7}{11}$ $0.\overline{63}$ **j.** $\frac{5}{13}$ $0.\overline{384615}$

Any fraction can be expressed as an equivalent decimal by applying the definition of a fraction, $\frac{a}{b} = a \div b,\ b \neq 0$.

6. Find the decimal equivalent. 2.375

a. $3\frac{1}{4}$ 3.25 **b.** $7\frac{1}{16}$ 7.0625 **c.** $5\frac{2}{9}$ $5.\overline{2}$ **d.** $8\frac{3}{4}$ 8.75 **e.** $2\frac{3}{8}$

Before a repeating decimal can be used in computation, it must be rounded. The number of digits retained in rounding depends on the degree of precision required in the computation. Ordinarily, rounding to the nearest thousandth is sufficient unless you are otherwise instructed.

EXAMPLE

$\frac{1}{11} = 0.\overline{09}$

≈ 0.1 to the nearest tenth

≈ 0.09 to the nearest hundredth

≈ 0.091 to the nearest thousandth

≈ 0.0909 to the nearest ten-thousandth

≈ 0.09091 to the nearest hundred-thousandth

We use the symbol $\approx$ to indicate the approximation because the symbol $=$ is used to indicate that the two numerals are names for the same number.

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7. Find the decimal equivalent, and then write each equivalent rounded to the nearest thousandth. 11.190

a. $\frac{5}{18}$ 0.278 **b.** $3\frac{6}{7}$ 3.857 **c.** $\frac{5}{11}$ 0.455 **d.** $7\frac{5}{36}$ 7.139 **e.** $11\frac{4}{21}$

Proportions can be useful if an applied problem contains two equal ratios in which one of the terms is missing.

EXAMPLE

The White Sox won 35 of their first 54 games. If the team continues to win at the same rate, how many of their 162 games can they expect to win this season? This problem may be outlined as a proportion:

$$\frac{(\text{games won})_1}{(\text{games played})_1} = \frac{(\text{games won})_2}{(\text{games played})_2} \text{ or } \frac{W_1}{P_1} = \frac{W_2}{P_2}$$

Note: Subscripts indicate first situation and second situation.

Let x represent the games they will win: $\frac{35}{54} = \frac{x}{162}$ Solve and check. 105

You can also use proportions in problems involving distance.

EXAMPLE

An airplane flew from San Francisco to Portland, a distance of 552 air miles, in 1.2 hours. At that rate how long would it take the airplane to fly from San Francisco to New York, a distance of 2587 air-miles?

$$\frac{(\text{hours})_1}{(\text{miles})_1} = \frac{(\text{hours})_2}{(\text{miles})_2} \text{ or } \frac{h_1}{m_1} = \frac{h_2}{m_2}$$

Let x represent the time required to make the flight to New York.

$$\frac{1.2}{552} = \frac{x}{2587}$$ Solve and check. 5.6 hr.

Proportions are useful to solve applied problems of purchasing and of earning money.

EXAMPLE

Henry picked 17 bushels of apples in 2.5 hours. He is paid 22¢ per bushel. At that rate how much can he earn in an 8-hour day?

$$\frac{(\text{earnings})_1}{(\text{hours})_1} = \frac{(\text{earnings})_2}{(\text{hours})_2} \text{ or } \frac{e_1}{h_1} = \frac{e_2}{h_2}$$

Then $\frac{374}{2.5} = \frac{x}{8}$ Solve and check. Your answer will be in cents. $11.97

You may notice that the examples compared the elements of a first (given) situation to the elements of the second situation in the same order. What was the meaning of

$$\frac{h_1}{m_1} \qquad \frac{W_1}{P_1} \qquad \frac{e_2}{h_2} \quad ?$$ See front.

You could solve the second example by first finding the plane's air speed on the flight to Portland, and then dividing 2587 by that number. This method introduces two possible difficulties: (1) There is an additional computation and (2) the number of miles per hour may be awkward as a divisor. You will see the value of proportions by comparing both methods (the proportion and a two step approach) in Exercises 1 and 2.

Use a proportion to solve each of the following problems.

1. An automobile traveled 220 miles on 15 gallons of gasoline. At this rate how far should it travel on a full tank of 20 gallons? Can you use $\frac{g_1}{m_1} = \frac{g_2}{m_2}$? Would $\frac{m_1}{m_2} = \frac{g_1}{g_2}$ yield the same result? $293\frac{1}{3}$; yes

2. Harriett purchased 8 yards of material for curtains for \$3.82. She needs 5 yards more. What will it cost? $\frac{y_1}{d_1} = \frac{y_2}{d_2}$ \$ 2.39

3. Henry says his dog weighs $\frac{3}{8}$ as much as he does. His dog weighs 57 pounds. How much does Henry weigh? $\frac{d}{H} = \frac{3}{8}$ 152 lb.

4. A certain truck traveled 191 miles in 4 hours. At that rate how long will it take to travel 573 miles? 12 hr.

5. Sugar is advertised at 4 pounds for 37¢. At that rate how much will 32 pounds of sugar cost? \$ 2.96

6. Oscar earned \$4.20 in 2.5 hours picking berries for Mr. Anderson. At that rate how much can he earn in an 8-hour day? \$ 13.44

7. A given automobile wheel makes 180 revolutions in traveling 450 feet. How many revolutions does it make per mile? 2112 r.p.m.

8. On a road map, a segment 4 inches long represents a distance of 100 miles. How many miles does a segment 2.6 inches long represent? 65 mi.

9. On the first three days of its voyage, a ship burns 1000 tons of fuel. At this rate how much fuel should be on board for a 10-day voyage? 3333 tons

10. A salesman receives a commission of \$45 on sales amounting to \$750. What should be allowed, at that rate, on sales amounting to \$2250? \$ 135

11. Mr. Odell, in trying out a new tractor on his farm, plowed 15 acres in 6.5 hours. How long would it take him to plow his 80-acre field if he can maintain this rate? 34.7 hr.

41 gal.

12. After driving 168 miles, Mr. Jones found it took 12 gallons to fill his gasoline tank, which was full when he started. At that rate how much gasoline would he expect to use on a trip of 574 miles?

13. At the start of a trip, the pilot of an airplane found that the plane was traveling at the rate of 92 miles in 12 minutes. How long should a 2000-mile trip take at this rate? 4 hr. 21 min.

14. If 80 feet of wire weigh 27 pounds, what will 250 feet of the same wire weigh? $84\frac{3}{8}$ lb.

15. A manufacturing company has developed a machine that can produce 3125 nails in 25 minutes. How many nails can this machine produce in an 8-hour day? 60,000

16. Helen can read 25 pages of a novel in 35 minutes. How long will it take her, at this rate, to read a whole book of 325 pages? 7 hr. 35 min.

17. A steel rail 4 feet long weighs 425 pounds. What would a 30-foot rail of the same steel weigh? 3187.5 lb.

18. The Boy Scouts hiked 10 miles in 3.5 hours. How long would it take them to hike 17.5 miles at that rate? $6\frac{1}{8}$ hr.

19. Mabel typed seven pages of manuscript in 50 minutes. How long should it take her at this rate to type a manuscript of 245 pages?

29 hr. 10 min.

20. One way of measuring the height of an object is to measure its shadow. At any given time of day the ratio of the height of an object compared to the length of its shadow is the same for any two objects. If the height of one is known, the height of the other can be calculated from the proportion:

$$\frac{\text{known height}_1}{\text{shadow}_1} = \frac{\text{unknown height}_2}{\text{shadow}_2} \text{ or } \frac{h_1}{s_1} = \frac{h_2}{s_2}$$

Note: The subscripts refer to first object and second object.

When a 42-foot flagpole casts an 18-foot shadow, the shadow of a tree in the school yard is 15 feet long. How high is the tree? 35 ft.

21. The shadow of an office building is 54 feet long at the same time that an 8-foot fence post casts a 6-foot shadow. How tall is the building?

72 ft.

22. A flagpole casts a shadow 55 feet long when George, who is 5′10″ tall, casts a shadow 8′2″ long. How tall is the flagpole? $39\frac{2}{7}$ ft.

23. The shadow of a tree is 52 feet long when the shadow of a yardstick is 2 feet long. How tall is the tree? 78 ft.

Part One

Add:

1.	2.	3.	4.	5.
$14\frac{3}{4}$	$11\frac{2}{5}$	$4\frac{3}{4}$	$14\frac{1}{2}$	$27\frac{3}{8}$
$7\frac{5}{6}$	$6\frac{3}{8}$	$8\frac{5}{6}$	$24\frac{3}{4}$	$13\frac{7}{16}$
$12\frac{2}{3}$	$13\frac{1}{4}$	$17\frac{2}{3}$	$8\frac{3}{8}$	$15\frac{1}{2}$
$9\frac{7}{12}$	$11\frac{7}{10}$	$9\frac{1}{2}$	$11\frac{5}{16}$	$17\frac{9}{24}$
$44\frac{5}{6}$	$42\frac{29}{40}$	$40\frac{3}{4}$	$58\frac{15}{16}$	$73\frac{11}{16}$

Subtract:

6.	7.	8.	9.	10.
$42\frac{3}{4}$	$35\frac{1}{8}$	$25\frac{1}{6}$	$38\frac{1}{2}$	$13\frac{7}{10}$
$31\frac{7}{8}$	$17\frac{3}{4}$	$15\frac{2}{3}$	$19\frac{7}{8}$	$7\frac{3}{8}$
$10\frac{7}{8}$	$17\frac{3}{8}$	$9\frac{1}{2}$	$18\frac{5}{8}$	$6\frac{13}{40}$

Multiply:

11.	12.	13.	14.	15.
375	50	48	$3\frac{1}{8}$	$3\frac{5}{8}$
$15\frac{1}{2}$	$12\frac{2}{5}$	$9\frac{5}{8}$	$\frac{9}{10}$	$7\frac{2}{3}$
$5812\frac{1}{2}$	620	462	$2\frac{13}{16}$	$27\frac{19}{24}$

Divide:

16. $\frac{5}{6} \div 1\frac{5}{8}$ $\frac{20}{39}$

17. $9\frac{3}{8} \div 1\frac{1}{24}$ 9

18. $145 \div 2\frac{1}{2}$ 58

19. $6\frac{7}{8} \div 1\frac{2}{3}$ $4\frac{1}{8}$

Express as equivalent decimals:

20. $\frac{7}{16}$ 0.4375

21. $\frac{5}{32}$ 0.15625

22. $\frac{6}{7}$ $0.\overline{857142}$

23. $3\frac{1}{8}$ 3.125

24. $\frac{5}{9}$ $0.\overline{5}$

25. $\frac{1}{4}$ 0.25

Multiply:

26. 3.6×12.5 45.0

27. 28.7×0.06 1.722

28. 5.02×0.9 4.518

29. 3.16×1.295 4.0922

30. 7.5×6.4 48.0

31. 28.9×7.25 209.525

Divide:

32. $5489.1 \div 5.13$ 1070.0

33. $3.1209 \div 3.09$ 1.01

34. $4.7766 \div 0.057$ 83.8

35. $59.295 \div 1.005$ 59

36. $1.01525 \div 26.2$ $.03875$

37. $684.477 \div 9.09$ 75.3

38. *Solve:*

a. $n + 6 = 13$ 7

b. $24 \div n = 3$ 8

c. $7n = 42$ 6

d. $n \div 8 = 3$ 24

e. $17 - n = 4$ 13

f. $n - 12 = 23$ 35

Part Two

1. Write a mathematical sentence.

a. If 7 is increased by 4, the sum is 11. $7+4=11$

b. The product of 8 and 3 is greater than 25. $8 \cdot 3 > 25$

c. If a certain number is divided by 9, the quotient is 7. $n \div 9 = 7$

d. If twice a certain number is decreased by 8, the result is 4. $2n-8=4$

e. Seven-eighths of 96 is less than 90. $\frac{7}{8} \cdot 96 < 90$

f. The product of 8 and 7 is equal to the sum of 54 and 2. $8 \cdot 7 = 54+2$

2. In Exercise 1 find

a. three true sentences a, e, f **b.** two conditional sentences c, d

3. *Solve:*

a. $\frac{x}{7} = \frac{5}{14}$ 2.5 **d.** $\frac{x}{12} = \frac{6}{18}$ 4

b. $\frac{3}{15} = \frac{9}{x}$ 45 **e.** $\frac{4}{5} = \frac{x}{100}$ 80

c. $\frac{8}{x} = \frac{16}{50}$ 25 **f.** $\frac{5}{9} = \frac{x}{27}$ 15

4. Write the multiplicative inverse of the following:

a. 5 $\frac{1}{5}$ **b.** $\frac{2}{3}$ $\frac{3}{2}$ **c.** $\frac{13}{9}$ $\frac{9}{13}$ **d.** $3\frac{1}{2}$ $\frac{2}{7}$ **e.** $\frac{2}{19}$ $\frac{19}{2}$ **f.** $2\frac{5}{7}$ $\frac{7}{19}$

5. Indicate the value of the number named by each digit using exponents to indicate powers of 10: See front.

a. 40,705 **b.** 389,765 **c.** 2,050,706

6. Simplify. Write improper fractions as mixed numerals.

a. $\frac{15}{3}$ 5 **c.** $\frac{18}{6}$ 3 **e.** $\frac{87}{12}$ $7\frac{1}{4}$ **g.** $\frac{55}{13}$ $4\frac{3}{13}$

b. $\frac{13}{91}$ $\frac{1}{7}$ **d.** $\frac{24}{8}$ 3 **f.** $\frac{29}{7}$ $4\frac{1}{7}$ **h.** $\frac{100}{12}$ $8\frac{1}{3}$

7. Examine the proportion at the right.

$$\frac{15}{x} = \frac{10}{16}$$

a. Which terms are the extremes? 15, 16

b. Which terms are the means? 10, x

c. What is the product of the means? 10x

d. What is the product of the extremes? 240

e. Solve and check your answer. 24

8. Which of these sets contain only natural numbers? A

$A = \{1, 3, 5, 7\}$ $B = \{0, 2, 4, 8\}$ $C = \{\frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}\}$

9. Which set is made up of like fractions? C

$A = \{\frac{3}{5}, \frac{3}{7}, \frac{3}{9}\}$ $B = \{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}\}$ $C = \{\frac{1}{8}, \frac{3}{8}, \frac{5}{8}\}$

10. Which set is made up of equivalent fractions? B

$A = \{\frac{1}{7}, \frac{3}{7}, \frac{5}{7}\}$ $B = \{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}\}$ $C = \{\frac{8}{9}, \frac{8}{7}, \frac{8}{5}\}$

11. Which set of fractions is in simplest form? *C*

$A = \{\frac{8}{9}, \frac{8}{8}, \frac{8}{7}\}$ $B = \{\frac{1}{2}, \frac{2}{4}, \frac{4}{8}\}$ $C = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$

12. Which fraction is the multiplicative inverse of $5\frac{3}{8}$? *e*

a. $\frac{53}{8}$ **b.** $8\frac{3}{5}$ **c.** $3\frac{5}{8}$ **d.** $\frac{43}{8}$ **e.** $\frac{8}{43}$ **f.** $\frac{8}{53}$

Part Three

Write the conditional sentence and solve:

1. Sixteen pieces of sheet copper are piled on a shelf. The pile is 12 inches high. What is the thickness of each piece of copper? *$\frac{3}{4}$ in.*

2. Last year there were 36 members in the Camera Club. This year there are 48 members. What is the ratio of this year's membership to that of last year? *$\frac{4}{3}$*

3. A certain car traveled 368 miles on 23 gallons of gasoline. At that rate how many gallons of gasoline will the car use on a trip of 800 miles? *50 gal.*

4. If furniture is advertised at 0.15 off the regular price, what is the sale price of a recliner regularly priced at $240? *$ 204*

5. Henry purchased 12.7 gallons of gasoline at 36.9¢ per gallon. To the nearest cent, what did the gasoline cost him? *$ 4.69*

6. A plane made a flight of 3450 miles in 6.25 hours. What is its rate per hour? *552 m.p.h.*

7. Jim purchased a motor boat for $360. He paid 0.35 of the price in cash, and the rest in nine equal monthly payments. How much was each monthly payment? *$ 26*

8. A 12-foot board is cut into pieces each 28.8 inches long. Disregarding the width of the saw-cut, how many pieces can be obtained? *5*

9. What will 30 cans of milk cost at the rate of 12 cans for $1? *$ 2.50*

10. George earned $12.30 last week by selling 820 newspapers. At this rate how much would he earn by selling 1200 newspapers? *$ 18*

11. A contractor charges $1050 for pouring a concrete foundation that contains 20 cubic yards of concrete. At this rate what should he charge for pouring a foundation that contains 85 cubic yards? *$ 4462.50*

12. A jet plane traveled at a speed of 560 miles per hour for 3.75 hours. How far did it travel in that time? *2100 mi*

13. How high is a telephone pole that casts a 49-foot shadow when a 4-foot fence casts a shadow 3′6″ long? *56 ft.*

RATIONAL NUMBERS

WORDS TO WATCH FOR

absolute value
additive inverse
dense
exponent
exponential form
integer
negative number
per cent
period
positive number
rational number
reciprocal
repeating decimal
signed number
terminating decimal

In the previous chapter we learned that a set of numbers is closed under an operation if, when an operation is performed on two members of the set, the resulting number is also a member of the set. The set that we have been working with up to this point is the set, F_a, the fractional numbers of arithmetic. Let us examine this set, F_a, to see if it is closed under the four fundamental operations. We will consider each operation separately starting with addition. *yes*

1. Is F_a closed under addition? Find each of these sums and decide if it is a member of F_a.

124 **a.** $35 + 89$
1650 **b.** $796 + 854$
996 **c.** $823 + 173$
$2\frac{7}{8}$ **d.** $\frac{3}{8} + 2\frac{1}{2}$
$1\frac{5}{18}$ **e.** $\frac{5}{6} + \frac{4}{9}$
$6\frac{1}{2}$ **f.** $2\frac{1}{5} + 4\frac{3}{10}$
$867\frac{1}{3}$ **g.** $627\frac{1}{3} + 240$
$1431\frac{5}{9}$ **h.** $1059 + 372\frac{5}{9}$
$1093\frac{3}{16}$ **i.** $117\frac{3}{16} + 976$

2. Can you think of any two members of F_a whose sum is not a member of the set? This, of course, does not prove that the set is closed under addition. But if you could find a pair of addends whose sum is not a member of the set, this would prove that the set is not closed under addition. Since we cannot investigate all possible combinations, let us assume that the set is closed under addition. *no*

3. Why is it reasonable to assume that if a set is closed under addition, it is also closed under multiplication? *Multiplication is continued addition.*

4. Find each product and verify that it is a member of F_a.

a. 25×810 *20,250*
b. $\frac{2}{3} \times \frac{9}{10}$ $\frac{3}{5}$
c. $\frac{5}{6} \times \frac{12}{25}$ $\frac{2}{5}$
d. $\frac{3}{8} \times \frac{56}{75}$ $\frac{7}{25}$
e. 91.2×17 *1550.4*
f. 1.34×100 *134*
g. 2.37×10 *23.7*
h. 5.1×1000 *5100*
i. 2.3×6.9 *15.87*
j. $4\frac{1}{3} \times 1\frac{1}{26}$ $4\frac{1}{2}$
k. $2\frac{3}{4} \times \frac{1}{33}$ $\frac{1}{12}$
l. 0.2×0.8 *0.16*

5. Consider the question of closure of F_a under division. Is each of the following quotients a fractional number? Find each quotient and see.

a. $650 \div 25$ *26*
b. $16.38 \div 1.8$ *9.1*
c. $\frac{5}{8} \div \frac{5}{6}$ $\frac{3}{4}$
d. $2\frac{4}{5} \div \frac{11}{12}$ $3\frac{3}{55}$
e. $97.5 \div 7.5$ *13.0*
f. $19.5 \div 2.6$ *7.5*

6. Can you think of two members of F_a whose quotient is not a member of the set? This does not prove that the set is closed under division. However, if you could find two members whose quotient is not a member of the set, this would prove that the set is not closed under division. Let us assume that the set is closed under division. *no, with the exception of division by zero which has no meaning*

7. Is the set of fractional numbers of arithmetic closed under subtraction? Which of the following differences cannot be named by a number in the set? *e, g, h*

a. $\begin{array}{r} 18 \\ \underline{9} \end{array}$
c. $\begin{array}{r} 28 \\ \underline{27} \end{array}$
e. $\begin{array}{r} 19 \\ \underline{20} \end{array}$
g. $\begin{array}{r} 35 \\ \underline{35\frac{1}{3}} \end{array}$

b. $\begin{array}{r} \frac{3}{5} \\ \underline{\frac{3}{10}} \end{array}$
d. $\begin{array}{r} 4\frac{1}{2} \\ \underline{2\frac{3}{4}} \end{array}$
f. $\begin{array}{r} 5\frac{1}{8} \\ \underline{4\frac{1}{2}} \end{array}$
h. $\begin{array}{r} 12\frac{1}{9} \\ \underline{14\frac{4}{15}} \end{array}$

In general there is no number n in the set, F_a, to identify the difference in the operation:

$$n = a - b,\ b > a$$

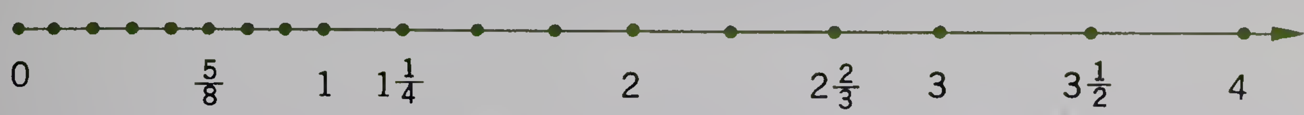
Do we ever need such a number? Consider this problem: On a certain date at a point on the coast of Maine, high tide was 24 feet above mean sea level. Low tide was 45 feet below this level of high tide. How many feet below mean sea level was low tide? *21 ft.*

The common practice is to identify mean sea level as 0. We can set up an equation:

	$n = 24 - 45$	$a_1 = s - a_2$
Then	$n + 45 = 24$	$a_1 + a_2 = s$

You could readily figure out the answer by making a sketch, but it is clear that we need to extend the set of numbers we have been using to define n so that when added to 45 it gives 24 as the sum.

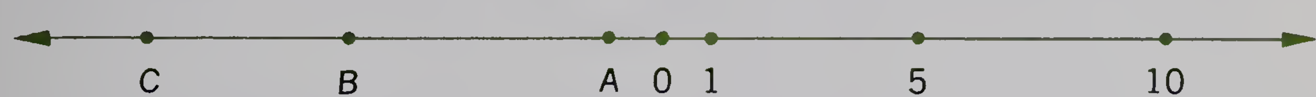
The set of fractional numbers of arithmetic F_a has been represented by a number line with the origin identified by zero at the left. All the other numbers in the set of fractional numbers are greater than zero. Any number represented on this line is greater than all numbers to its left, and less than any number to its right.



The point associated with 3, for example, is to the right of the point associated with $2\frac{2}{3}$.

Then $3 > 2\frac{2}{3}$. In like manner $4 > 3\frac{1}{2}$ and $\frac{5}{8} > 0$.

We can use this same relationship when we extend the number line to the left. Zero is now being used as a point of reference. What can you say about the value of any number named on the number line that is to the left of zero? It is less than zero.



The distance of a point from 0 determines the *absolute value* of a number. This is its numerical value, disregarding its direction from 0. The point A on the number line above is the same distance to the left of 0 as the point at 1 is to the right of zero. Then the numbers associated with the two points have the same absolute value, which we denote as $|1|$. What is the absolute value of B on the number line above? Is it correct to write $C < B < A$? Explain. $|5|$; determined by its distance from 0

Since A is to the left of 0, the value of the number associated with A is less than 0. Thus, $A < 0 < 1$. To denote this difference in value, numbers to the left of 0 are called *negative numbers* and are written with a symbol similar to the minus sign. Thus the number associated with A is $^{-}1$ and the number associated with B is $^{-}5$. The numbers named to the right of 0 are called *positive numbers* and may be written with the plus sign, as $^{+}1$. A numeral written with no symbol is understood to name a positive number.

Thus, for each fractional number of arithmetic, except zero, there is a "mate" associated with a point to the left of zero on the number line. The mate has the same absolute value as the fractional number of arithmetic, but it is a negative number. The set of fractional numbers of arithmetic and their mates form the rational numbers. The positive and negative numbers are symmetrically arranged around 0 on the number line.

With this extension of the number system we can return to the solution of the equation:

$$n = 24 - 45 \qquad a_1 = s - a_2$$

Note that on the number line, if you represent subtraction of one number from another, you start from the sum (s) and move *to the left* the number of units indicated by the known addend (a_2). Thus, by moving 45 units to the left of 24, you locate $^{-}21$, the value for n.

1. Numerals written with + or − symbols name *signed numbers.* Write the numerals associated with B and C as signed numbers. $^{-}5; ^{-}10$

2. Using signed numbers, write and solve equations for each exercise.

a. At 10:00 P.M. the thermometer recorded the temperature as 18°. During the next eight hours the temperature fell 25°. What was the thermometer reading at 6:00 A.M.? $^{-}7°$

b. In the next six hours the temperature as recorded at 6:00 A.M. rose 25°. What was the thermometer reading at noon? $^{+}18°$

c. On its first down a football team gained 6 yards. On the next down the team lost 8 yards. How far is the team from the original line of scrimmage? $^{-}2$ yd.

d. On Tuesday the value of a share of stock increased \$2.50. The next day its value fell \$3.75. What was its relative value compared to that of two days before? $^{-}$\$1.25

3. We have defined the absolute value of a number as the distance from zero on the number line to the point associated with the number. Absolute value of a number can also be defined as the magnitude of the number, disregarding its sign. Essentially these two definitions have the same meaning. Explain and illustrate this fact using positive and negative numbers. See front.

4. The highest point in the United States is Mount McKinley, whose altitude measures 20,320 feet above mean sea level. The lowest point is in Death Valley, where the measure of the distance of the valley floor is 282 feet below sea level. The measure of the altitude of Mount McKinley is how much greater than the measure of the distance of the floor of Death Valley below mean sea level? 20,602 ft.

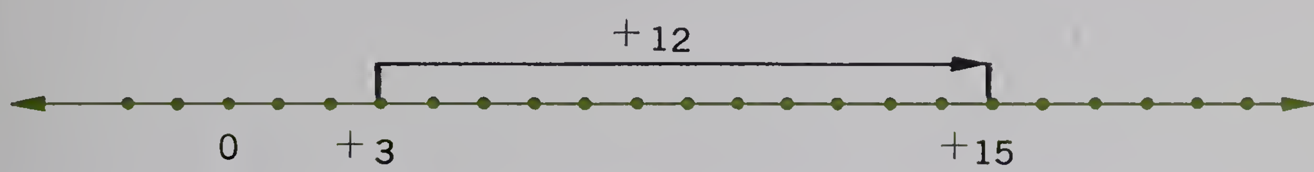
Note: The statement: $+20{,}320 - (^{-}282) = n$ represents the difference in altitudes. Find n.

5. The deepest point in the ocean measured to date is the Marianas Trench in the Pacific, which is 36,198 feet below sea level. The highest point is Mount Everest, whose peak measures 29,028 feet above sea level. The measure of the depth of the Marianas Trench is how much greater than the measure of the altitude of Mount Everest? 7170 ft.

Examine this operation on the number line.

$$n = {}^+3 + ({}^+12) \qquad s = a_1 + a_2$$

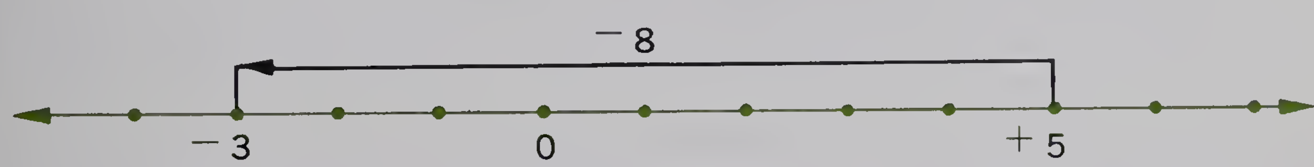
The equation instructs you to start at $a_1 = {}^+3$ and to go 12 units to the right. This is the *direction* indicated by the + sign of a_2. You are accustomed to this procedure, although it has not been important until now. As you expect, this brings us to $s = 15$.



Now let us consider the case when the addend is a negative number.

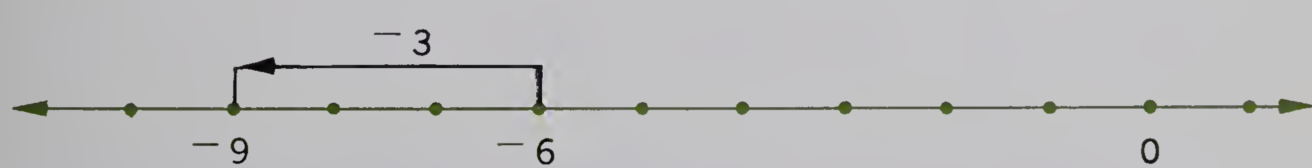
$$n = {}^+5 + ({}^-8) \qquad s = a_1 + a_2$$

The equation instructs us to start with $a_1 = {}^+5$ as before. But the *direction* of a_2 is to the left since its sign is negative. The line indicating $^-8$ is 8 units long so it brings us to $s = {}^-3$.



Let us examine an equation in which both addends are negative numbers.

$$n = {}^-6 + ({}^-3) \qquad s = a_1 + a_2$$



Explain how we obtain $n = {}^-9$ using the number line.
Move 6 units to the left of $^-3$.

1. Sketch a number line as in the examples above and perform the following additions.

a. $^+8 + ({}^+5)$ *+13*
b. $^+6 + ({}^-9)$ *−3*
c. $^-12 + ({}^-2)$ *−14*
d. $^-7 + ({}^+10)$ *+3*
e. $^-8 + ({}^-11)$ *−19*
f. $^+16 + ({}^-7)$ *+9*
g. $^+8 + ({}^-12)$ *−4*
h. $^-9 + ({}^-5)$ *−14*
i. $^+5 + ({}^+4)$ *+9*

2. Find in Exercise 1 as many examples as you can that illustrate:
a,c,e,h,i

To add two signed numbers with like signs, add their absolute values, and give the result the common sign.

3. Find as many examples in Exercise 1 as you can that illustrate:
b, d, f, g

To add two signed numbers with unlike signs, find the difference of their absolute values, and give the result the sign of the number with the larger absolute value.

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4. Use the principles stated in Exercises 2 and 3 to perform these additions.

a. $^{+}15 + (^{-}8)$ $+7$
b. $^{-}7 + (^{-}3)$ $^{-}10$
c. $^{-}18 + (^{+}18)$ 0
d. $^{+}55 + (^{-}55)$ 0
e. $^{+}25 + (^{+}31)$ $^{+}56$
f. $^{-}13 + (^{-}7)$ $^{-}20$
g. $^{+}21 + (^{-}9)$ $+12$
h. $^{+}17 + (^{+}13)$ $^{+}30$
i. $^{-}24 + (^{-}24)$ $^{-}48$
j. $^{-}36 + (^{+}41)$ $^{+}5$
k. $^{+}13 + (^{-}20)$ $^{-}7$
l. $^{-}19 + (^{-}13)$ $^{-}32$
m. $^{+}17 + (^{+}31)$ $^{+}48$
n. $^{-}18 + (^{+}33)$ $^{+}15$
o. $^{+}25 + (^{-}19)$ $^{+}6$

5. On page 96 Exercise 3 contained a discussion of absolute value. It is important that you understand the definition of absolute value. Determine if the following statements are *True* or *False*.

T **a.** The number $^{+}8$ is greater than $^{-}12$.
F **b.** The number $|^{+}8|$ is greater than $|^{-}12|$.
T **c.** If $x + |^{-}13| = 13$, then $x = 0$.
T **d.** If $x + |18| = 11$, then $x = {}^{-}7$.
F **e.** If $|12| + |^{-}18| = x$, then $x = {}^{-}6$.
T **f.** If $|^{-}12| + |^{-}18| = x$, then $x = 30$.
T **g.** If $|^{-}18| + |12| = x$, then $x = 30$.
T **h.** If $x + |^{-}12| = 18$, then $x = 6$.
T **i.** If $|x| + |^{-}12| = 18$, then $x = 6$ or $x = {}^{-}6$.
T **j.** If $|x| + |24| = 29$, then $x = 5$ or $x = {}^{-}5$.
T **k.** If $^{-}8 + |^{-}12| = x$, then $x = 4$.
T **l.** The absolute value of $^{-}81$ equals the absolute value of $^{+}81$.

6. Compare the absolute values of the numbers in each of the following pairs, using the symbol $<$ or $>$ to make a true statement.

EXAMPLES

If $(^{-}19, {}^{+}15)$ is given, then $|^{-}19| > |^{+}15|$.
If $(^{-}3, {}^{+}12)$ is given, then $|^{-}3| < |^{+}12|$.

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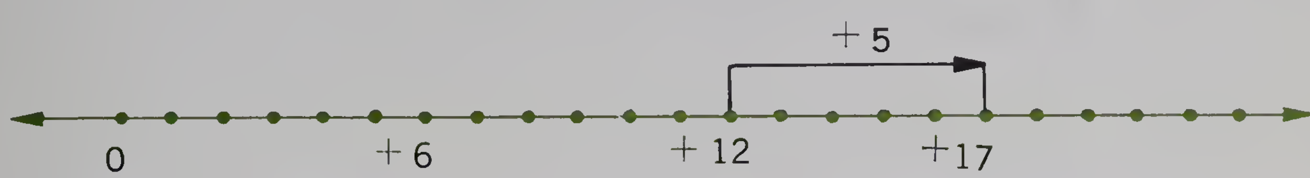
a. $(^{+}15, {}^{+}9)$ $>$
b. $(^{-}27, {}^{+}15)$ $>$
c. $(^{+}35, {}^{-}30)$ $>$
d. $(^{+}27, {}^{+}40)$ $<$
e. $(^{+}41, {}^{-}50)$ $<$
f. $(^{-}31, {}^{-}35)$ $<$
g. $(^{+}31, {}^{+}30)$ $>$
h. $(^{-}39, {}^{+}25)$ $>$
i. $(^{-}33, {}^{-}32)$ $>$
j. $(^{-}43, {}^{+}50)$ $<$
k. $(^{+}53, {}^{-}52)$ $>$
l. $(^{+}60, {}^{-}61)$ $<$

Let us examine an equation calling for subtraction with signed numbers.

$$n = {}^{+}17 - ({}^{+}12) \qquad a_1 = s - a_2$$

Referring to the number line, you see that the question is: What number added to $^{+}12$ gives $^{+}17$ as the sum? That is:

$$n + {}^{+}12 = {}^{+}17 \qquad a_1 + a_2 = s$$



As the number line shows, we start at 12 and move to the right. Because its *direction* is *to the right*, the sign of the difference is + and the answer is $^{+}5$.

Suppose the equation had read:

$$n = {}^{+}17 + ({}^{-}12)$$

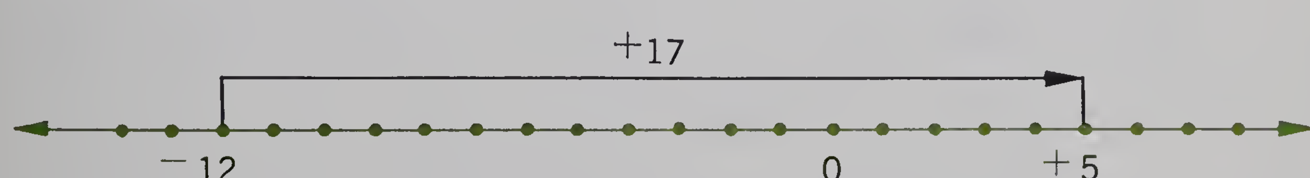
Would the number line above have been different? Would the value for n have been the same? Therefore, the equations *no ; yes*

$$n = {}^{+}17 - ({}^{+}12) \text{ and } n = {}^{+}17 + ({}^{-}12)$$

are equivalent.

Let us examine an equation calling for subtraction with a negative subtrahend.

$$n = {}^{+}5 - ({}^{-}12) \text{ or an equivalent equation } ({}^{-}12) + n = {}^{+}5$$



The question is, how far is it and in what direction from $^{-}12$ to $^{+}5$?

Examining the number line, you can see that the absolute value of the difference is 17 units. Since the direction is to the right, the value of n is $^{+}17$, or simply 17.

Again let us ask: If the equation had been $n = 5 + 12$, would the value for n have been the same? How would the number line have been different? Therefore, the equations *yes ; in no way*

$$n = 5 - ({}^{-}12) \text{ and } n = 5 + 12$$

are equivalent. You will notice further that the equations $n = 7 - ({}^{-}10)$ and $n = 7 + 10$ are equivalent in the same manner.

1. Sketch a number line for reference in performing these subtractions.

a. $13 - (^{-}9) = n$ $^{+}22$

b. $^{-}6 - (^{+}7) = n$ $^{-}13$

c. $^{-}9 - (^{+}15\frac{1}{2}) = n$ $^{-}24\frac{1}{2}$

d. $^{-}8\frac{1}{2} - (^{-}11) = n$ $^{+}2\frac{1}{2}$

e. $^{-}15 - (^{-}10\frac{2}{3}) = n$ $^{-}4\frac{1}{3}$

f. $^{+}13\frac{2}{3} - (^{+}10\frac{1}{4}) = n$ $^{+}3\frac{5}{12}$

2. Since addition and subtraction are inverse processes, it appears reasonable that subtraction might be replaced by an equivalent addition. In Exercise 1

$s - (^{+}a) = s + (^{-}a)$ represents parts b, c, f

$s - (^{-}a) = s + (^{+}a)$ represents parts a, d, e

Go back over Exercise 1 and show that the replacement gives you the same answers.

The principle of Exercise 2 may be stated

In subtracting signed numbers, change the sign of the subtrahend and use the rules for addition of signed numbers.

Solve:

3. $a = {}^{+}17 - (^{-}9)$ $^{+}26$

4. $b = {}^{-}15\frac{1}{2} - (^{+}16)$ $^{-}31\frac{1}{2}$

5. $c = {}^{-}23 - (^{-}25)$ $^{+}2$

6. $d = {}^{+}17.2 - (^{-}18.3)$ $^{+}35.5$

7. $f = {}^{+}19\frac{1}{5} - (^{+}33\frac{3}{10})$ $^{-}14\frac{1}{10}$

8. $g = {}^{-}12 - (^{-}12)$ 0

9. $h = {}^{+}25 - (^{+}18\frac{2}{5})$ $^{+}6\frac{3}{5}$

10. $i = {}^{+}34 - (^{-}16)$ $^{+}50$

11. $j = {}^{-}17\frac{5}{6} - (^{-}32\frac{1}{2})$ $^{+}14\frac{2}{3}$

12. $k = {}^{-}25 - (^{-}25)$ 0

13. $l = 25.9 - (^{-}19.4)$ $^{+}45.3$

14. $m = {}^{-}13.1 - (^{+}15)$ $^{-}28.1$

15. $n = {}^{-}22.6 - (^{-}28.7)$ $^{+}6.1$

16. $o = {}^{+}23\frac{1}{6} - (^{-}34\frac{1}{3})$ $^{+}57\frac{1}{2}$

17. Is there a number n such that $16 + n = 0$?

If	$16 + n = 0$	$a_1 + a_2 = s$
Then	$n = 0 - 16$	$a_2 = s - a_1$
or	$n = 0 + (^{-}16)$	$a_2 = s + (^{-}a_1)$
and	$n = {}^{-}16.$	

Does $16 + (^{-}16) = 0$? yes

18. Is there a number n such that $^{-}18 + n = 0$?

If	$^{-}18 + n = 0$	$a_1 + a_2 = s$
Then	$n = 0 - (^{-}18)$	$a_2 = s - a_1$
or	$n = 0 + (^{+}18)$	$a_2 = s + (^{-}a_1)$
and	$n = {}^{+}18$	

Does $^{-}18 + (^{+}18) = 0$? yes This leads us to consider the additive inverse.

If the sum of any two rational numbers is zero, then the two rational numbers are *additive inverses* of one another. If a is any positive number, what is its additive inverse?

$$a + n = 0$$
$$n = 0 - a = {}^-a$$

That is, if a is any positive number, its additive inverse is its negative "mate." Since each is the additive inverse of the other, the additive inverse of a negative number is its positive "mate."

1. Write the additive inverse of each of the following:

$^-15$ **a.** 15	$^+7$ **e.** $^-7$	^+y **i.** ^-y	$\frac{^-2x}{3}$ **m.** $\frac{2x}{3}$	^+3y **q.** ^-3y
$^-12$ **b.** 12	$\frac{^-n}{2}$ **f.** $\frac{n}{2}$	$^-16$ **j.** 16	^+5n **n.** ^-5n	$^+14$ **r.** $^-14$
$^+4$ **c.** $^-4$	^+b **g.** ^-b	^+n **k.** ^-n	^-12x **o.** $12x$	$\frac{^-5n}{8}$ **s.** $\frac{5n}{8}$
$^+9$ **d.** $^-9$	$^+19$ **h.** $^-19$	^-4x **l.** $4x$	$\frac{^-3a}{7}$ **p.** $\frac{3a}{7}$	^+3a **t.** ^-3a

2. Sketch a number line and identify three pairs of additive inverses. What can you say about the absolute value of a number and its additive inverse?

3. How does the direction of a number from zero relate to that of its additive inverse?

4. Subtraction may be defined in terms of the additive inverse.

In subtracting signed numbers, add to the minuend the additive inverse of the subtrahend.

Use this definition to help you in the following exercises. Remember that $n = a - b$ and $n = a + ({}^-b)$ are equivalent equations.

$^+8$ **a.** $^+16 + (^-8)$	$^+8$ **i.** $^+11 - (^+3)$	$^-26$ **q.** $^-17 + (^-9)$
$^+6$ **b.** $^+8 - (^+2)$	0 **j.** $^-13 + (^+13)$	$^-15$ **r.** $^-12 - (^+3)$
$^+12$ **c.** $^+9 + (^+3)$	$^-2$ **k.** $^+8 - (^+10)$	0 **s.** $^+14 + (^-14)$
$^+23$ **d.** $^+16 - (^-7)$	$^+20$ **l.** $^+17 - (^-3)$	$^-4$ **t.** $^-17 - (^-13)$
$^-2$ **e.** $^+9 + (^-11)$	$^-2$ **m.** $^+14 + (^-16)$	$^-20$ **u.** $^-7 + (^-13)$
$\frac{^+1}{9}$ **f.** $^-\frac{5}{9} + (^+\frac{2}{3})$	$^-5\frac{1}{8}$ **n.** $^-1\frac{5}{8} - (^+3\frac{1}{2})$	$^+4\frac{1}{2}$ **v.** $^-5\frac{1}{2} - (^-10)$
$\frac{^+1}{2}$ **g.** $^+\frac{4}{5} - (^+\frac{3}{10})$	$\frac{^-3}{4}$ **o.** $^+3\frac{1}{3} + (^-4\frac{1}{12})$	$^-2\frac{1}{6}$ **w.** $^+2\frac{1}{6} + (^-4\frac{1}{3})$
$^-1\frac{7}{8}$ **h.** $^-1\frac{1}{8} + (^-\frac{3}{4})$	$^+3\frac{1}{4}$ **p.** $^-2\frac{1}{4} - (^-5\frac{1}{2})$	$^-7\frac{1}{2}$ **x.** $^-4\frac{1}{2} - (^+3)$

The additive inverse becomes a powerful device to use in solving equations. As you know, an equation is solved when the variable with a coefficient 1 is alone on one side, and a constant is alone on the other side of the equation. Sometimes there is a variable or a constant that must be "removed" from one side of the equation before this can be done. This "removal" can be done by adding the additive inverse of the constant or variable (or both) to both sides of the equation. Remember!

If the same number is added to both sides of an equation, the solution of the equation is not changed.

EXAMPLES

1. *Solve:* $2x + 5 = 17$ — We need to "remove" the 5 to get terms containing the variable alone.

$$2x + 5 + (^{-}5) = 17 + (^{-}5)$$ Add the additive inverse of 5.

$$2x = 12 \qquad f_1 f_2 = p$$

$$x = 12 \div 2 = 6 \qquad p \div f_1 = f_2$$

Check: $2 \times 6 + 5 = 17$ or $12 + 5 = 17$, $17 = 17$

2. *Solve:*

$$2n - 5 = n + 25$$

$$2n + (^{-}5) = n + 25 \qquad \text{Subtraction definition}$$

Then $2n + (^{-}5 + 5) = n + 25 + 5$ — Explain.

or $2n = n + 30$

Then $2n + (^{-}n) = n + (^{-}n) + 30$ — Explain.

$$n = 30$$

Check: $(2 \times 30) - 5 = 30 + 25$ and $60 - 5 = 55$

3. *Solve:*

$$3.5x - 7 = 1.5x + 1$$

$$3.5x + (^{-}1.5x) + (^{-}7) + (^{+}7) = 1.5x + (^{-}1.5x) + 1 + (^{+}7)$$

or $(3.5x - 1.5x) + (^{-}7 + 7) = (1.5x - 1.5x) + (1 + 7)$

$$2x = 8 \qquad f_1 f_2 = p$$

$$x = 8 \div 2 = 4 \qquad p \div f_1 = f_2$$

Check: $(3.5 \times 4) - 7 = (1.5 \times 4) + 1$ or $14 - 7 = 6 + 1$, $7 = 7$

In the following exercises use the additive inverse when necessary to solve each equation. Explain each step, as in the examples.

1. $3x + 5 = 17$ 4

2. $5x + 5 = 35$ 6

3. $x - 5 = 17$ 22

4. $\frac{x}{2} + 3 = 4$ 2

5. $x - 6 = 12 - x$ 9

6. $7x = 63$ 9

7. $6x = 24 - 2x$ 3

8. $\frac{5x}{7} = 35$ 49

9. $\frac{5x}{2} + 6 = 9.5$ 1.4

10. $17 - x = 13$ 4

11. $4x + 12 = x + 27$ 5

12. $\frac{x}{3} - 5 = \frac{x}{6} - 1$ 24

13. $\frac{4x}{3} + 2\frac{2}{3} = 2x - 2$ 7

14. $\frac{5x}{4} - 1.5 = x$ 6

15. $\frac{6x}{7} = 84$ 98

16. $\frac{5x}{4} + 2 = 13.5 - \frac{x}{4}$ $7\frac{2}{3}$

17. $\frac{3x}{4} + 3 = \frac{x}{2} + 9$ 24

18. $2x - 5 = x - 1$ 4

19. $\frac{x}{2} + 6 = 11$ 10

20. $2x - 1 = x - 6$ $^{-}5$

21. $18 - x = 7$ 11

22. $6x + 7 = 5x + 9$ 2

23. $\frac{3x}{5} = 54$ 90

24. $2x - 2 = 8 - 2x$ 2.5

25. $84 = 35 + x$ 49

26. $\frac{7x}{4} + \frac{1}{2} = x + 2\frac{1}{4}$ $2\frac{1}{3}$

27. $\frac{5x}{4} = 3.5 - \frac{x}{2}$ 2

28. $\frac{13x}{20} = 91$ 140

29. $\frac{13x}{4} - \frac{1}{2} = \frac{5x}{2} + 2$ $3\frac{1}{3}$

30. $x - 19 = 31$ 50

The following exercises are similar to Example 3 on page 102.

31. $4.5x - 9 = 1.5x + 3$ 4

32. $8.5x - 11 = 2.5x + 7$ 3

33. $9.5x - 17 = 1.5x + 7$ 3

34. $3.5x + 6 = 1.5x + 18$ 6

35. $5.5x + 11 = 2.5x + 23$ 4

36. $7.5x + 19 = 6.5x + 25$ 6

37. $4.2x - 9 = 2.2x + 3$ 6

38. $5.7x - 14 = 3.7x - 8$ 3

39. $17.6x - 19 = 4.6x + 7$ 2

40. $12.3x + 4 = 9.3x + 13$ 3

41. $19.2x - 6 = 5.2x + 36$ 3

42. $27.8x - 14 = 10 + 19.8x$ 3

43. $9.6x + 18 = 18 + 5.6x$ 0

44. $23.7x - 15 = 10 + 18.7x$ 5

45. $14.9x + 13 = 32 - 4.1x$ 1

46. $33.6x - 14 = 98 - 12.4x$ $2\frac{10}{23}$

Write an equation for each of these statements. Solve for the variable and check your solution by substituting it in the original statement.

$\frac{s}{2} + 3 = 8$; 10

1. If one half of a certain number s is increased by 3, the sum is 8.

$4n + 4 = 16$; 3

2. If four times a certain number n is increased by 4, the sum is 16.
3. If $4\frac{1}{3}$ is added to $\frac{2}{3}$ of a certain number y, the sum is $12\frac{1}{3}$. 12
4. If $1\frac{1}{3}$ is added to twice a certain number m, the sum is $12\frac{1}{3}$. 5.5
5. If 3 is subtracted from $\frac{5}{2}$ of a certain number t, the result is twice the number. $\frac{5t}{2} - 3 = 2t$; 6
6. If 5 is subtracted from twice a certain number b, the result is the same as if 2 were subtracted from the number. $2b - 5 = b - 2$; 3
7. Thirteen times a certain number x is 52. $13x = 52$; 4
8. A certain number n decreased by 25 is equal to 15. $n - 25 = 15$; 40
9. Six less than four times a certain number a, decreased by 5, is equal to 1. $4a - 6 - 5 = 1$; 3
10. Twice a certain number a, decreased by 5, is equal to 1. $2a - 5 = 1$; 3
11. If a certain number x is added to 26, the sum is 54. $26 + x = 54$; 28
12. If twice a certain number y is increased by 4, the result is the same as if 3 times the same number is decreased by 3. $2y + 4 = 3y - 3$; 7
13. Three times a certain number m is equal to 2 decreased by the same number. $3m = 2 - m$; $\frac{1}{2}$

$\frac{x}{2} + 3 = 4$; 2

14. One half of a certain number x, increased by 3, is equal to 4.
15. If 15 is subtracted from 5 times a certain number x, the result is 5 more than the same number. $5x - 15 = x + 5$; 5
16. If 6 times a certain number n is decreased by 4, the result is equal to 2 more than 4 times the same number. $6m - 4 = 4m + 2$; 3
17. 15 times a certain number x is 225. $15x = 225$; 15
18. 84 divided by a certain number c is 6. $84 \div C = 6$; 14
19. One and one-half times a certain number n is 21. $\frac{3n}{2} = 21$; 14
20. If 15 times a certain number a is decreased by 2, the result is the same as if ten times the same number is increased by 18.

$15a - 2 = 10a + 18$; 4

21. If 8 times a certain number n is increased by 9, the result is the same as if 13 times the same number is decreased by 26.

$8n + 9 = 13n - 26$; 7

One of the most important steps in solving a problem is Step 5, when you set up the equation. This not only reveals clearly the mathematical relationships in the problem, but also removes the irrelevant details that might lead to confusion. Let us see how the equation is set up in a complex problem.

EXAMPLE

Jim and Jerry together earned $530 last summer. Jim earned $6 more than 3 times as much as Jerry. How much did each of the boys earn?

Let x represent the number of dollars Jerry earned.

NOTE: Jerry earned the least.

Then $3x + 6$ represents the number of dollars Jim earned.

Then $x + (3x + 6)$ or $4x + 6$ represents the number of dollars they earned together.

We know that 530 is the number of dollars they both earned.

Therefore, $4x + 6 = 530$.

Solve the equation and check. *Jim, $ 399; Jerry, $ 131*

STEPS FOR SOLVING APPLIED PROBLEMS

1. Understand the problem.
2. Note what the problem asks for.
3. Look for hidden questions.
4. Estimate a reasonable answer.
5. Set up and solve the conditional sentence(s).
6. Check your answer.

Follow the same procedure to solve the following problems. Be sure to specify what the variable represents.

1. The sum of three numbers is 111. The second number is 6 more than the first. The third is 12 more than the first. What is each number? HINT: Which number is the smallest? *31; 37; 43*

2. Harry and his father traveled 320 miles by car on a recent trip. Harry drove 30 miles more than his father. How far did each drive? HINT: Which person drove the lesser amount? *father, 145 mi.; Harry, 175 mi.*

3. Three times a certain number, increased by 16, is 64. What is the number? *16*

4. A farmer owned 1600 acres of land. He divided it among his three sons. The first received twice as many acres as the second, and the third 200 acres fewer than the first. How many acres did each of the sons receive? HINT: Which son received the fewest?
first son, 720 A.; second son, 360 A.; third son, 520 A.

5. The measure of the length of a rectangle is 4 inches greater than the measure of the width. The perimeter is 40 inches. What are the dimensions? HINT: Which measure is smaller? w = 8 in.; l = 12 in.

6. The perimeter of a field is 960 rods. The measure of the length is 3 times the measure of the width. What is the measure of each side?
w = 120 rd.; l = 360 rd.

7. Mabel and Helen together have \$40. Helen has \$10 less than Mabel. How much has each? Helen, \$15; Mabel, \$25

8. If twice a certain number is increased by 9, the result is the same as if 3 times the number were subtracted from 29. What is the number? 4

9. Jim and Henry together sold 115 subscriptions for a magazine. If Jim had sold 5 more subscriptions, he would have sold twice as many as Henry. How many did each sell? Jim, 75; Henry, 40

10. The perimeter of a rectangular field is 480 rods. The measure of the length is twice that of the width. What are the dimensions of the field? w = 80 rd.; l = 160 rd.

11. Jerry has an 18-foot board that he plans to saw in two so that one piece will be twice as long as the other. How long should each piece be? 12 ft.; 6 ft.

12. The measure of the length of a room is 6 feet more than twice that of the width. The perimeter of the room is 72 feet. What are the dimensions of the room? w = 10 ft.; l = 26 ft.

13. The perimeter of a field is 612 rods. The measure of the width is 6 feet more than $\frac{1}{2}$ of the measure of its length. What are the dimensions of the field? w = 106 rd.; l = 200 rd.

14. The measures of the equal sides of an isosceles triangle are each 3 times the measure of the base. The perimeter of the triangle is 56 inches. How long is each of the sides? base, 8 in.; sides, 24 in.

15. The length of each of the sides of equal measure of an isosceles triangle is 9 inches greater than that of the base. The perimeter of the triangle is 93 inches. What is the length of each side of the triangle? base, 25 in.; sides, 34 in.

16. In a scalene triangle, the longest side measures 3 times that of the shortest side. The third side measures 9 inches more than the shortest side. The perimeter is 34 inches. How long is each side?
longest side, 15 in.; shortest side, 5 in.; third side, 14 in.

In the previous chapter you had experience in analyzing the meaning of the digits in a numeral in powers of 10, as indicated by exponents. The use of *negative exponents* enables us to extend this analysis to decimals. When a power of 10 is indicated by an exponent, it is said to be in exponential form. Thus, in exponential form $100 = 10^2$, $1000 = 10^3$, and so forth.

1. Write these numerals, which are now in *standard form,* in exponential form.
a. 1,000,000 10^6 **b.** 10,000 10^4 **c.** 1,000,000,000 10^9

2. Write these numerals in standard form.
a. 10^5 100,000 **b.** 10^8 100,000,000 **c.** 10^1 10

3. As an exponent of 10 increases by 1, the previous number is multiplied by 10. As an exponent of 10 decreases by 1, the previous number is decreased by a factor of 10. Write these quotients and products in standard form.
a. $10,000 \times 10$ 100,000 **c.** $10^4 \times 10$ 100,000 **e.** $10^6 \div 10$ 100,000
b. $10,000 \div 10$ 1000 **d.** $10^4 \div 10$ 1000 **f.** $10^5 \times 10$ 1,000,000

4. Write the numerals expressing the value of the quotient $10^1 \div 10$ in exponential form and in standard form. 10^0, 1

5. What is the value of each of the following?
a. 5×10^0 5 **b.** 8×10^0 8 **c.** 15×10^0 15

6. To carry the rule expressed in Exercise 3 one step further, if we divide 10^0 by 10, the quotient should be 10^{-1}. That is, $10^0 \div 10 = 10^{-1}$. Express this division without using exponents. What is the value of 10^{-1}? $1 \div 10 = \frac{1}{10}$; $\frac{1}{10}$ or 0.1

7. Write these numerals as decimals.
a. 7×10^{-1} 0.7 **b.** 3×10^{-1} 0.3 **c.** 8×10^{-1} 0.8 **d.** 13×10^{-1} 1.3

8. Write these numerals in exponential form.
a. 0.4 4×10^{-1} **b.** 0.8 8×10^{-1} **c.** 0.3 3×10^{-1} **d.** 1.5 15×10^{-1} **e.** 11.7 117×10^{-1} **f.** 13.2 132×10^{-1}

9. To determine the meaning of 10^{-2}, we can divide:

$$10^{-1} \div 10 = 10^{-2}$$

Write these numerals without use of exponents, stating the value of 10^{-2}. $0.1 \div 10 = .01$; $10^{-2} = .01$ or $\frac{1}{100}$

10. Write these numerals as decimals.
a. 16×10^{-2} 0.16 **c.** 4×10^{-2} .04 **e.** 32.1×10^{-2} 0.321
b. 29×10^{-2} 0.29 **d.** 113×10^{-2} 1.13 **f.** 3.21×10^{-2} .0321

11. Using the rule of Exercise 3, you can determine the meaning of 10^{-3} by dividing 10^{-2} by 10. Write the division using the exponential form and also in standard form. $10^{-2} \div 10 = 10^{-3}$; $.01 \div 10 = .001$

12. Write each of the following as numerals with negative exponents.
a. 0.135 **b.** 0.5 **c.** 0.17 **d.** 0.117 **e.** 0.23 See below.

13. Write each of the following as decimals.
a. 10^{-4} .0001 **b.** 10^{-5} .00001 **c.** 10^{-6} .000001

14. Write each of the following in exponential form. See below.
a. 0.1625 **b.** 0.35125 **c.** 0.075 **d.** 0.0025

15. Using negative exponents, it is possible to analyze decimals to indicate the value of the number named by each digit. Thus 257.8123 can be analyzed as follows:

(ex. 12)
a. 135×10^{-3}
b. 5×10^{-1}
c. 17×10^{-2}
d. 117×10^{-3}
e. 23×10^{-2}

$$\begin{array}{rcrcr} 2 \times 10^2 &=& 2 \times 100 &=& 200 \\ 5 \times 10^1 &=& 5 \times 10 &=& 50 \\ 7 \times 10^0 &=& 7 \times 1 &=& 7 \\ 8 \times 10^{-1} &=& 8 \times .1 &=& 0.8 \\ 1 \times 10^{-2} &=& 1 \times .01 &=& 0.01 \\ 2 \times 10^{-3} &=& 2 \times .001 &=& 0.002 \\ 3 \times 10^{-4} &=& 3 \times .0001 &=& 0.0003 \\ \hline &&&& 257.8123 \end{array}$$

(ex. 14)
a. 1625×10^{-4}
b. 35125×10^{-5}
c. 75×10^{-3}
d. 25×10^{-4}

Using this procedure, analyze each of these numerals. See front.
a. 12.375 **b.** 22.222 **c.** 183.07025

16. Any natural number raised to a negative power names a fractional number. You can see this if you apply the rule.
a. What is the value of 6^0? 1
b. What is the value of $6^0 \div 6$? $\frac{1}{6}$
c. Express without exponents: 6^{-1}; 6^{-2}. $\frac{1}{6}$; $\frac{1}{36}$

17. If a is any natural number, then a^{-1} is the multiplicative inverse of a. Show that this is the case by writing each of these numerals as a fraction.
a. 7^{-1} $\frac{1}{7}$ **b.** 15^{-1} $\frac{1}{15}$ **c.** 2^{-1} $\frac{1}{2}$ **d.** 13^{-1} $\frac{1}{13}$ **e.** 25^{-1} $\frac{1}{25}$

18. What do you think $20x^{-2}$ equals if $x = 10$? 0.2

19. If $x = 10$, what is the value of $100x^{-2}$? 1

20. If $x = 10$, what is the value of $10x^{-2}$? 0.1

21. If $x = 10$, what is the value of $(10x)^{-2}$? .0001

22. If $x = 10$, what is the value of $(100x^{-2})^{-2}$? 1

23. If $x = 10$, what is the value of $\frac{10}{x^2}$? 0.1

24. If $x = 10$, what is the value of $\frac{100}{x^2}$? 1

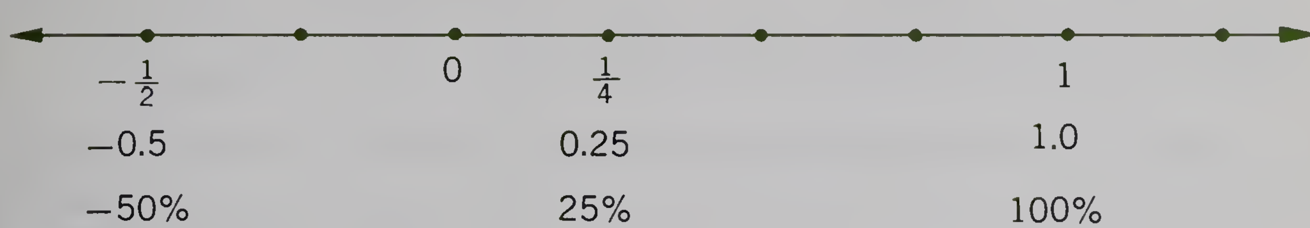
25. If $x = 10$, what is the value of $50x^{-2}$? 0.5

The term *per cent* is derived from the Latin *per centum,* meaning *out of a hundred.* Thus, if it is reported that 95 per cent of the students in a school are present on a certain day, this means that if 100 students were enrolled, 95 were present. If more or less are enrolled, the ratio $\frac{95}{100}$ remains the same. The symbol for per cent is %. Thus 95 per cent is written 95%. The symbol may be interpreted as "times 0.01." Thus the statement about school attendance may be written in three ways, all with the same meaning.

95% of the students are present.
0.95 of the students are present.
$\frac{95}{100}$ or $\frac{19}{20}$ of the students are present.

A rational number can be named in three ways:

25%   0.25 $\frac{1}{4}$ 100%   1.0   1



1. Write two other names associated with
   - **a.** $\frac{3}{4}$ 0.75; 75%
   - **b.** 80%   0.8; $\frac{4}{5}$
   - **c.** 0.65   65%; $\frac{13}{20}$
   - **d.** $2\frac{1}{2}$ 2.5; 250%
   - **e.** 1.75 $1\frac{3}{4}$; 175%
2. Write each of these statements using a fraction in simplest form; then write each one using a decimal.
   - **a.** A dime is 10% of a dollar. $\frac{1}{10}$; 0.1
   - **b.** The baseball team has won 60% of its games. $\frac{3}{5}$; 0.6
   - **c.** 85% of the class is present today. $\frac{17}{20}$; 0.85
   - **d.** A quart is 25% of a gallon. $\frac{1}{4}$; 0.25
   - **e.** About 10% of the families in this country live in rural areas. $\frac{1}{10}$; 0.1
   - **f.** About 50% of high school graduates go on to college. $\frac{1}{2}$; 0.5
   - **g.** Mabel saved 60% of what she earned last summer. $\frac{3}{5}$; 0.6
   - **h.** Jim solved 85% of the problems correctly. $\frac{17}{20}$; 0.85
   - **i.** The population of the city has increased 120% in the past 20 years. $1\frac{1}{5}$; 1.2
   - **j.** The average family income today is 500% of what it was 30 years ago. 5; 5.0

SPECIAL PROJECT

Find statements in newspapers and periodicals using per cent. Bring some of the more interesting items to class and be ready to display them. Change the per cents to decimal and fractional form.

Express the answer to each of these problems in three ways: as a common fraction in simplest form, as a decimal rounded to the nearest thousandth, and as per cent to the nearest tenth of 1%.

1. Mr. Jensen earns $6800 a year. His family spends $1700 a year for food. What part of their income is spent for food?
$\frac{1}{4}$; 0.250; 25.0%

2. In a class of 32 pupils, 4 pupils were absent last Monday. What part of the class was absent? $\frac{1}{8}$; 0.125; 12.5%

3. On a test of 20 questions, George had 4 wrong. What part did he have correct? $\frac{4}{5}$; 0.800; 80.0%

4. Mabel has been earning $15 a week. She has received a raise of $3 a week. What part of her former salary is the increase?
$\frac{1}{5}$; 0.200; 20.0%

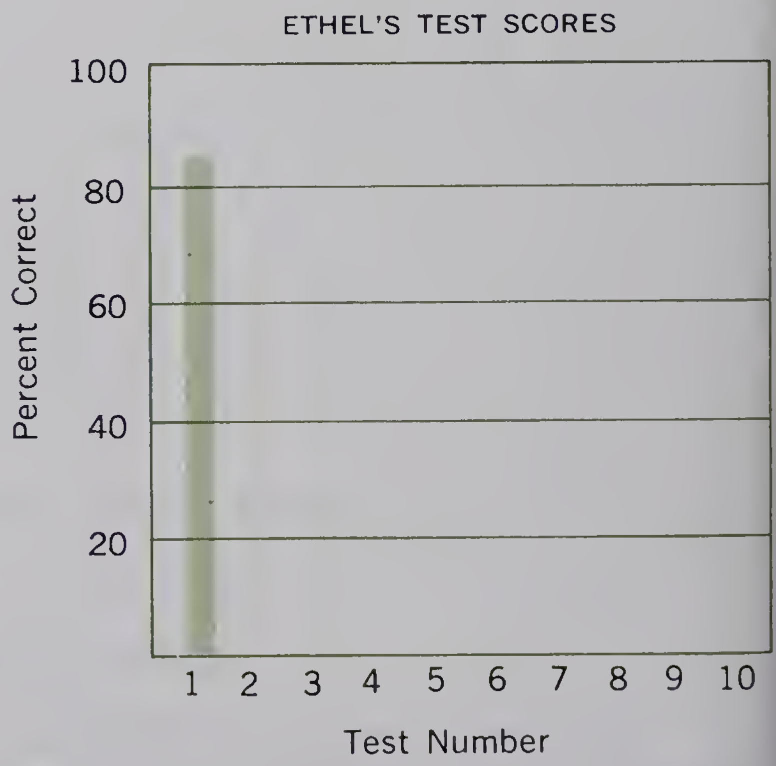
5. Jim bought for $2.80 a book that was regularly priced at $3.50. What part of the regular price did he save? $\frac{1}{5}$; 0.200; 20.0%

6. In 1940 Centerville had a population of 2400. In 1950 the population was 2760. The increase was what part of the 1940 population?
$\frac{3}{20}$; 0.150; 15.0%

7. The president of the music club announced recently that of the 60 members, 54 had paid their dues. What part of the members have paid their dues? $\frac{9}{10}$; 0.900; 90.0%

8. What part of her problems did Ethel have correct in each weekly test? Copy and complete the graph below to show the record of Ethel's ten test scores.

Week		*Number of Problems*	*Number Correct*
1	84%	25	21
2	90%	20	18
3	62.5%	40	25
4	76%	25	19
5	86%	50	43
6	90%	30	27
7	96%	25	24
8	90%	30	27
9	90%	40	36
10	96%	25	24



9. Can Lucy obtain an 85 test-average for the first six weeks if her first five-week test scores are 80, 90, 70, 100, 80?
yes, if she receives a score of 90 or above

1. A basketball team won 22 games and lost three games.
- **a.** How many games has the team played? 25
- **b.** What is the ratio of the games won to the games played, expressed as a fraction? $\frac{22}{25}$
- **c.** What is the ratio expressed as a decimal? 0.88
- **d.** What is the ratio expressed as a per cent? 88%

2. Mary works as a secretary afternoons and Saturdays, 12 hours a week, at $1.50 an hour. She saves $4.50 a week.
- **a.** How much does Mary earn a week? $ 18
- **b.** What per cent of her earnings does she save? 25%
- **c.** How much can she save in a year (52 weeks)? $ 234

3. A rectangular field is 60 rods wide. Its perimeter (distance around) is 320 rods.
- **a.** How many rods is the sum of the measures of two longer sides? 200 rd.
- **b.** What is the number of rods in the length of the field? 100 rd.
- **c.** Express the ratio of width to length in three ways. $\frac{3}{5}$; 0.6 ; 60%

4. The Lakeside basketball team has won 8 games and lost 4. The Hillcrest team has won 10 games and lost 6.
- **a.** How many games has the Lakeside team played? 12
- **b.** Express the ratio of games won to games played as a decimal rounded to the nearest thousandth. 0.667
- **c.** How many games has the Hillcrest team played? 16
- **d.** Express the ratio of games won to games played as a decimal rounded to the nearest thousandth. 0.625
- **e.** Which team has won the greater fraction of its games? Lakeside

5. The sales tax in Baldwin is 3% of the cost of an article.

3% is $\frac{3}{100}$ or .03.

- **a.** If a hat is priced at $4.00, what is the sales tax? 12 ¢
- **b.** What will be the total price of the hat? $ 4.12
- **c.** If the sales tax Mary paid on a dress she bought was 30¢, what was the price of the dress? $ 10
- **d.** Kevin is buying a pair of skates which are listed at $9.00. What is the total amount he will need to buy them? $ 9.27
- **e.** If a sport shirt is priced at $2.98, what is the sales tax? 9 ¢
- **f.** If the sales tax Joan had to pay for a blouse she bought was 9¢, what was the price of the blouse? $ 3
- **g.** The Jones family were planning to buy a television set priced at $199. What will be the total price of the set? $ 204.97

6. The ratio of the width of a field to its length is $\frac{5}{16}$. The width is 40 rods.
   a. Remember, two fractions are equivalent if they are identical when written in simplest form. What fraction with numerator 40 is equivalent to $\frac{5}{16}$? $\frac{40}{128}$
   b. What is the length of the field? 128 rd.
   c. What is the ratio of the length of the field to its width? $\frac{16}{5}$

7. John is 5′ 9″ tall. Mike is 6′ tall.
   a. What is the height of each boy in inches? 69″, 72″
   b. Express the ratio of John's height to Mike's as a fraction in simplest form. $\frac{23}{24}$
   c. Express the ratio of Mike's height to John's as a fraction in simplest form. $\frac{24}{23}$

8. The quarterback for the Rams attempted 16 forward passes and completed 10. The quarterback for the Eagles attempted 12 passes and completed 9.
   a. What per cent of his forward passes did each quarterback complete? Rams, 62.5%; Eagles, 75%
   b. Which quarterback completed the greater per cent of his passes? Eagles
   c. The number of passes completed by the Eagles' quarterback was what per cent of the number completed by the Rams' quarterback? 90%
   d. The number of passes attempted by the Rams' quarterback was what per cent of the number attempted by the Eagles' quarterback? 133.3%

9. Last Monday 48 pupils were absent from the Wilson High School. This is 4% of the total enrollment.
   a. Express 4% as a fraction with denominator 100. $\frac{4}{100}$
   b. What fraction with numerator 48 is equivalent to this fraction that you have just written? $\frac{48}{1200}$
   c. How many pupils are enrolled in the Wilson High School? 1200
   d. How many pupils were present last Monday? 1152

10. The Fairview Sport Shop is having a sale at which all prices are reduced 20%.
   a. What is the selling price of a coat regularly priced at \$25? \$20
   b. Jim bought a baseball glove at the sale for \$16. This was what per cent of the regular price? 80%
   c. Express this per cent as a fraction with denominator 100. $\frac{80}{100}$
   d. What fraction with numerator 16 is equivalent to this fraction you just wrote? What is the regular price of the glove? $\frac{16}{20}$; \$20
   e. What is the selling price of a glove regularly priced at \$19? \$15.20

In many mathematical sentences the ratio between two numbers is expressed as a per cent. In these sentences you can always find two equal ratios as follows:

1. The per cent, which is expressed as a fraction with denominator 100.
2. The ratio of the other two numbers is also expressed as a fraction. The number to which the other is compared will be used as the denominator.

EXAMPLE

Write the proportion expressed in this statement:

160% of 90 is 144 — 144 is compared to 90

Since $160\% = \frac{160}{100}$

then $\frac{160}{100} = \frac{144}{90}$

How do you know that 90 should be the denominator? Does the product of the means equal the product of the extremes? Yes, both products are 14,400.

1. Write the proportion expressed in each of these statements. Check each by seeing if the product of the means is equal to the product of the extremes.

a. 24 is 60% of 40 $\frac{60}{100} = \frac{24}{40}$

b. 15% of 80 is 12 $\frac{15}{100} = \frac{12}{80}$

c. 150% of 70 is 105 $\frac{150}{100} = \frac{105}{75}$

d. 20 is 50% of 40 $\frac{50}{100} = \frac{20}{40}$

e. 60 is 40% of 150 $\frac{40}{100} = \frac{60}{150}$

f. 120% of 600 is 720 $\frac{120}{100} = \frac{720}{600}$

g. 16% of 40 is 6.4 $\frac{16}{100} = \frac{6.4}{40}$

h. 150% of 60 is 90 $\frac{150}{100} = \frac{90}{60}$

i. 36 is 150% of 24 $\frac{150}{100} = \frac{36}{24}$

j. 144 is 120% of 120 $\frac{120}{100} = \frac{144}{120}$

If one of the terms in the statement is unknown, the proportion becomes a conditional equation. The missing term may be found by solving the proportion.

EXAMPLES

1. 56% of 45 is x

Since $56\% = \frac{56}{100}$,

$$\frac{56}{100} = \frac{x}{45}$$

Then $100x = 2520$ $\quad f_1 \times f_2 = p$

Complete the solution. 25.2

2. $y\%$ of 36 is 45

Since $y\% = \frac{y}{100}$,

$$\frac{y}{100} = \frac{45}{36}$$

Then $36y = 4500$ $\quad f_1 \times f_2 = p$

Complete the solution. 125

Check your answers following the procedure in Exercise 1.

Write and solve the proportions to find the missing term in each statement. Check by substituting the value for the variable in the original statement.

1. 36% of 150 is n 54
2. 27 is x% of 20 135 %
3. 2.4 is 30% of y 8
4. 4.2 is 60% of y 7
5. x% of 25 is 6 24 %
6. 9.5 is 19% of n 50
7. 54 is 40% of y 135
8. 36 is n% of 45 80%
9. 27% of 5.2 is y 1.404
10. x% of 60 is 24 40 %
11. 750% of n is 9.3 1.24
12. 150% of n is 7.5 5
13. 1.5% of n is 3.6 240
14. x% of 55.5 is 44.4 80%
15. 9% of 82.5 is y 7.425
16. y is 83% of 700 581
17. 18% of x is 3.24 18
18. 87% of 46 is n 40.02
19. 16.8 is 400% of y 4.2
20. 135% of 6.4 is x 8.64
21. 225% of x is 10.8 4.8
22. 179% of 360 is x 644.4
23. 32.9 is 175% of y 18.8
24. n is 0.5% of 800 4
25. n is 6.5% of 8000 520
26. 5.04 is y% of 1.12 450 %
27. 14.5 is 30% of x 48.333
28. 3.2% of x is 0.8 25
29. .05% of 360 is n 0,18
30. 1.75% of 60 is y 1.05

A systematic procedure for solving problems in which ratios are expressed in per cent is as follows:
Write a percentage equation, omitting all dollar signs and other measures. Then set up the proportion, and solve to find the unknown term.

EXAMPLE

Jim is saving his money to purchase a used car. He has saved \$378 and says that this is 45% of what he needs to buy the car. What is the price of the car?

Let n represent the price of the car.

378 is 45% of n percentage statement

$\frac{45}{100} = \frac{378}{n}$ proportion

Solve and check. \$ 840

On the following two pages, we will apply this procedure.

STEPS FOR SOLVING APPLIED PROBLEMS

1. Understand the problem.	**2.** Note what the problem asks for.	**3.** Look for hidden questions.
6. Check your answer.	**5.** Set up and solve the conditional sentence(s).	**4.** Estimate a reasonable answer.

Solve each of these problems, using a percentage equation and proportion.

1. Mr. Adams planted 25% of his farm in corn. He has a 240-acre farm. How many acres are planted in corn? 60 A.
2. The winning team in a World Series received $251,000, which was divided among the players. Each player received 3.7% of the total as his share. How much did each receive? $9287
3. Mike had 85% of the problems correct on a recent test. He had 17 problems correct. How many problems did he miss? 3
4. Tom can buy a new trumpet for $300 or a used one for $180. The cost of the used trumpet is what per cent of the cost of the new one? 60%
5. The Jones family budgets 13% of the family income for clothing. Last year they spent $910 for clothing. If this represented the allotted per cent, how much was their income? $ 7000
6. Mr. Jensen's income is $7500 a year. He pays $120 a month for rent. What per cent of his income is used for rent? 19.2%
7. A basketball team won 12 games this year, which is 75% of the games it played. How many games did the team lose? 4
8. The Lockwood Department Store advertised a dinette set that regularly sells for $120 at a sale price of $90. The reduction is what per cent of the regular price? 25%
9. Mary purchased a coat priced at $40 at a reduction of 20%. What did she pay for the coat? $ 32
10. A $60 bicycle is sold for $54. The reduction is what per cent of the regular price? 10 %
11. Harry sells magazines for 25¢ each. He receives 20% of his sales as commission. What is his commission on 150 copies? $ 7.50

12. Jim received \$12 commission for selling newspaper subscriptions. His commission was 15% of his sales. What was the amount of his subscriptions sales? *\$ 80*

13. The secretary of the photography club reported that 20% of the members were absent. What is the total membership if 36 members were present? *45*

14. A department store reported that 15% of its expenses last year were for advertising. If advertising cost \$12,600, what were the store's total expenses? *\$ 84,000*

15. When a bus for Portland had gone 180 miles from San Francisco, the driver said they had covered 25% of the distance. By this route, how far is it from San Francisco to Portland? *720 mi.*

16. Jim had 80% of his problems correct on last week's test. He missed four problems. How many did he have correct? *16*

17. Last year the basketball team won 60% of the games played. They lost 8 games. How many did they win? *12*

18. Harry says his dog weighs 30% as much as he does. The dog weighs 54 pounds. How much does Harry weigh? *180 lb.*

19. Jim purchased a car with a \$210-down payment. This was 30% of the price. What was the total price? *\$ 700*

20. Mr. Henderson plans to save 11.5% of his income. If his annual income is \$9500, how much should he save per year? *\$ 1092.50*

21. A real estate company sold a lot for \$6400. The commission on the sale was \$960. What per cent of the selling price was the commission? *15 %*

22. Mr. Olsen's income the year before last was \$10,400. Last year it was 8% less. How much was his income last year? *\$ 9568*

23. A wholesale house allows a discount of 5% if bills are paid within 10 days. How much is the discount on a bill of \$1575? *\$ 78.75*

24. Mary plans to save 20% of her income. If she saved \$85 per month according to plan, what was her monthly income? *\$ 425*

25. Last season the football team lost 40% of the games it played. They won 6 games. How many did they lose? *4*

26. Mr. Kenyon obtained a 15% discount on a fishing pole listed at \$40. How much did he pay? *\$ 34*

27. Patricia saved \$84 which was 12% of her monthly salary. Find her monthly salary. *\$ 700*

A. *Solve:*

1. $\frac{n}{3} = \frac{4}{6}$ 2
2. $\frac{3}{7} = \frac{n}{21}$ 9
3. $\frac{75}{25} = \frac{n}{6}$ 18
4. $\frac{18}{32} = \frac{27}{n}$ 48
5. $\frac{3}{29} = \frac{6}{n}$ 58
6. $\frac{15}{25} = \frac{27}{n}$ 45

B. Write each as a per cent to the nearest tenth of 1%.

1. $\frac{9}{10}$ 90.0%
2. $\frac{5}{16}$ 31.3%
3. $\frac{3}{4}$ 75.0%
4. $\frac{4}{5}$ 80.0%
5. $\frac{4}{7}$ 57.1%
6. $\frac{2}{3}$ 66.7%

C. Find the value for n that makes each a true statement.

1. 75% of 62.4 is n 46.8
2. 16.8 is n% of 8.4 200%
3. 0.5% of 420 is n 2.1
4. 35% of n is 140 400
5. n% of 8.4 is 2.1 25%
6. 15 is n% of 7.5 200%
7. 5.5% of 840 is n 46.2
8. 5% of n is 1.6 32
9. 125% of 9.2 is n 11.5
10. 35% of n is 7.07 20.2

D. *Solve:*

1. $6n = 72$ 12
2. $n + 3 = 12$ 9
3. $72 \div n = 24$ 3
4. $2n + 3n = 36 - 11$ 5
5. $3n - 4 = 11$ 5
6. $n \div 16 = 4$ 64

E. *Add:*

1. $^{+}16 + (^{-}4)$ $^{+}12$
2. $^{-}32 + (^{+}8)$ $^{-}24$
3. $^{-}18 + (^{-}3)$ $^{-}21$
4. $^{+}72 + (^{-}24)$ $^{+}48$
5. $^{+}12 + (^{-}18)$ $^{-}6$
6. $^{-}16 + (^{-}16)$ $^{-}32$

F. *Subtract:*

1. $^{+}15 - (^{+}7)$ $^{+}8$
2. $^{+}18 - (^{-}19)$ $^{+}37$
3. $^{-}16 - (^{+}3)$ $^{-}19$
4. $^{-}17 - (^{-}17)$ 0
5. $^{+}19 - (^{-}19)$ $^{+}38$
6. $^{-}25 - (^{-}24)$ $^{-}1$

If you need further practice, turn to the Practice Exercises on page 487. If you do not need further practice, you may work in the Experts' Corner on the next page.

Fractions equivalent to repeating decimals

Each fraction has an equivalent decimal. To find the decimal equivalent to $\frac{3}{4}$, for example, we use the definition $\frac{3}{4} = 3 \div 4$, and find that $\frac{3}{4} = 0.75$. This is a *terminating decimal;* that is, the remainder is 0. If a fraction has a denominator with prime factors of only 2 or 5, the fraction will be equivalent to a terminating decimal. Decimals equivalent to other fractions will not terminate. In finding the decimal equivalent to $\frac{1}{3}$, as we have seen, there will always be a remainder of 1, and the 3's in the quotient will repeat indefinitely.

Thus it is not correct, mathematically, to write $\frac{1}{3} = 0.3$ or 0.33. The convention, as you learned in Chapter 2, is to write:

$$\frac{1}{3} = 0.\overline{3}$$

The bar over the 3 shows that this is a repeating decimal and that 3 is the *period* that repeats. Other examples are:

$$\frac{1}{6} = 0.1\overline{6} \text{ (one digit in the period)}$$

$$\frac{1}{11} = 0.\overline{09} \text{ (two digits in the period)}$$

$$\frac{1}{7} = 0.\overline{142857} \text{ (six digits in the period)}$$

Each repeating decimal has an equivalent fraction. A simple computation can reveal the fractional equivalent of a given decimal. Let us develop a procedure to show that $0.1\overline{6} = \frac{1}{6}$

Let $n = 0.1\overline{6}$

Then $10n = 1.6\overline{6}$ — If both sides of an equation are multiplied by the same number, the solution remains unchanged.

Subtract:

$$\begin{array}{rcl} 10n &=& 1.6\overline{6} \\ -\ n &=& 0.1\overline{6} \\ \hline 9n &=& 1.5 \end{array}$$

If the same number is subtracted from both sides of an equation, the solution is unchanged.

$$n = \frac{1.5}{9.0} = \frac{15}{90} = \frac{1}{6}$$

Review the procedure again before you continue.

Note that it was necessary to write $n = \frac{1.5}{9.0}$, rather than $1.5 \div 9$. This latter division would result in another repeating decimal.

Explain how $\frac{1.5}{9.0}$ is reduced to $\frac{1}{6}$.

Let us examine a repeating decimal with a longer period. Explain each step in the solution.

EXAMPLE

Find the fraction equivalent to $0.\overline{63}$.

Let $n = 0.\overline{63}$ Then $100n = 63.\overline{63}$

$$\begin{aligned} 100n &= 63.\overline{63} \\ n &= 0.\overline{63} \\ \hline 99n &= 63.00 \\ n &= \frac{63}{99} = \frac{7}{11} \end{aligned}$$

Check:
$$\begin{array}{r} .63 \\ 11\overline{)7.00} \\ \underline{66} \\ 40 \\ \underline{33} \\ 7 \end{array}$$

(ex. 3)
$\frac{1}{7} = 0.\overline{142857}$
$\frac{2}{7} = 0.\overline{285714}$
$\frac{3}{7} = 0.\overline{428571}$
$\frac{4}{7} = 0.\overline{571428}$
$\frac{5}{7} = 0.\overline{714285}$
$\frac{6}{7} = 0.\overline{8571428}$

How do you know what power of 10 both sides of the equation should be multiplied by in the first step? Note that you wish to make the difference a whole number. If the period contained one digit, we multiplied the decimal by 10^1, which is simply 10. In the example above, the period contains two digits; thus, we multiplied by 10^2, which is 100. For a repeating decimal that has three digits in the period, we would multiply by 10^3, which is 1000. Do you see the pattern? If the period contains four digits, what will be the exponent of 10? 4 Thus, if the repeating decimal is $0.\overline{0142857}$, we would multiply by 10^7 (10,000,000).

1. First state which of these fractions have terminating decimal equivalents. Then find the decimal equivalent of each.

a. $\frac{2}{3}$ $0.\overline{6}$ $0.\overline{5}$ **c.** $\frac{5}{9}$ $0.\overline{428571}$ **e.** $\frac{3}{7}$ 0.625 **g.** $\frac{5}{8}$
b. $\frac{5}{6}$ $0.8\overline{3}$ 0.75 **d.** $\frac{3}{4}$ 0.32 **f.** $\frac{8}{25}$ $0.\overline{90}9$ **h.** $\frac{10}{11}$

2. Find the fraction equivalent to each of these repeating decimals.

a. $0.\overline{3}$ $\frac{1}{3}$ $\frac{8}{11}$ **d.** $0.\overline{72}$ $\frac{5}{11}$ **g.** $0.\overline{45}$
b. $0.\overline{5}$ $\frac{5}{9}$ $\frac{35}{37}$ **e.** $0.\overline{945}$ $\frac{13}{99}$ **h.** $0.\overline{13}$
c. $0.\overline{36}$ $\frac{4}{11}$ $\frac{2}{33}$ **f.** $0.0\overline{6}$ $\frac{115}{333}$ **i.** $0.\overline{345}$

3. Calculate the periods of $\frac{1}{7}$, $\frac{2}{7}$, $\frac{3}{7}$, $\frac{4}{7}$, $\frac{5}{7}$, and $\frac{6}{7}$. Can you sense a pattern? Examine the digits carefully. See above.
All decimals have the same digits and the same order.

The operations of arithmetic have certain properties that can be very useful to you once you clearly understand them. They are called the commutative, associative, and distributive properties.

The *commutative property of addition* tells us $a + b = b + a$ where a and b represent rational numbers.

For example: $2\frac{5}{7} + 6\frac{3}{8} = 6\frac{3}{8} + 2\frac{5}{7}$ illustrates that: the *order* in which two addends are combined does not affect the sum.

The *commutative property of multiplication* tells us $ab = ba$ where a and b represent rational numbers.

For example: $3\frac{7}{8} \times 2\frac{5}{11} = 2\frac{5}{11} \times 3\frac{7}{8}$ illustrates that: the *order* in which two factors are multiplied does not affect the product. You are familiar with this property. It permits you to multiply the factors in the most convenient order. Thus, if you have to find the product 8×137, you will multiply $\begin{array}{r} 137 \\ \times\ \ 8 \\ \hline \end{array}$ rather than $\begin{array}{r} 8 \\ \times 137 \\ \hline \end{array}$. In combination with other properties the commutative property has further important uses.

1. The operations of subtraction and division are called *inverse operations* because they undo the operations of addition and multiplication respectively. Does the commutative property hold for these operations? If we can find one example where it does not hold, we can say the property does not hold. Use rational numbers to find out if these are true statements: no
   a. $a - b = b - a$ where a and b are rational numbers.
   b. $a \div b = b \div a$ where a and b are rational numbers.

The *associative property for addition* tells us that $(a + b) + c = a + (b + c)$ where a, b, and c are rational numbers.

For example: $(5\frac{3}{4} + 2\frac{1}{4}) + 3\frac{7}{8} = 5\frac{3}{4} + (2\frac{1}{4} + 3\frac{7}{8})$ illustrates two possible ways to group these addends without changing the order. You will notice that one group is easier to add than the other.

Since only two addends may be combined in one step, it is important to know that we have a choice.

EXAMPLE

$$\text{Add: } 27\tfrac{2}{3} + 83\tfrac{1}{4} + 72\tfrac{1}{3}$$

Suppose we try $(27\frac{2}{3} + 83\frac{1}{4}) + 72\frac{1}{3} = 110\frac{11}{12} + 72\frac{1}{3} = 183\frac{1}{4}$

For most of us this computation would require use of pencil and paper. But if we use the commutative and associative property,

$$83\tfrac{1}{4} + (27\tfrac{2}{3} + 72\tfrac{1}{3}) = 83\tfrac{1}{4} + 100 = 183\tfrac{1}{4}$$

the computation does not require a pencil.

2. In each of the following select a grouping that permits you to perform the computation without a pencil. Copy the exercise using parentheses to indicate the grouping. Then write the answer.

a. $36 + 14 + 48$ 98
b. $25 + 28 + 22$ 75
c. $17 + 13 + 48$ 78
d. $29 + 12 + 38$ 79
e. $49 + 11 + 38$ 98
f. $1\frac{1}{6} + \frac{5}{6} + \frac{3}{4}$ $2\frac{3}{4}$
g. $2\frac{1}{2} + \frac{3}{4} + \frac{1}{4}$ $3\frac{1}{2}$
h. $4\frac{1}{5} + \frac{3}{8} + 1\frac{5}{8}$ $6\frac{1}{5}$
i. $\frac{5}{9} + 1\frac{4}{9} + 2\frac{7}{9}$ $4\frac{7}{9}$
j. $\frac{3}{8} + \frac{7}{8} + \frac{1}{8}$ $1\frac{3}{8}$

The commutative and associative properties enable you to regroup addends for more convenient computation. Thus:

$$\tfrac{5}{8} + (1\tfrac{7}{8} + \tfrac{3}{8}) = \tfrac{5}{8} + (\tfrac{3}{8} + 1\tfrac{7}{8}) = (\tfrac{5}{8} + \tfrac{3}{8}) + 1\tfrac{7}{8} = 1 + 1\tfrac{7}{8} = 2\tfrac{7}{8}$$

3. Rearrange as necessary. Then indicate a convenient grouping, using parentheses, and write the answer.

a. $58 + 37 + 42$ 137
b. $89 + 27 + 11$ 127
c. $56 + 45 + 55$ 156
d. $60 + 31 + 19$ 110
e. $72 + 39 + 28$ 139
f. $1\frac{5}{7} + \frac{3}{14} + \frac{2}{7}$ $2\frac{3}{14}$
g. $\frac{3}{8} + \frac{5}{6} + \frac{5}{8}$ $1\frac{5}{6}$
h. $1\frac{3}{4} + \frac{5}{6} + \frac{5}{4}$ $2\frac{3}{4}$
i. $3\frac{3}{8} + \frac{5}{6} + \frac{5}{8}$ $4\frac{5}{6}$
j. $\frac{7}{9} + \frac{2}{9} + 3\frac{1}{2}$ $4\frac{1}{2}$

After you study each property, try to describe and illustrate it using examples as well as variables.

The *associative property of multiplication* tells us that

$$(a \times b) \times c = a \times (b \times c)$$

where a, b, and c represent rational numbers.

In stating the principle less formally, we say that the manner in which the factors are taken in pairs does not affect the product. This property may be used to simplify an operation that can be rather difficult following the order indicated in the problem.

For example,

$$19 \cdot 25 \cdot 4 \text{ can be solved by writing } (19 \cdot 25) \cdot 4 =$$

$$475 \cdot 4 = 1900 \text{ or } 19 \cdot (25 \cdot 4) = 19 \cdot 100 = 1900.$$

The first computation might require the use of paper and pencil while the second would not. In the following exercises select a grouping that enables you to perform the computation mentally.

4. Copy the exercise indicating the grouping with parentheses, and write the answer.

a. $25 \times 4 \times 7$ *700*
b. $19 \times 20 \times 5$ *1900*
c. $13 \times 4 \times 250$ *13,000*
d. $25 \times 4 \times 15$ *1500*
e. $50 \times 2 \times 39$ *3900*
f. $\frac{2}{3} \times 6 \times 8$ *32*
g. $7 \times \frac{4}{5} \times 5$ *28*
h. $8 \times \frac{3}{4} \times 9$ *54*
i. $\frac{5}{8} \times 16 \times \frac{3}{5}$ *6*
j. $\frac{3}{7} \times 14 \times \frac{2}{3}$ *4*

You can use the commutative and associative properties to rearrange the factors for effective grouping.

$$24 \times (1\tfrac{2}{5} \times \tfrac{5}{6}) = 24 \times (\tfrac{5}{6} \times 1\tfrac{2}{5}) = (24 \times \tfrac{5}{6}) \times 1\tfrac{2}{5} = 20 \times 1\tfrac{2}{5} = 28$$

Sometimes an effective grouping is made possible by renaming a number as the product of factors: $4 \times 275 = 4 \times (25 \times 11) = (4 \times 25) \times 11 = 1100$. Also $75 \times 368 = (\frac{3}{4} \times 100) \times 368 = (100 \times \frac{3}{4}) \times 368$ or $100 \times (\frac{3}{4} \times 368) = 100 \times 276 = 27{,}600$. Describe what properties were used to obtain these products.

5. Rename numbers using factors as necessary to provide effective grouping. Copy, showing the grouping with parentheses, and write the answers.

a. 25×32 *800*
b. 75×12 *900*
c. 16×375 *6000*
d. $5 \times 13 \times 20$ *1300*
e. $25 \times 35 \times 4$ *3500*
f. $\frac{3}{8} \times 17 \times 16$ *102*
g. $\frac{5}{6} \times 11 \times 12$ *110*
h. $18 \times 15 \times \frac{1}{9}$ *30*
i. $3\frac{1}{2} \times \frac{4}{7} \times 14$ *28*
j. $\frac{2}{5} \times 25 \times 17$ *170*

6. Does the associative property hold for inverse operations? *no* Use rational numbers to determine if these statements are true. Then write a statement answering the question above.

a. $(a - b) - c = a - (b - c)$ **b.** $(a \div b) \div c = a \div (b \div c)$

Multiplication is distributive with respect to addition. Thus

$$a \times (b + c) = (a \times b) + (a \times c)$$

where a, b, and c are rational numbers.

EXAMPLE

$$8 \times (12 + 7) = (8 \times 12) + (8 \times 7) = 96 + 56 = 152$$

This property is the basis for the procedure used in multiplication. Notice that multiplying 7×2896 can also be described as $7 \times (6 + 90 + 800 + 2000) = 42 + 630 + 5600 + 14{,}000 = 20{,}272$

$$\begin{array}{r} 2896 \\ \times\ 7 \\ \hline 42 \\ 630 \\ 5600 \\ 14000 \\ \hline 20{,}272 \end{array}$$

1. Use the distributive property to find the products with as little written work as possible.

a. 9×53 477 NOTE: $9 \times 53 = (9 \times 50) + (9 \times 3)$

b. 8×75 600 **d.** 7×83 581 **f.** $\frac{3}{4} \times 4\frac{4}{5}$ $3\frac{3}{5}$

c. 4×6.4 25.6 **e.** $5 \times 7\frac{3}{5}$ 38 **g.** $9 \times 2\frac{2}{3}$ 24

Multiplication is also distributive with respect to subtraction. That is: $a(b - c) = (a \times b) - (a \times c)$.

This property is not so widely used as the distributive property with respect to addition. However, it provides a useful computation in some instances.

For example: $15 \times 99 = (15 \times 100) - (15 \times 1) = 1485$

2. Use this procedure to find these products with as little written work as possible.

a. 7×99 693 **c.** 8×98 784 **e.** 4×998 3992

b. 8×999 7992 **d.** $6 \times 4\frac{2}{3}$ 28 **f.** $9 \times 8\frac{8}{9}$ 80

Division is distributive with respect to addition.

$$(b + c + d) \div a = (b \div a) + (c \div a) + (d \div a)$$

where a, b, c, and d represent rational numbers, $a \neq 0$.

Illustrating with numbers: $2534 \div 7 =$
$(2100 \div 7) + (420 \div 7) + (14 \div 7) = 300 + 60 + 2 = 362$

Note that we separated 2534 into the largest multiple of 7 and 100, the largest multiple of 7 and 10, and the largest multiple of 7 that it contained.

Note also that this is the basis for the procedure of long division. In dividing $29574 \div 9$, we separate 29574 into $(27000 \div 9) + (1800 \div 9) + (720 \div 9) + (54 \div 9)$

$$\begin{array}{r} 3286 \\ 9\overline{)29574} \\ \underline{27000} \\ 2574 \\ \underline{1800} \\ 774 \\ \underline{720} \\ 54 \\ \underline{54} \end{array}$$

3. Use the distributive property to find the quotients with as little work as possible.

a. $3616 \div 8$ 452 **c.** $3645 \div 9$ 405 **e.** $4755 \div 15$ 317
b. $768 \div 6$ 128 **d.** $5208 \div 7$ 744 **f.** $3048 \div 12$ 254

4. State whether each of the following statements illustrates: A. The commutative property alone. B. The associative property alone. C. The distributive property. D. The associative and commutative properties.

A **a.** $(\frac{7}{8} + 1\frac{2}{3}) = (1\frac{2}{3} + \frac{7}{8})$
B **b.** $(\frac{3}{5} + \frac{1}{6}) + \frac{5}{6} = \frac{3}{5} + (\frac{1}{6} + \frac{5}{6})$
D **c.** $(\frac{5}{6} + \frac{3}{5}) + \frac{1}{6} = (\frac{5}{6} + \frac{1}{6}) + \frac{3}{5}$
C **d.** $5 \times (4 + 7) = (5 \times 4) + (5 \times 7)$
C **e.** $(16 + 24) \div 4 = (16 \div 4) + (24 \div 4)$
D **f.** $(5 + 7) + 8 = (5 + 8) + 7$
A **g.** $(\frac{3}{8} \times \frac{5}{9}) = (\frac{5}{9} \times \frac{3}{8})$
C **h.** $(7 \times 384) = (7 \times 300) + (7 \times 80) + (7 \times 4)$
C **i.** $(6 \times 8\frac{1}{3}) = (6 \times 8) + (6 \times \frac{1}{3})$
C **j.** $(2652 \div 6) = (2400 \div 6) + (240 \div 6) + (12 \div 6)$

5. In this chapter you have performed the operations of addition and subtraction with rational numbers. Later on we shall perform the operations of multiplication and division with rational numbers. Because the operations of addition and subtraction must be performed with negative as well as positive numbers, the operations take into account the *absolute value* of the numbers. What is the absolute value of each of these numbers?

a. $+9$ 9 **b.** -15 15 **c.** $+45$ 45 **d.** $+18$ 18 **e.** -27 27

6. List the numbers of Exercise 5 in order from the least to the greatest. On a separate line list the absolute values of each number in order from the least to the greatest. $^-27$, $^-15$, $^+9$, $^+18$, $^+45$

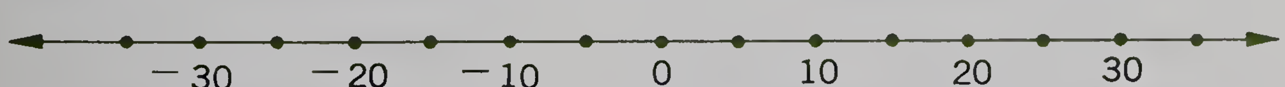
$|^+9|$, $|^-15|$, $|^+18|$, $|^-27|$, $|^+45|$

One of the most interesting areas of mathematics is concerned with systems of numeration and the properties of operation. These we ordinarily take for granted. When we explore them, we find them neither so simple nor so ordinary as we had supposed.

In this chapter you have used several different sets of numbers that were discovered independently. The natural numbers N $\{1,2,3, \ldots\}$ came into use as soon as men needed to count. Fractions were needed later when land, food, or other commodities had to be measured. Interestingly enough, zero was not conceived of as a number until much later when the Hindus invented it to make possible the positional system of numeration that we still use. Today zero is an element in the set of whole numbers W $\{0,1,2, \ldots\}$.

The fractional numbers of arithmetic, which include the whole numbers as a subset, were adequate to describe magnitudes; that is, how many or how much. For many purposes, however, we needed to indicate direction as well as amount. Temperatures fall below zero and we say it is $-12°$. The stock market fluctuates, and we say the change is -4 points. The German merchants of the 15th century used the symbols $+$ and $-$ to show an excess or shortage in the weight of a cask in relation to the nearest hundredweight. Mathematicians borrowed these symbols to designate direction, as well as magnitude, and thus developed the use of positive and negative numbers.

In this way, each number to the right of zero on a number line acquired a "mate" to the left of zero, equal in magnitude, but opposite in direction. Rather than an endpoint of a ray, zero became a point of reference on a number line that extends in either direction. The set of rational numbers can be associated with points on the number line. These points will be integers or on the line between the integers.



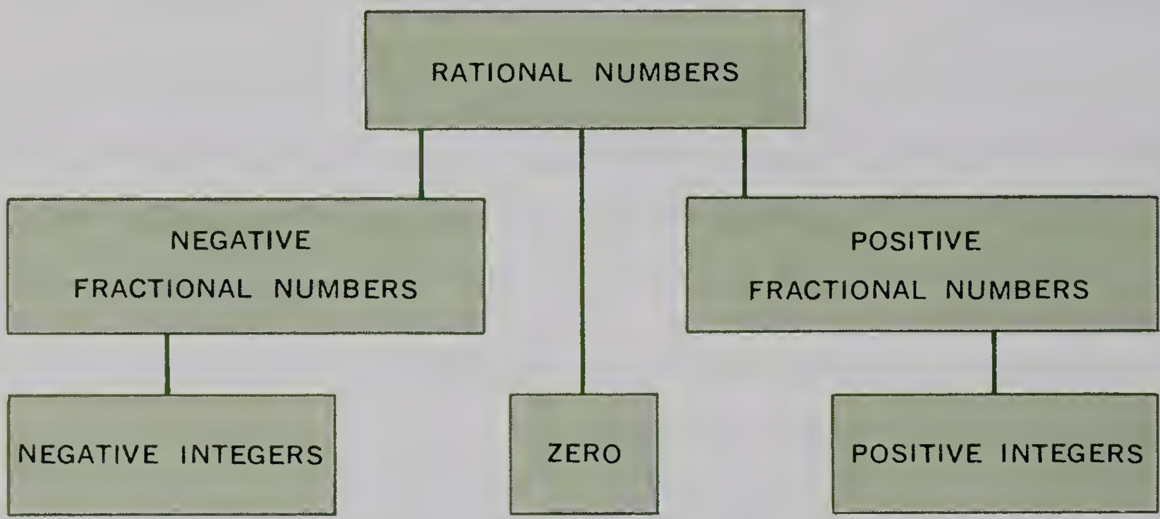
1. The set of numbers associated with points on this number line includes several subsets that you are already familiar with. On which side of zero are the points associated with the natural numbers? *right*
2. What do you call the set of numbers that includes zero and the natural numbers? *whole numbers*
3. The set of integers includes the whole numbers and the additive inverses of the natural numbers. The set of positive integers is identical with the natural numbers. The set of negative integers are their additive inverses. What whole number is neither a positive or negative integer? *zero*

4. The set of fractional numbers of arithmetic, together with their additive inverses, comprises the set of rational numbers. Any rational number can be expressed as $\frac{a}{b}$ where a is an integer and b is a natural number. What subset does a rational number belong to if a is positive and b is a factor of a?

5. Is $\frac{a}{b}$ a rational number if $a = 0$ and $b \neq 0$? *yes*

6. Starting with $\frac{5}{0} = n$, show why, if $\frac{a}{b}$ is a rational number, then $b \neq 0$. (If you cannot do it after a good try, refer back to page 40 and study the explanation again. Then try again.)

7. The relation of the set of rational numbers to its subsets may be diagramed as follows:



Which subset in this diagram is another name for the set of natural numbers? *positive integers*

8. Let us label the various sets that we have discussed as follows:

I is the set of integers
N is the set of natural numbers
R is the set of rational numbers
W is the set of whole numbers
Z is zero
I_n is the set of negative integers
F_a is the set of fractional numbers of arithmetic
F_n is the set of negative fractional numbers
F_p is the set of positive fractional numbers
I_c is the set of positive integers

What is another name for the I_c? *natural numbers*

9. Copy and complete these statements.

a. $F_p \cup Z \cup F_n = \square$ *R*
b. $N \cup Z \cup I_n = \square$ *I*
c. $N \cup Z = \square$ *W*
d. $Z \cup I_c = \square$ *W*
e. $F_p \cap I_c \cup Z = \square$ *W*
f. $I_n \cup I_c \cup Z = \square$ *I*

Part One

1. Find the value for each of the following.

   a. $^{+}7 + (^{-}8)$ $^{-}1$
   b. $^{+}15 + (^{+}\frac{1}{3})$ $^{+}15\frac{1}{3}$
   c. $^{+}\frac{5}{6} - (^{+}\frac{5}{12})$ $\frac{^{+}5}{12}$
   d. $^{+}2\frac{1}{2} + (^{-}\frac{5}{8})$ $^{+}1\frac{7}{8}$
   e. $^{+}15 + (^{-}11\frac{1}{2})$ $^{+}3\frac{1}{2}$
   f. $^{+}\frac{3}{4} - (^{+}\frac{2}{3})$ $\frac{^{+}1}{12}$
   g. $^{+}3\frac{3}{4} - (^{-}\frac{5}{16})$ $^{+}4\frac{1}{16}$
   h. $^{+}7\frac{1}{2} - (^{-}3\frac{3}{4})$ $^{+}11\frac{1}{4}$

2. Rearrange, if necessary, to provide effective grouping for computation without pencil and paper. Copy, showing the grouping with parentheses, and write the product.

   a. $25 \times 17 \times 4$ 1700
   b. $5 \times 19 \times 20$ 1900
   c. $25 \times 4 \times 15$ 1500
   d. $50 \times 29 \times 2$ 2900
   e. $4 \times 25 \times 13$ 1300
   f. $\frac{2}{3} \times 8 \times 9$ 48
   g. $\frac{3}{4} \times 7 \times 8$ 42
   h. $\frac{5}{8} \times \frac{3}{5} \times 16$ 6
   i. $\frac{1}{2} \times 15 \times \frac{4}{5}$ 6
   j. $\frac{2}{9} \times 18 \times \frac{3}{4}$ 3

3. Which property of division is illustrated by this statement? distributive
   $3756 \div 6 = (3600 \div 6) + (120 \div 6) + (30 \div 6) + (6 \div 6)$

4. Write a similar statement to show how you can use the same property to find this quotient: $5068 \div 7$. Write the quotient.
   $(4900 \div 7) + (140 \div 7) + (28 \div 7) = 724$

5. Rewrite one or both of the numbers in each of the following as the product of its factors, and regroup as necessary to perform the computation without pencil and paper. Copy, showing the grouping with parentheses, and write the product.

   a. 12×75 900
   b. 32×25 800
   c. 75×40 3000
   d. 150×12 1800
   e. 25×16 400
   f. 8×375 3000

6. Write as a per cent rounded to the nearest tenth of 1%.

   a. $\frac{3}{16}$ 18.8%
   b. $\frac{2}{3}$ 66.7%
   c. $\frac{5}{9}$ 55.6%
   d. $\frac{5}{8}$ 62.5%
   e. $\frac{3}{7}$ 42.9%
   f. $\frac{4}{5}$ 80%
   g. $\frac{7}{8}$ 87.5%
   h. $\frac{3}{4}$ 75%
   i. $\frac{3}{10}$ 30%
   j. $\frac{5}{6}$ 83.3%

7. Given: Set $A = \{^{-}4, ^{-}3\frac{1}{2}, ^{-}3, ^{-}2\frac{1}{2}, ^{-}2, ^{-}1\frac{1}{2}, ^{-}1, 0, ^{+}\frac{1}{2}, ^{+}1\}$.
   List the members of each of these subsets.

   a. The negative integers $^{-}4, ^{-}3, ^{-}2, ^{-}1$
   b. The whole numbers $0, 1$
   c. The fractional numbers of arithmetic $0, \frac{^{+}1}{2}, ^{+}1$
   d. The negative fractional numbers $^{-}4, ^{-}3\frac{1}{2}, ^{-}3, ^{-}2\frac{1}{2}, ^{-}2, ^{-}1\frac{1}{2}, ^{-}1$
   e. The positive integers $^{+}1$
   f. The whole number that is not a positive integer 0

8. What number in set A above has the greatest absolute value? the least absolute value? $^{-}4, 0$

Part Two

1. Write the multiplicative inverse for each of the following.

a. 5 $\frac{1}{5}$ **b.** $\frac{1}{8}$ 8 **c.** $\frac{5}{9}$ $\frac{9}{5}$ **d.** 45 $\frac{1}{45}$ **e.** $2\frac{3}{4}$ $\frac{4}{11}$

2. Write the additive inverse for each of the following.

a. 3 $^-3$ **b.** $^-7$ $^+7$ **c.** $\frac{2x}{5}$ $\frac{-2x}{5}$ **d.** ^-7x ^+7x **e.** $\frac{^-5}{3}$ $\frac{+5}{3}$

3. Solve the following proportions.

a. $\frac{3}{5} = \frac{n}{20}$ 12 **d.** $\frac{n}{4} = \frac{1}{2}$ 2 **g.** $\frac{8}{n} = \frac{2}{3}$ 12

b. $\frac{4}{7} = \frac{16}{n}$ 28 **e.** $\frac{6}{8} = \frac{n}{24}$ 18 **h.** $\frac{n}{5} = \frac{12}{15}$ 4

c. $\frac{5}{n} = \frac{15}{30}$ 10 **f.** $\frac{7}{8} = \frac{21}{n}$ 24 **i.** $\frac{7}{9} = \frac{n}{27}$ 21

4. Write a proportion to find the value for n in each statement, and then solve it.

a. 152 is $n\%$ of 95 160
b. 175 is $n\%$ of 50 350
c. 37.5% of 96 is n 36
d. 0.2% of 600 is n 1.2
e. 135 is 90% of n 150
f. 65 is $n\%$ of 50 130
g. 180% of n is 135 75
h. 4.2 is $n\%$ of 5.6 75
i. 6.5% of 88 is n 5.72
j. $n\%$ of 6.4 is 4.8 75

5. Write an equivalent equation (if necessary) that has the variable alone on one side of the equation. Solve and check your answer by substituting in the original equation.

a. $n + 7 = 23$ 16
b. $17x = 85$ 5
c. $28 \div x = 7$ 4
d. $54 \div y = 9$ 6
e. $54 \div y = 18$ 3
f. $25 + n = 33$ 8
g. $19x = 114$ 6
h. $n \div 18 = 4$ 72
i. $n = 19 + 37$ 56
j. $13y = 91$ 7
k. $x - 15 = 10$ 25
l. $y \div 13 = 3$ 39
m. $72 \div n = 12$ 6
n. $x = 98 \div 14$ 7
o. $36n = 144$ 4
p. $48 - n = 25$ 23
q. $y + 32 = 48$ 16
r. $42 - 17 = n$ 25
s. $7n = 63$ 9
t. $y \div 8 = 9$ 72

6. Solve for x, using the additive inverse as required.

a. $\frac{5x}{4} = x + 1.5$ 6
b. $7x + 3 = 5x + 13$ 5
c. $\frac{x}{6} + 1 = 2\frac{2}{3}$ 10
d. $2x - 30 = x + 27$ 57
e. $\frac{x}{2} - 3 = 6 - \frac{x}{2}$ 9
f. $2x - 5 = x - 2$ 3
g. $5x + 6 = 19$ $2\frac{3}{5}$
h. $3x + 36 = 2x + 60$ 24

7. Copy the letter of each of the following exercises and after the letter write the answer to the exercise.

a. $9 \cdot (3 \cdot 8) = (9 \cdot 3) \cdot 8$ illustrates the _?_ property of multiplication. *associative*

b. $(64 \div 8) \neq (8 \div 64)$ shows that the _?_ property does not hold for division. *commutative*

c. $(35 + 15) \div 5 = (35 \div 5) + (15 \div 5)$ illustrates the _?_ property for division. *distributive*

d. $(7 + 9) + 12 = 7 + (9 + 12)$ illustrates the _?_ property for addition. *associative*

e. $(9 - 7) - 3 \neq 9 - (7 - 3)$ shows that the _?_ property does not hold for subtraction. *associative*

f. $(9 \cdot 498) = (9 \cdot 500) - (9 \cdot 2)$ illustrates the _?_ property for multiplication. *distributive*

g. $(17 - 9) \neq (9 - 17)$ shows that the _?_ property does not hold for subtraction. *commutative*

h. $6 \cdot (7 \cdot 9) = (6 \cdot 7) \cdot 9$ illustrates the _?_ property for multiplication. *associative*

i. With 7 and 9 as factors, write a statement illustrating the commutative property of multiplication. $7 \cdot 9 = 9 \cdot 7$

j. With 7 and 12 as addends write a statement illustrating the commutative property of addition. $7 + 12 = 12 + 7$

8. Compare the absolute values of the following pairs of numbers in the order given, using the symbol < or > to make a true statement.

a. $|{}^{+}8| \square |{}^{+}6|$ >

b. $|{}^{+}17| \square |{}^{-}18|$ <

c. $|{}^{-}29| \square |{}^{+}21|$ >

d. $|{}^{+}25| \square |{}^{-}18|$ >

e. $|{}^{-}27| \square |{}^{-}19|$ >

f. $|{}^{-}28| \square |{}^{+}27|$ >

g. $|{}^{-}3\frac{3}{4}| \square |3\frac{3}{8}|$ >

h. $|{}^{-}5\frac{1}{3}| \square |5\frac{1}{6}|$ >

i. $|{}^{-}2\frac{3}{8}| \square |2\frac{5}{16}|$ >

j. $|{}^{-}32\frac{5}{9}| \square |32\frac{5}{11}|$ >

9. Write each of these statements using a fraction in simplest form.

a. A quarter is 25% of a dollar $\frac{1}{4}$

b. The football team won $87\frac{1}{2}\%$ of its games $\frac{7}{8}$

c. 65% of the class is present today $\frac{13}{20}$

d. A pint is $12\frac{1}{2}\%$ of a gallon $\frac{1}{8}$

e. Jane solved 75% of the problems correctly $\frac{3}{4}$

f. A gallon is 400% of a quart 4

g. The population of this city has increased 175% in the past 30 years $1\frac{3}{4}$

h. The average markup on a line of plastic items is 80% of the cost $\frac{4}{5}$

10. Write these numerals as decimals.

a. 32×10^{-2} 0.32

b. 837×10^{-1} 83.7

c. 28×10^{-1} 2.8

d. 9164×10^{-2} 91.64

e. 318×10^{-2} 3.18

f. 183×10^{-3} 0.183

Part Three

Write an equation for each problem before you solve the problem.

STEPS FOR SOLVING APPLIED PROBLEMS

1. Understand the problem.
2. Note what the problem asks for.
3. Look for hidden questions.
4. Estimate a reasonable answer.
5. Set up and solve the conditional sentence(s).
6. Check your answer.

1. Three times a certain number increased by 16 is 25. *3*
2. Three less than $2\frac{1}{2}$ times a certain number equals twice the same number. *6*
3. If a plane flies 1375 miles in 2.5 hours, how long will it take it to fly 3300 miles at the same rate? *6 hr.*
4. A car travels 88 miles on 5.5 gallons of gasoline. How far will it travel at the same rate of consumption on 20 gallons? *320 mi.*
5. The perimeter of a triangle is 45 inches. The measure of the longest side is 10 inches greater than that of the shortest side. The measure of the third side is 5 inches less than that of the longest side. What are the dimensions? *10", 20", 15"*
6. If an 8-foot fence post casts a 6-foot shadow, what is the height of a building that is casting a 96-foot shadow? *128 ft.*
7. The measure of the length of a field lacks 95 rods of being 3 times that of the width. The number of rods of fencing required to enclose the field is 290. What are the dimensions? *w = 60 rd. l = 85 rd.*
8. What is the per cent of reduction from the regular price when a bicycle marked \$60 is sold for \$51? *15 %*
9. Mary had 75% of her problems correct on last week's test. She missed 5 problems. How many did she have correct? *15*
10. Jim purchased a used motorcycle for \$200, paying 30% of the price in cash. The rest he is to pay in five equal monthly payments. How much is each payment? *\$ 28*
11. The president of the mathematics club announced that of the 64 members 58 had paid their dues. What per cent of the membership had not paid their dues? *9.375 %*

Part One

A. *Add:*

1. 49	**2.** 94	**3.** 539	**4.** 4719	**5.** 1500
57	78	925	74	3048
35	453	56	1830	290
98 *239*	70 *695*	89 *1609*	7783 *14406*	77 *4915*

6. $\frac{5}{8}$	**7.** $\frac{9}{16}$	**8.** $3\frac{2}{3}$	**9.** $11\frac{5}{6}$	**10.** $13\frac{15}{16}$
$\frac{5}{6}$	$3\frac{5}{8}$	$6\frac{1}{2}$	$17\frac{5}{8}$	$21\frac{3}{4}$
$\frac{2}{3}$	$1\frac{1}{2}$	$9\frac{5}{6}$	$7\frac{3}{4}$	$22\frac{5}{8}$
$2\frac{1}{8}$	*$5\frac{11}{16}$*	*20*	*$37\frac{5}{24}$*	*$58\frac{5}{16}$*

B. *Subtract:*

1. 24,305	**2.** 15,631	**3.** 17,500	**4.** 19,587	**5.** 8,003
15,283 *9022*	9,700 *5931*	11,801 *5699*	12,337 *7250*	2,876 *5127*

6. $6\frac{13}{16}$	**7.** $23\frac{9}{10}$	**8.** $4\frac{5}{8}$	**9.** $19\frac{5}{6}$	**10.** $27\frac{4}{5}$
$3\frac{5}{8}$	$14\frac{1}{5}$	$2\frac{3}{4}$	$17\frac{2}{3}$	$19\frac{3}{10}$
$3\frac{3}{16}$	*$9\frac{7}{10}$*	*$1\frac{7}{8}$*	*$2\frac{1}{6}$*	*$8\frac{1}{2}$*

C. *Multiply:*

1. $3.9 \times .07$ *0.273*
2. 5.56×3.8 *21.128*
3. $16 \times 1\frac{3}{8}$ *22*
4. $24 \times 6\frac{2}{3}$ *160*
5. 0.17×1.05 *0.1785*
6. 6.9×8.3 *57.27*
7. $\frac{3}{8} \times \frac{4}{15}$ *$\frac{1}{10}$*
8. $\frac{5}{9} \times \frac{18}{25}$ *$\frac{2}{5}$*
9. 7.16×3.05 *21.838*
10. 0.15×9.5 *1.425*
11. $4\frac{1}{2} \times 3\frac{1}{3}$ *15*
12. $16\frac{2}{3} \times 8\frac{4}{5}$ *$146\frac{2}{3}$*

D. *Divide:*

1. $62.5 \div .025$ *2500*
2. $33.3 \div 3.7$ *9.0*
3. $52.8 \div 0.48$ *110.0*
4. $3.43 \div 0.49$ *7.0*
5. $16.25 \div 3.2$ *5.078125*
6. $56.25 \div 1.875$ *30.0*
7. $\frac{9}{10} \div \frac{3}{4}$ *$1\frac{1}{5}$*
8. $\frac{15}{16} \div \frac{7}{8}$ *$1\frac{1}{14}$*
9. $3\frac{1}{5} \div 3\frac{5}{9}$ *$\frac{9}{10}$*
10. $1\frac{3}{7} \div 6\frac{2}{3}$ *$\frac{3}{14}$*
11. $16\frac{1}{4} \div 15\frac{7}{8}$ *$1\frac{3}{127}$*
12. $2\frac{1}{7} \div 2\frac{1}{2}$ *$\frac{6}{7}$*

E. *Perform the indicated operation:*

1. $^-6 + (^-7)$ *$^-13$*
2. $^+15 + (^-16)$ *$^-1$*
3. $^+8 - (^+7)$ *$^+1$*
4. $^+16 - (^-19)$ *$^+35$*
5. $^+18 + (^-9)$ *$^+9$*
6. $^-19 + (^+15)$ *$^-4$*
7. $^-19 - (^-19)$ *0*
8. $^-21 - (^+19)$ *$^-40$*
9. $^-27 + (^+28)$ *$^+1$*
10. $^+50 + (^-49)$ *$^+1$*
11. $^+35 - (^+30)$ *$^+5$*
12. $^-17 - (^-14)$ *$^-3$*

Part Two

Write the ratio of the first number to the second in each of the following pairs in three ways: as a fraction in lowest terms, as a decimal, and as per cent.

1. 5 to 8 $\frac{5}{8}$; 0.025; 62.5%
2. 16 to 5 $3\frac{1}{5}$; 3.2; 320%
3. 19 to 4 $4\frac{3}{4}$; 4.75; 475%
4. 4 to 25 $\frac{4}{25}$; 0.16; 16%
5. 6 to 24 $\frac{1}{4}$; 0.25; 25%
6. 15 to 30 $\frac{1}{2}$; 0.5; 50%
7. 120 to 48 $2\frac{1}{2}$; 2.5; 250%
8. 9 to 20 $\frac{9}{20}$; 0.45; 45%
9. 3 to 15 $\frac{1}{5}$; 0.2; 20%

Part Three

Given: set $M = \{^-3, ^-2, ^-1, 0, ^+1, ^+2\}$ $O = \{^-3, ^-2, ^-1, 0\ ^+1, ^+2, ^+3, ^+4\}$
set $N = \{^+1, ^+2, ^+3, ^+4\}$ $P = \{^+1, ^+2\}$

Write the elements in set O and set P if $O = M \cup N$ and $P = M \cap N$

Part Four

Solve for the indicated variable.

A.

1. $\frac{3}{8} = \frac{n}{16}$ 6
2. $\frac{6}{n} = \frac{3}{29}$ 58
3. $\frac{5}{n} = \frac{25}{35}$ 7
4. $\frac{12}{18} = \frac{n}{45}$ 30
5. $\frac{n}{6} = \frac{7}{3}$ 14
6. $\frac{14}{n} = \frac{21}{27}$ 18

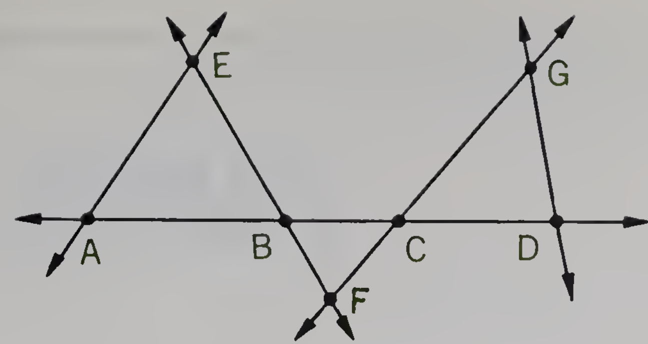
B.

1. 120% of n is 42 35
2. 15 is n% of 25 60
3. n% of 380 is 19 5
4. 211% of 48 is n 101.28
5. 0.7% of 480 is n 3.36
6. 68 is 50% of n 136
7. 13.3% of 80 is n 10.64
8. 49 is n% of 21 233.3
9. 312.5% of 48 is n 150
10. 42.8 is 10% of n 428
11. 96 is n% of 24 400
12. 325% of n is 13 4

C.

1. $x + 17 = 25$ 8
2. $5n - 9 = 41$ 10
3. $4y = 96$ 24
4. $96 \div x = 16$ 6
5. $13y = 117$ 9
6. $3x + 7 = 40$ 11
7. $\frac{3n}{4} = 3.75$ 5
8. $n \div 15 = 7$ 105
9. $\frac{5n}{6} = 45$ 54
10. $45 \div x = 9$ 5

Part Five



1. Refer to the drawing above and answer the following:

   a. $AE \cap AC = \square$ *A*   d. $\overrightarrow{CG} \cup \overrightarrow{CD} = \square$ *∠GCD*   g. $C = \square \cap \square$ ($\overrightarrow{CA}$, $\overrightarrow{CF}$)

   b. $\overrightarrow{EF} \cap \overrightarrow{GF} = \square$ *F*   e. $\angle CDG = \square \cup \square$ ($\overrightarrow{CD}$, $\overrightarrow{DG}$)   h. $\overrightarrow{CA} \cap \overrightarrow{CF} = \square$ *C*

   c. $\overrightarrow{AB} \cup \overrightarrow{BC} = \square$ $\overrightarrow{AC}$ f. $\overline{AD} \cap \overline{FE} = \square$ *B*   i. $\overleftrightarrow{DG} \cap \overleftrightarrow{FG} = \square$ *G*

2. Draw a figure similar to the letter *S* that has point symmetry and locate the center of symmetry. Identify two corresponding points and label them *A* and *B*.
   Drawings may vary.

Part Six

1. Mrs. Baker bought six bars of soap at 3 bars for 34¢. How much change should she receive from a \$1 bill? *32¢*

2. A freight truck is transporting machinery to a city 1950 miles away. The driver expects to make the trip in five days. How many miles should he average per day? *390 mi.*

3. Jim worked as a gas station attendant during spring vacation. One week he worked the following hours: Monday, $8\frac{1}{2}$ hours; Tuesday, 7 hours; Wednesday, $9\frac{1}{4}$ hours; Thursday, $8\frac{3}{4}$ hours; Friday, $9\frac{3}{4}$ hours, and Saturday, 4 hours. At \$1.20 per hour, how much did he earn during the week? *\$ 56.70*

4. Last summer Ellen earned \$360 doing clerical work. She spent \$216. What per cent of her earnings were left? *40%*

5. Elliot purchased a typewriter, paying 50% of the price in cash. After 30 days he paid 50% of the balance. What per cent of the purchase price was left to pay at the end of 60 days, when it was due? *25%*

6. A plane cruising at 320 miles per hour used 60 gallons of gasoline per hour. At this rate, how many gallons of gasoline will be used on a 2080-mile flight? *390 gal.*

7. When one of the 9-foot posts around the tennis court casts a 15-foot shadow, one of the trees in the school yard casts a shadow 60 feet long. How tall is the tree? *36 ft.*

FINANCIAL PLANNING

WORDS TO WATCH FOR

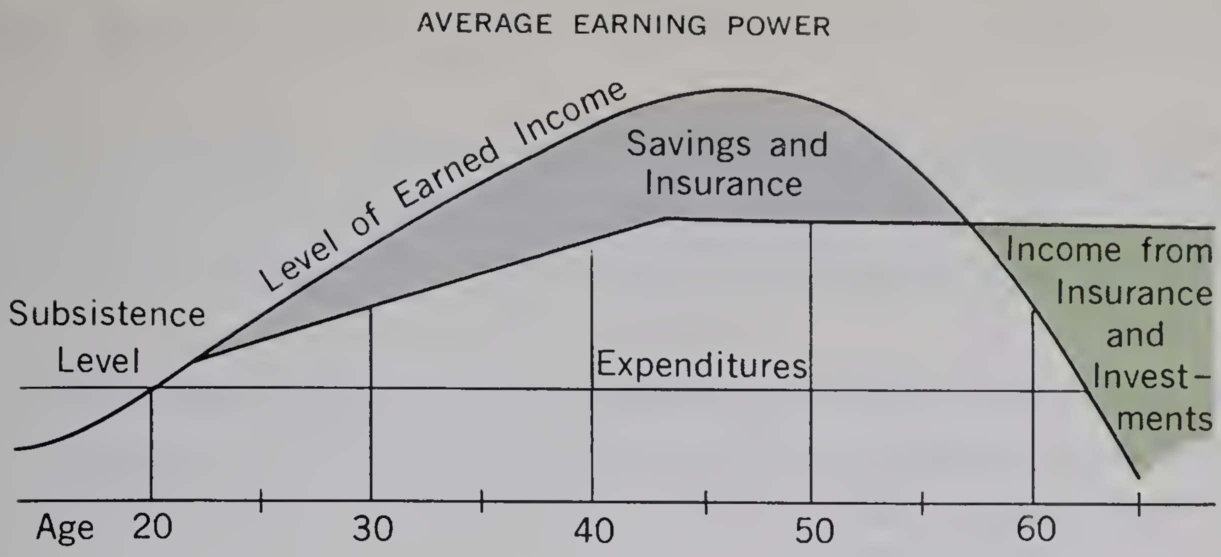
accrued interest
beneficiary
bond
budget
capital stock
date of maturity
endowment
face value
market value
rate of return
shareholder
stock exchange
stock quotation
term
yield

There are many purposes for which you might want to set aside a part of your earnings. Savings provide for both foreseeable and emergency needs. You may plan to use your savings later in purchasing something such as a camera or a bicycle. You may be saving to meet part or all of your expenses when you go to college.

A family may have a savings plan for various reasons too, such as a special holiday at Christmas time, a vacation trip next summer, or a new car or appliance. Often, in making a purchase, a better deal can be made if a substantially large down payment is made. A family must also consider the unexpected events such as accidents, illness, or loss of income through unemployment. Since it is also true that the earning power of the head of the family will increase to a point, and then level off or decline, some form of savings plan for retirement has to be considered. In a very general way the rise and fall of income can be illustrated in a graph. Examine the graph on the following page.

TOPICS FOR DISCUSSION

1. The graph shows the period of maximum earnings at the age of 47. Will this age be the same for all vocations? Explain.



2. Find the average age at which an individual:
   a. Becomes self-supporting.
   b. Saves the greatest proportion of his income.
   c. Ceases to save.
   d. Ceases to balance expenditures with earned income.

3. Consider persons engaged in the following vocations: auto mechanics, salesmanship, secretarial work, engineering, business executive, medicine, law.
   a. Which is likely to rise above the level of dependence earliest?
   b. Which is likely to reach his level of maximum earnings earliest?
   c. Which is likely to drop to the level of dependence earliest?
   d. What is the effect of a period of training, such as is required for professions, on the age at which one rises to the level of dependence?

4. After a worker retires in his sixties, his income is provided by savings and insurance bought in earlier years. What common forms of savings plans do you know about?

5. Investigate the Social Security program to determine its purpose and benefits. The enacted Medicare legislation is designed to assist the elderly meet the rising costs of medical treatment. What are the major provisions of this legislation that relate to savings plans that an individual has or may undertake?

6. Find out what "fringe benefits" are provided by employers in your community. Determine whether these would make certain places of employment more desirable than others. Do you feel that these benefits affect the workers' morale and efficiency? How would membership in a pension plan affect one's willingness to move to a job with a different employer?

7. What are some of the factors that determine the amount of pension that a retiree gets? If possible, find illustrations from actual cases.

Teen-agers constitute one of our country's important social and economic groups. What they earn and spend, and the rate of their employment and unemployment are receiving a great deal of attention in the news media.

1. A recent report from the Census Bureau stated that there were 23 million teen-agers in this country. By 1970 the number was expected to increase by 58%. What is the expected 1970 teen-age population? 36.34×10^6

2. Teen-age girls buy 20% of all women's clothing sold. Total sales of women's clothing amount to $15 billion annually. What is the amount of purchases by teen-age girls? $ 3 billion

3. Half of the boys and a third of the girls in high school have part time jobs. If we assume equal numbers of each in the high school population, what fraction of high school pupils do not have part-time jobs? $\frac{5}{12}$

4. These 23 million teen-agers spent $15 billion dollars of their own money annually on such items as sports equipment, records, phonographs, theater tickets, and sports clothing. To the nearest dollar, this was how much per teen-ager? $ 652

   $ 358.60

5. Teen-agers earned about 55% of what they spent. The rest was from allowances and other sources. To the nearest dollar, what was the amount earned by teen-agers at the time of this report?

6. Many teen-agers purchase and pay for the upkeep of a car. Operating a car costs about $60 a month. How many hours must one work per month at $1.25 an hour to pay for operating a car? By choosing to save $60 a month toward college instead of having a car, how much could be saved the last two years of high school?

   48 hr. ; $ 1440

7. About half of the high school graduates each year plan to go on to college. Recently 2 million students graduated from high school; 980,000 of that number entered college. What per cent entered?

   49 %

8. Approximately 100,000 found the colleges of their choice were filled. What per cent were disappointed in this way? 10.2%

9. Out of 2 million high-school graduates, 49% attend college, 8.1% attend special schools (technical, business, and so on), and the rest, for the most part, seek employment. How many seek employment?

   42.9 % or 858,000

10. If 14% of those seeking work remain unemployed, how many find themselves in this situation? 120,120

Jack Wheeler showed his father some advertisements from large corporations seeking people trained as business managers, personnel workers, journalists, and as experts in electronics, computer technology, and aerospace engineering. Jack asked his father, "What do I have to be able to do to get into this kind of work?"

Mr. Wheeler looked over the clippings and replied, "First of all most of these jobs require ability to handle mathematics and college training. College will broaden your interests and expand your understanding of what is going on in the world around us. You will be better prepared to take part in determining what should be going on. You will get to know many fine people with interests similar to yours. It is also worth taking into account that many interesting and profitable fields of employment are open only to college graduates. A college graduate earns, on the average, 55% more per year than a high school graduate. It is also true that many positions open to high school graduates are not open to dropouts."

1. Mr. Wheeler then showed Jack some figures on annual earnings of people with different amounts of schooling.

A college graduate	$8000
A high school graduate	5300
An eighth grade dropout	4000

How much more does the college graduate expect to earn per year than a high school graduate? than someone who did not go to high school? $ 2700 ; $ 4000

2. A person with two years of college preparation may expect to earn an average of $7000 a year. How much will he earn if he works regularly from age 20 to age 65 at this rate? $ 315,000

3. If a college graduate worked regularly from age 22 until retirement at 65, with the average annual salary listed above, what would his lifetime earnings amount to? $ 344,000

4. If a high school graduate worked regularly from age 18 to age 65, what would his lifetime earnings amount to? $ 249,100

5. Assume that an eighth grade dropout works regularly from age 16 to age 65 at the annual earnings listed above. How much less will his lifetime earnings be than those of a high school graduate?
As you consider the information in this topic, you will see that the decision to become a dropout cannot be taken lightly. The long-term effect is easily seen through lifetime earnings. $ 53,000

6. A Census Bureau report reveals the following average lifetime earnings for persons with various amounts of schooling.

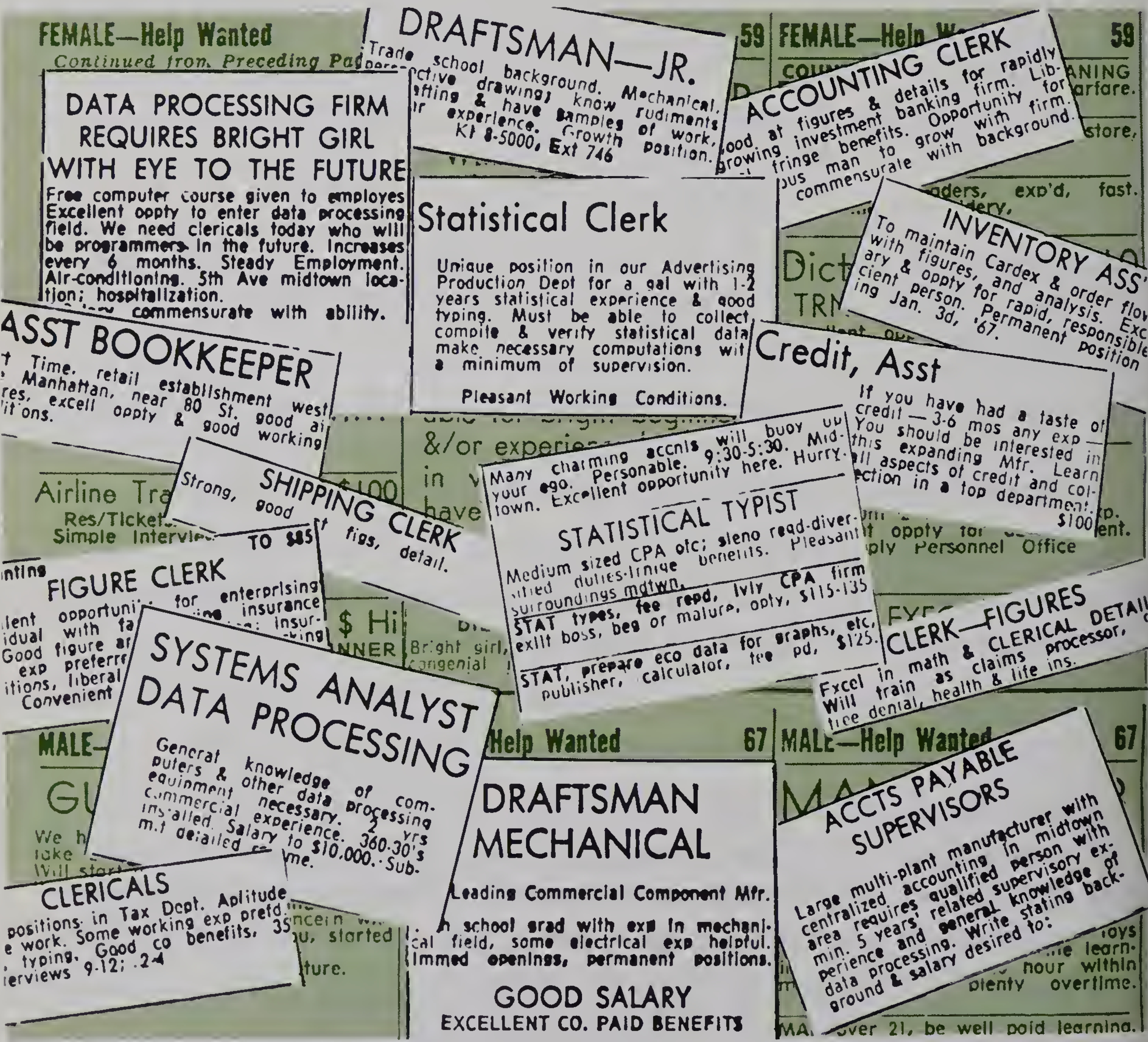
Less than eighth grade	$143,000	High school graduate	$247,000
Eighth grade	184,000	College graduate	385,000

College graduates include groups of professional workers who do not have compulsory retirement at age 65. Thus, their lifetime earnings extend over a longer period. The lifetime earnings of persons with five years of college preparation are reported as averaging $455,000. This group includes workers in many branches of engineering. The average income of the professional groups varies from $11,000 a year to $14,000 a year. Name some professions that require study beyond 4 years of college preparation.

(varies among states) teaching, medicine, lawyer, dentist, etc.

COMMITTEE ASSIGNMENTS

What kinds of jobs are open to high school graduates? college graduates? Select some interesting occupations and find what educational background is required for each.



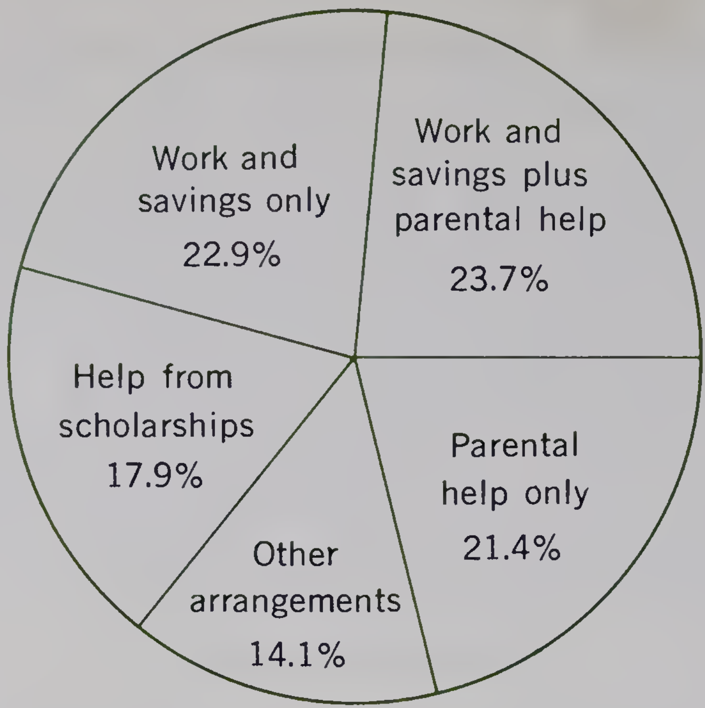
Jack's friends are thinking about the future. Perry wants to be a mechanic and have his own garage. May wants to be a teacher; John, a dentist; Bert, an architect; and Janet, a lawyer. Jack was thinking about how they would prepare for these different kinds of work and wondered how many of them would be planning for college.

"I know that different careers demand college training," said Jack. "I wonder how many people go to college?" "It's hard to tell," said his father. "There are so many kinds of colleges and many boys and girls don't go to college immediately after graduation. They may enroll in night school courses or begin college after a few years. However, this information might answer your question."

Enrollment in school

Elementary school	29,400,000
High school	15,700,000
College	4,200,000

1. What is the total enrollment in school at all levels? 49,300,000
2. How many more students are there in high school than in college? 11,500,000
3. How many more pupils are in elementary school than high school? 13,700,000
4. Jack found a clipping from the Office of Education posted in the guidance office at school which listed the average yearly cost of attending a private college at $2800, and the average yearly cost of attending a state college or university as $1600. How much more would it cost per year to attend a private college? $ 1200
5. Janet Martin decided to take four years of pre-law at a state university, and then attend the university's law school for three years. What will Janet's college education cost? $ 11,200
6. Perry, who wants to have his own garage, found that he will need some technical courses, so that he will be able to keep up to date and service new cars. He will also need some business and accounting courses. A state college has a two-year course that is ideal for Perry. How much will it cost? $3200
7. May could get her four years of teacher training at a nearby state college or at a private college. How much more would it cost her to get her training at a private college? How much can John expect to pay for his three years of pre-dental schooling at a private college and his four-year dental course at a state school? $ 4800; $ 14,800



8. Most students in college pay part or all of their expenses with their own earnings or with scholarships. The results of a survey are summarized in the circle graph. If 4,200,000 students were in college at the time the survey was made, how many were paying all their own expenses? 961,800 were receiving help from scholarship grants? 751,800

9. Jack wanted to study engineering. He found that the State Agricultural and Mechanical College offered a fine engineering program. His father thought that the family could provide \$500 a year for Jack's college education. How much would Jack have to earn each year for himself? \$1100

10. Jack was interested in the track team, and he knew that he couldn't have a part-time job during the three-month track season. How much would he have to earn each of the other six months if he earned \$350 during the summer? \$125

11. Jack's father explained that Jack would have to take a part-time job while he went to college. If these jobs paid about \$1.25 per hour, how many hours would he need to work each month to supply the extra money he needed? 100 hr.

SPECIAL PROJECTS

a. Fill out a table for colleges in your area, listing such things as tuition, cost, distance from home, type of college, and so on.
b. Consult your teachers or counsellors to learn what college scholarship grants are available and what the requirements are.
c. Develop local statistics on costs such as we covered in the Exercises.

A. Express as per cent, to the nearest tenth of 1%:

1. $\frac{5}{8}$ 62.5%
2. $\frac{3}{16}$ 18.8%
3. $\frac{3}{20}$ 15%
4. $\frac{3}{5}$ 60%
5. $\frac{2}{7}$ 28.6%
6. $\frac{4}{25}$ 16%
7. $\frac{5}{9}$ 55.6%
8. $\frac{5}{3}$ 166.7%
9. $\frac{17}{50}$ 34%
10. $\frac{7}{8}$ 87.5%
11. $\frac{9}{5}$ 180%
12. $\frac{45}{16}$ 281.3%

B. Express as per cent:

1. 0.57 57%
2. 3.5 350%
3. 2.51 251%
4. .09 9%
5. 1.8 180%
6. 0.001 0.1%
7. 4.1 410%
8. .065 6.5%
9. 0.003 0.3%
10. 0.18 18%
11. .005 0.5%
12. .0025 0.25%
13. .047 4.7%
14. 4.23
15. 1.9 190%
16. .08 8%

(ex. 14) 423%

C. Find the value for N:

1. 5 is $N\%$ of 40 12.5
2. $N\%$ of 68 is 17 25
3. 375% of 40 is N 150
4. 85% of N is 34 40
5. 275% of 48 is N 132
6. 6.1% of N is 48.8 800
7. 16% of 85 is N 13.6
8. N is 9.7% of 64 6.208
9. 62 is $N\%$ of 310 20
10. $N\%$ of 30 is 36 120
11. 35 is 70% of N 50
12. 35 is $N\%$ of 20 175

D. *Add:*

1. $^{+}5 + (^{+}9)$ $^{+}14$
2. $^{+}17 + (^{-}7)$ $^{+}10$
3. $^{-}13 + (^{-}9)$ $^{-}22$
4. $^{-}20 + (^{+}20)$ 0
5. $^{+}31 + (^{-}36)$ $^{-}5$
6. $^{-}19 + (^{+}25)$ $^{+}6$
7. $^{-}7 + (^{-}11)$ $^{-}18$
8. $^{-}16 + (^{+}16)$ 0
9. $^{+}15 + (^{+}27)$ $^{+}42$
10. $^{+}34 + (^{-}15)$ $^{+}19$

E. *Subtract:*

1. $^{+}17 - (^{-}4)$ $^{+}21$
2. $^{+}9 - (^{+}6)$ $^{+}3$
3. $^{-}19 - (^{+}19)$ $^{-}38$
4. $^{-}12 - (^{+}12)$ $^{-}24$
5. $^{-}17 - (^{-}15)$ $^{-}2$
6. $^{+}14 - (^{+}14)$ 0
7. $^{-}26 - (^{-}26)$ 0
8. $^{-}13 - (^{+}25)$ $^{-}38$
9. $^{-}12 - (^{-}12)$ 0

If you need more practice, turn to the Practice Exercises on page 497. If not, you may work in the Experts' Corner on the following page.

Magic Square

Draw a square divided into 25 cells. Instead of each letter, put in the answer to the corresponding exercise below.

a	b	c	d	e
f	g	h	i	j
k	l	m	n	o
p	q	r	s	t
u	v	w	x	y

a. 11 in. is what part of a foot? $\frac{11}{12}$

b. 30 in. is what part of a yd.? $\frac{5}{6}$

c. 1760 ft. is what part of a mi.? $\frac{1}{3}$

d. 23 in. is how many ft.? $1\frac{11}{12}$

e. Divide: $2\frac{1}{4} \div 1\frac{10}{17}$ $1\frac{7}{12}$

f. How many qts. is 3 pts.? $1\frac{1}{2}$

g. Divide: $\frac{4}{5} \div \frac{4}{5}$ 1

h. 16 qts. is what part of a bu.? $\frac{1}{2}$

i. 15 in. is what part of a yd.? $\frac{5}{12}$

j. 1 pt. is how many cups? 2

k. 25 in. is how many ft.? $2\frac{1}{12}$

l. 57 in. is how many yds.? $1\frac{7}{12}$

m. 39 in. is how many yds.? $1\frac{1}{12}$

n. Divide: $12\frac{1}{4} \div 21$ $\frac{7}{12}$

o. 1 cup is what part of 3 qts.? $\frac{1}{12}$

p. 6 in. is what part of a yd.? $\frac{1}{6}$

q. 7 pks. is how many bu.? $1\frac{3}{4}$

r. 20 in. is how many ft.? $1\frac{2}{3}$

s. Multiply: $1\frac{1}{2} \times \frac{7}{9}$ $1\frac{1}{6}$

t. 3520 ft. is what part of a mi.? $\frac{2}{3}$

u. 12 oz. is what part of a lb.? $\frac{3}{4}$

v. 500 lb. is what part of a ton? $\frac{1}{4}$

w. Divide: $2\frac{1}{5} \div 1\frac{1}{5}$ $1\frac{5}{6}$

x. 48 in. is how many yds.? $1\frac{1}{3}$

y. 10 pints is how many gals.? $1\frac{1}{4}$

Now check to see if you have a magic square. What is the total of each column? Of each row? Of each diagonal? If they are not all the same, find your mistake. $\frac{65}{12}$

Investigate the history of magic squares and write a report or prepare a talk on the subject.

The present century has seen more dramatic changes in employment and in educational requirements than has any similar period in our history. The change is continuing at an increasing rate. An effective way to portray such changes is by tables and graphs.

In this table you can see the shift in employment from farm and unskilled labor toward "white collar" occupations.

TRENDS IN OCCUPATIONAL DISTRIBUTION, 1900 TO 1975

	1900	*1950*	*1965*	*1975*[**]
WHITE COLLAR WORKERS	17.6%	36.6%	42.5%	46.6%
Professional, technical	4.3	8.6	11.3	14.0
Managers, proprietors	5.8	8.7	10.3	10.8
Clerical	3.0	12.3	14.4	14.4
Sales	4.5	7.0	6.5	7.4
MANUAL AND SERVICE WORKERS	44.9%	51.6%	49.9%	48.0%
Craftsmen, foremen	10.5	14.1	13.5	13.7
Machine operators	12.8	20.4	19.6	17.5
Industrial laborers	12.5	6.6	5.0	4.4
Service workers	9.0	10.5	11.8	12.4
FARM WORKERS	37.5%	11.8%	7.6%	5.3%
Farmers and farm managers	19.9	7.4	7.6	5.3
Farm laborers	17.7	4.4	—	—
NUMBER OF WORKERS—millions	29.0	59.0	73.5	86.9

[**]Projected

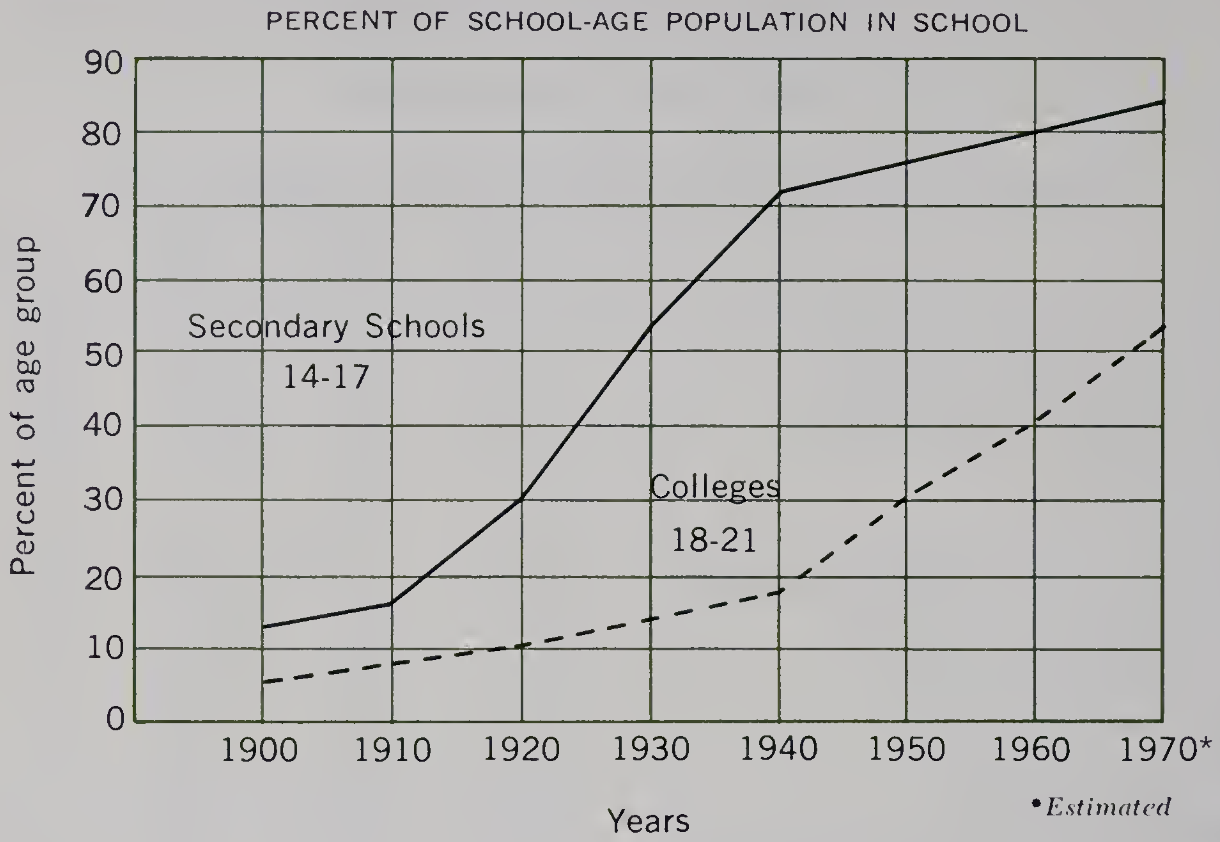
1. The total number employed in any given year is listed at the bottom of the table. From this you can calculate the number of workers in any group. For example, you can see that 4.3% of the workers in 1900 were professional, and technical. Note that there were a total of 29 million employed in that year. How many professional and technical workers were there? 1,247,000

2. What per cent of those employed in 1950 were professional and technical? What was the number of workers employed in that year? How many were professional and technical? 8.6%; 59 million; 5,074,000

3. How many professional and technical workers were there in 1965? predicted for 1975? 8,305,500 ; 12,166,000

4. The proportion of industrial laborers has steadily declined since 1900. Has the actual number declined? Calculate the number of industrial laborers for 1900, 1950, and 1965. Does the number decrease, increase, or remain about the same? Is the number predicted for 1975 greater or less than 1965? See front.

The more attractive vocational opportunities have higher educational requirements than in past years. As a result, more young people attend high school and college than ever before.



1. What per cent of youth of school age were in secondary school in 1940? 1960? 72 % ; 80 %
2. Why do you think the rate of increase slowed down in the 'forties? war
3. What per cent of high school age youth are expected to be in secondary schools in 1970? 84 %
4. The increasing demand for college training, by those companies which offer attractive positions, is reflected in college enrollments. What per cent of college age youth were in college in 1900? By how much did the per cent increase by 1940? Did the rate of increase in college enrollments become sharper or slower after 1940? What per cent of college age youth were in college in 1960? See front.
5. The number of college age youth in 1970 is predicted as 10,500,000. What college enrollment is predicted for that year? 5,460,000
6. One attractive vocational field that requires college preparation is engineering. In July, 1960 there were 822,000 engineers employed in this country. By 1970 it is expected that 1,375,000 will be needed. The increase in numbers is what per cent of the number of engineers in 1960? It is predicted that only 1,108,000 engineers will be available in 1970. What is the expected shortage? 67.3% ; 267,000

The number of unskilled workers in the nation's labor force has not changed greatly since the beginning of the century. To the nearest ten thousand, the number at the beginning of each decade has been as follows:

UNSKILLED WORKERS IN THE NATIONAL LABOR FORCE

1900	17,110,000	1940	18,470,000
1910	19,120,000	1950	17,010,000
1920	19,030,000	1960	17,210,000
1930	19,210,000		

These facts are also shown graphically in Figure 1.

While the number of unskilled workers has remained fairly constant, the proportion of unskilled workers in the labor force has declined rather sharply. This is shown in Figure 2.

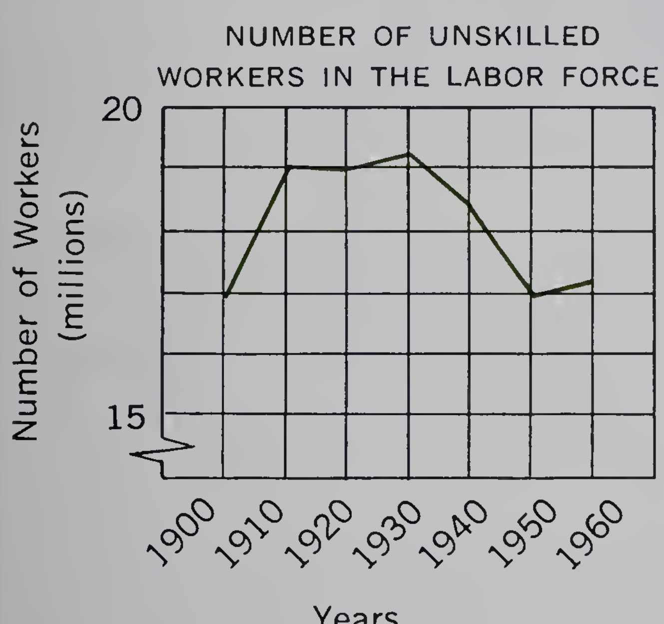


Figure 1

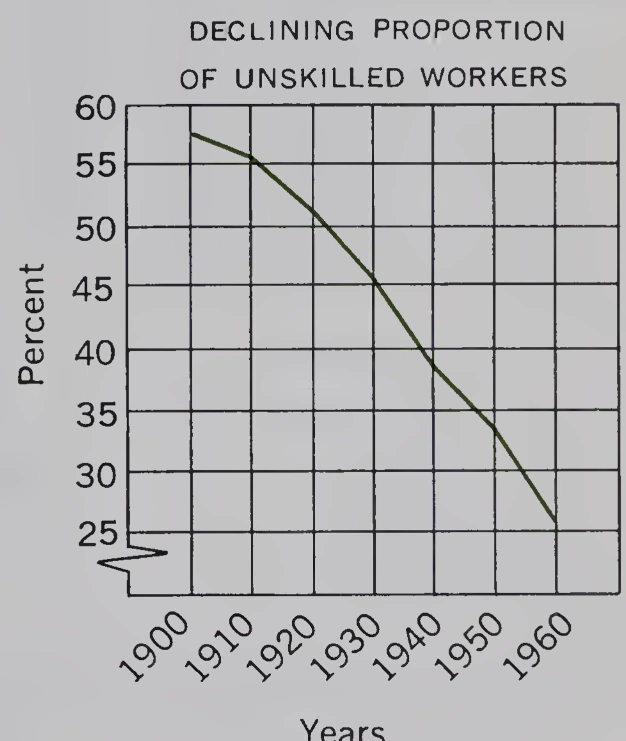


Figure 2

1. Does the table or the graph show the changes in the number of unskilled workers more clearly? *graph*

2. Which can be read more accurately—the graph or the table? *table*

3. Using the graph, list to the nearest whole per cent the employed workers who were in unskilled labor for each of the years shown. Then using the numbers in the table, calculate the number of employed workers in 1900 and 1960. *See front.*

The line graph, as you have seen, is especially useful to show the change in quantities over a period of time. The intervals of time are shown on the *horizontal axis*, and the quantities are shown on the *vertical axis*. Graph paper is very convenient in constructing a line graph. As you study the steps refer to the graphs on the previous page.

Step 1. Round the numbers so that the largest has no more than two or three non-zero digits.
Step 2. Mark the horizontal axis into convenient divisions to represent intervals of time.
Step 3. Mark the vertical axis into suitable intervals so that it will include the largest quantity to be shown.
Step 4. Label the horizontal and vertical axes.
Step 5. Locate the points on the graph that show the data you wish to present. Join the consecutive points by forming segments.
Step 6. Give the graph a title.

Construct line graphs to show the facts of the following exercises.

1. Expenditures for public schools during the first year of each decade in this century were as follows: *See front.*
1900–$ 214,965,000 1930–$2,316,790,000 1950–$ 5,837,643,000
1910–$ 426,250,000 1940–$2,344,049,000 1960–$15,075,000,000
1920–$1,036,151,000

2. To the nearest thousand, the number of college graduates in the first year of each decade of this century was as follows: *See front.*
1900–25,000 1930–125,000 1950–425,000
1910–37,000 1940–175,000 1960–500,000
1920–50,000

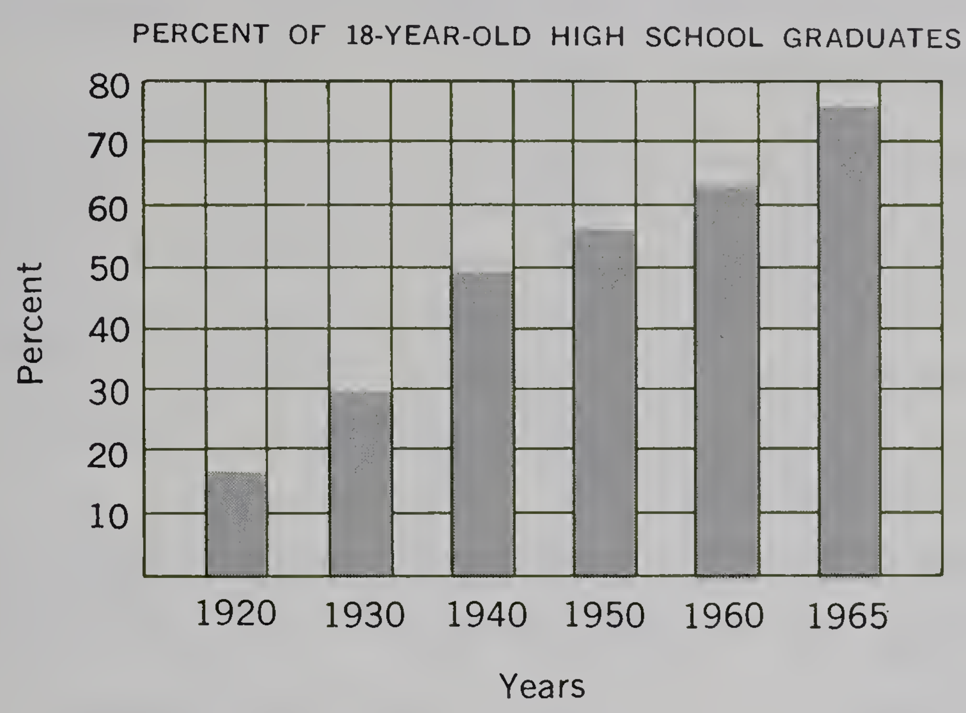
3. Line graphs are commonly used to show the rising cost of living. As you know, there is a tendency for the cost of most commodities to rise from year to year. During a period of inflation the increase may be quite rapid. During a recession, on the other hand, the trend may be reversed. These figures show the cost change of a commodity that could be purchased in 1939 for $1.00. *See front.*
1945–$1.25 1950–$1.70 1955–$1.71 1960–$1.90 1965–$1.99

4. Another way to show the rise in cost of living is to show the decrease in what a dollar will buy. If it took about $2.00 in 1965 to buy what could be bought for $1.00 in 1939, then the 1965 dollar is worth about 50¢ in 1939 dollars. The value of $1.00 in certain years, based on 1939 dollars, was as follows: *See front.*
1945–80¢ 1950–59¢ 1955–57¢ 1960–52¢ 1965–50¢

Not all people 18 years old are high school graduates. For one reason or another a number have dropped out along the way. Of 3,390,000 pupils who were in grade 5 in 1957-8, only 71% graduated from high school in the spring of 1965. How many graduated in 1965?

The proportion of students who complete high school is rapidly increasing, as you saw on Page 144. The increase is shown in another way in this graph. Try to interpret the graph before you continue to the exercises at the bottom of the page.

2,406,900



Study the graph and answer these questions about it.

1. To the nearest per cent, what per cent of 18-year olds graduated from high school in the years shown? 18%; 30%; 50%; 57%; 63%; 78%
2. Between which two years was the increase greatest? How much was the increase? 1930–1940; 20%
3. Between which two years was the increase least? How much was it? 1950-1960; 6%
4. Is the distance between the bars representing 1960 and 1965 less than the distance between those representing 1950 and 1960? no
5. Does a given distance on the horizontal axis represent a certain number of years? no
6. Does a given distance on the vertical axis represent a certain per cent? yes
7. Do you think the graph would be equally effective if it were arranged so that the bars were horizontal instead of vertical? yes

Here are the steps for constructing a bar graph. You will find graph paper convenient for the construction.

Step 1. Round the numbers so that each one has no more than two or three non-zero digits.
Step 2. Decide whether the bars should be vertical or horizontal. Then choose a convenient scale so that the largest quantity can be represented easily.
Step 3. Make the distance between the bars at least equal to the width of a bar. All bars should be the same width.
Step 4. Select appropriate labels and titles for the graph so that no further explanation is needed.

1. Even though the price of commodities has increased during the past thirty years, wages have increased even more. As a result, what a worker earns during an hour will purchase more today than in the past. In 1930 a worker on an average salary worked 25 minutes to earn enough to pay for a gallon of gasoline. In 1965 he earned enough to pay for a gallon of gasoline in 8 minutes. Show this fact in a bar graph. See front.

2. The length of time for a person on an average salary to earn enough to pay for certain commodities is indicated for 1950 and 1965.

Commodity	*Earning time required*	
	1950	*1965*
1 quart of milk	8 minutes	6 minutes
1 loaf of bread	6 minutes	6 minutes
1 dozen eggs	26 minutes	18 minutes
1 haircut (men)	43 minutes	48 minutes
1 house dress	1 hr. 48 min.	1 hr. 39 min.
1 pair shoes	5 hr. 43 min.	4 hr. 30 min.
1 man's suit	27 hr. 35 min.	29 hr. 10 min.

Were there any items that took longer to earn in 1965 than in 1950? What were they? haircut, man's suit

3. Which item(s) remained the same? Does this mean that the cost was the same in both years? loaf of bread ; no

4. Select one or two items that decreased by the greatest per cent in amount of time required to earn the cost and prepare a graph or graphs to present the information.

5. The average number of years of school completed by employed workers in this country was 9.3 in 1940 and 11.8 in 1960. Prepare a bar graph to present this fact. *See front.*

6. In 1965 women with an eighth-grade education who were full-time employees had an average annual income of $2600. With a high school education, women who were full-time employees had an average annual income of $4000. College graduates were earning $7200 annually. Prepare a bar graph to show these facts. *See front.*

7. A survey in 1960 revealed these facts about different classes of workers. See if you can design 2 bar graphs to show them clearly.
Have students make graphs. Designs may vary.

Class of workers	*Years of schooling*	*Annual income*
Professional and technical	16.2	$7788
Proprietors and managers	12.4	7112
Clerical and sales	12.5	6000
Skilled workers	11.0	6018
Semi-skilled	9.9	5150
Unskilled	8.6	4090

8. According to the Census Bureau, income of an individual depends both on his age and his education. It has published this information showing average incomes. Make bar graphs for several age groups.
Have students make graphs. Designs may vary.

Age	*H. S. 1-3 yr.*	*H. S. graduate*	*College graduate*
14-24	$ 748	$2469	$2878
25-34	4425	5175	6640
35-44	5245	6016	8878
45-54	5317	5989	9130
55-64	5201	5750	8510

9. The cost of medical services from 1948 to the present time has been steadily increasing. The medical costs include physicians, dentists, drugs, hospital and sanitarium bills, medical care and hospitalization insurance, etc. Graph this information: *See front.*
For selected years the cost was reported in billions as
8.5–1948 11.2–1953 17.8–1958 25.4–1963

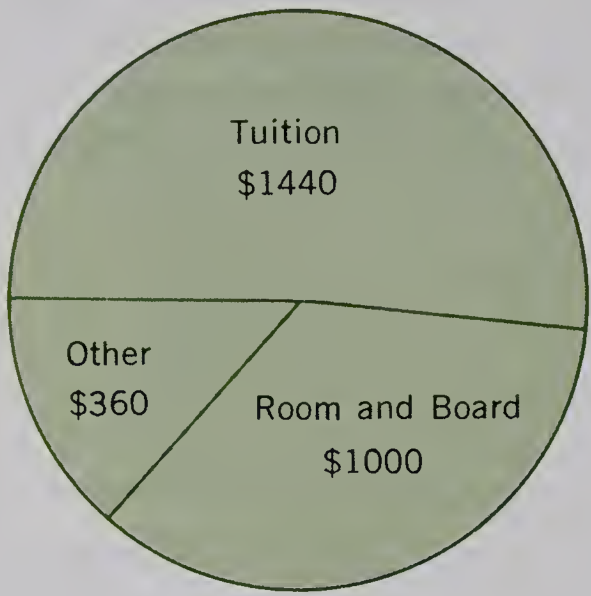
10. A recent report indicated that we spend the following amounts of time in recreation per year. Prepare a graph of this information.
See front.

Outdoor sports and recreation 13 days

Sightseeing	6 days	Walks and hikes	17 days
Swimming	$6\frac{1}{2}$ days	Pleasure driving	21 days

George is planning to go to college after graduation from high school. He has written to several colleges inquiring about cost of tuition. In general, it costs more to go to a private college than to a state college or university. The costs at one private college were estimated at $2800 a year. George prepared this graph:

COSTS AT A PRIVATE COLLEGE



1. In preparing his graph, George first listed the costs and then found what per cent each one was of the total. He rounded each to the nearest whole per cent. Copy and complete his table.

Item	*Cost*	*Per cent*
Room and board	$1000	36%
Tuition	1440	? 51 %
Other expenses	360	? 13%
	$2800	100%

2. Since room and board make up 36% of the total expenses, this item is represented by 36% of the circular region on the graph. The total measure of the angles having their vertices at the center of the circle is 360°. Then the angle of the sector representing room and board is 36% of 360°. To the nearest degree, this is how many degrees? Calculate the number of degrees to represent tuition and the other expenses. Round each to the nearest whole degree.

See front.

3. Does the sum of the three measures add to 360°? It should before the measures are rounded. If in rounding the sum becomes 359° or 361°, a degree should be added to or subtracted from the largest measure.

4. Complete the table for the items in three columns:

Item	*Cost*	*Per cent*	*Degrees*
Room and board	$1000	36%	? 130
Tuition	1440	? 51%	? 182
Other expenses	360	? 13%	? 46

Draw a circle and use a protractor to determine the angles at the center of the circle. Draw in the radii to define the sectors. See if your graph is similar to George's graph. If not, find your error.

5. In answer to George's inquiry, a state university reported that its average expenses for students were these: Tuition and fees, $350; room and board, $900; and other expenses, $350. Prepare a table, as shown in Exercise 4, for this information. Then draw the graph with labels and title. See front.

6. A census report showed the following information with regard to money income and the schooling of the head of the family.

Construct three circle graphs to show these facts.

	Family income		
Schooling, head of family	*Under $5000*	*$5000-$9999*	*Over $10,000*
8 years or less	59% 212°	35% 126°	6% 22°
High school graduate	38% 137°	52% 187°	10% 36°
College, 4 or more years	16% 58°	52% 187°	32% 115°

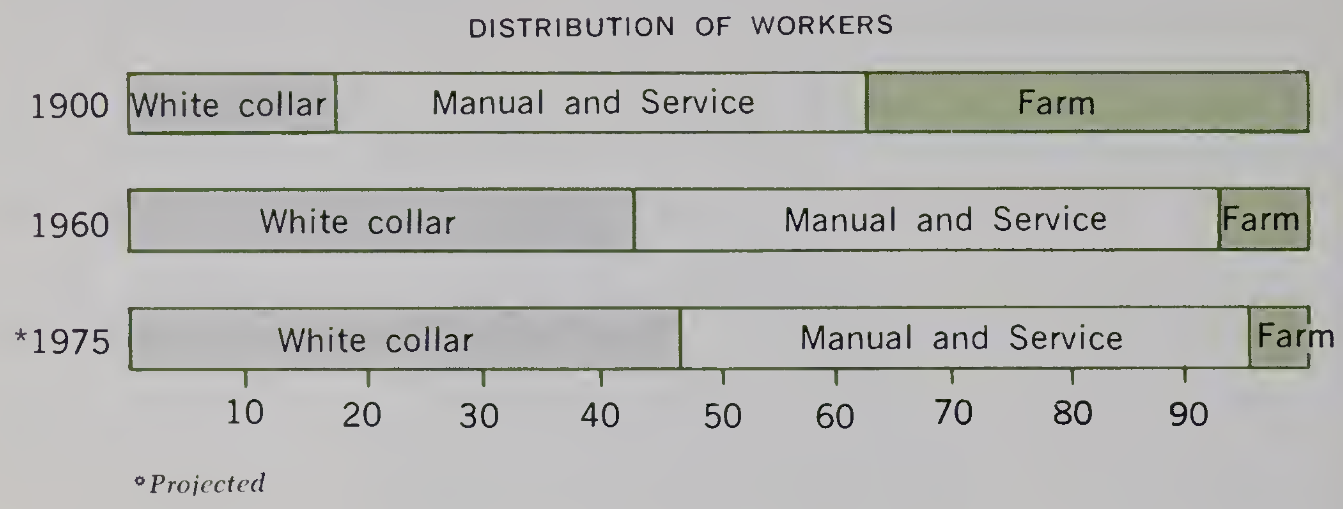
7. John, a college freshman, usually spent his daily time as indicated. Graph this information.

Class lecture	3 hours 45°	Sleep	8 hours 120°
Laboratory	2 hours 30°	Sports	2 hours 30°
Studying	6 hours 90°	Meals, etc.	3 hours 45°

8. Draw a graph to illustrate how Bill's friends spend their after-class time: Study 50%, Dinner 10%, Sports 30%, Free time 10%. 36°
180° 36° 108°

9. Collect some data about your class or school suitable for display on a circle graph and prepare one to show the data. For example:
   a. enrollment by grades
   b. what high school graduates of last year are doing now
   c. how students spend a typical school day
   d. favorite hobbies among members of your class
   e. results of a recent test or marking period
   f. favorite type of music

In this graph you see the per cent of workers in the three major classes of employment during certain years from the beginning of the century.



1. What unit is represented by each division of the bar? 10 %
What per cent of workers is represented by the entire bar? 100 %

2. Copy and complete this table, using per cents to the nearest whole per cent.

Kinds of workers	*Per cent of workers*		
	1900	*1960*	*1975*□□
White collar	? 17 %	? 43 %	? 47 %
Manual and service	? 45 %	? 50 %	? 48 %
Farm workers	? 38 %	? 7 %	? 5 %

□□Projected

3. Did the per cent of manual and service workers decrease or increase from 1900 to 1960? by what per cent? increase ; 3 %

4. Is the per cent of white collar workers expected to decrease or increase from 1960 to 1975? by what per cent? increase ; 4 %

The steps in constructing divided bar graphs are as follows:

Step 1. Select a convenient length bar to represent 100%.
You will find it handy to use graph paper.

Step 2. Prepare a table, as with the circle graph, with 3 columns:
1. The amount of each item
2. The per cent each amount is of the total
3. The length of bar needed to show each per cent

Step 3. Measure off the parts of the bar as indicated in the table.

Step 4. Provide the graph with necessary labels and title.

Use the steps to show the facts in each of these exercises.
Students should draw their own bar graphs for exercises 1-3.

1. The per cent of white collar workers in different types of work:

Kind of work	*Per cent of workers*		
	1900	*1960*	*1975*□□
Professional	25	27	30
Executive	33	24	26
Clerical	16	32	28
Sales	26	17	16

□□Projected

2. The manual and service class is also subdivided into groups that have changed through the years. The per cent of each classification is shown in this table.

Kind of worker	*Per cent of workers*		
	1900	*1960*	*1975*□□
Craftsmen and foremen	23	27	29
Machine operators	29	39	36
Services	20	24	26
Laborers	28	10	9

□□Projected

The services group includes barbers, hairdressers, filling station attendants, and so on.

3. The educational attainment of adults 25 years old and older for certain years has been as follows:

Educational attainment	*1940*	*1950*	*1960*	*1965*
Less than 5 years of school	10%	9%	7%	5%
5 to 8 years elementary	48%	37%	32%	27%
1 to 4 years high school	32%	42%	44%	50%
1 or more years college	10%	12%	17%	18%

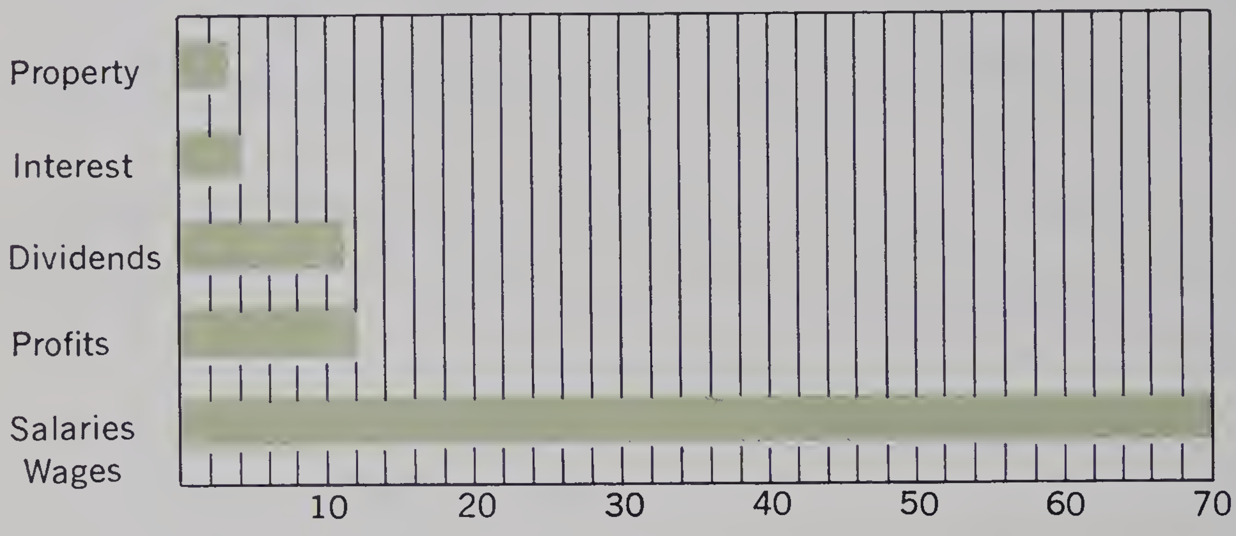
4. The annual take-home pay of the Owens family last year was \$7200. Draw a circle graph to represent their expenses.

Food	\$1800 90°	Home operation	\$1080 54°
Clothing	\$720 36°	Personal	\$720 36°
Savings	\$360 18°	Shelter	\$2520 126°

A family income is derived from a variety of sources. Usually it comes from salaries and wages, but frequently is supplemented by interest from savings or dividends from investments, and often by income from property ownership.

1. The sources of family income have been reported by a government source as shown in the bar graph. What per cent of income is from salaries and wages? interest? *70 % ; 4 %*

SOURCES OF FAMILY INCOME



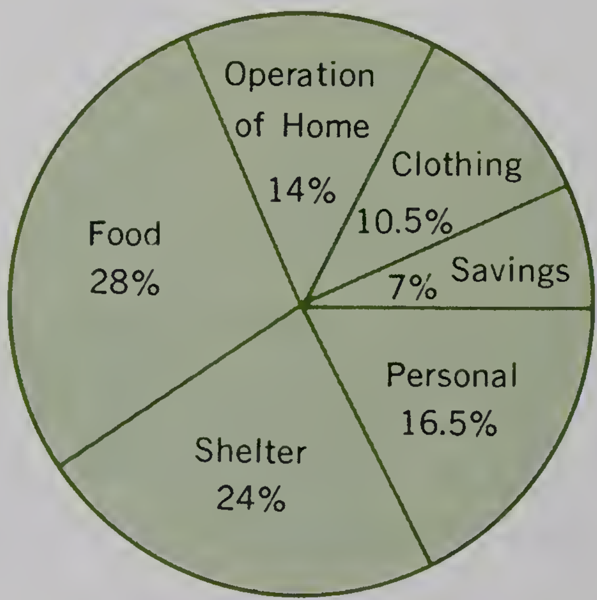
2. If the total personal annual income in this country was reported as being $510 billion and it was distributed as shown above, how much was the income from each source? *See below.*

3. If the population was 194 million, find the per capita income. *$ 2629*

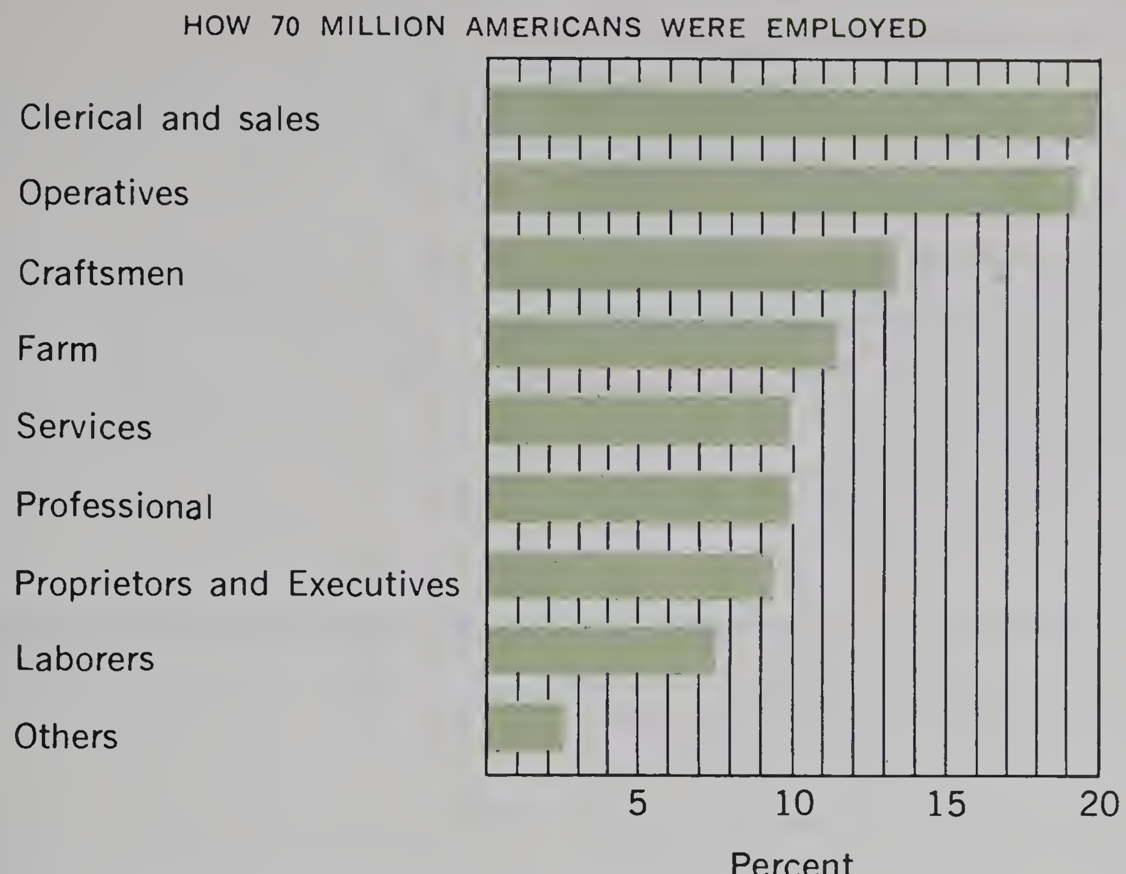
4. The average family income in the year represented was $6250. If spent as shown in the circle graph, how much was spent for food? shelter and operation of the home combined? *$ 1750 ; $ 2375*

(ex. 2)
Property, $ 15.3 billion
Interest, $ 20.4 billion
Dividends, $ 56.1 billion
Profits, $ 61.2 billion
Wages and Salaries, $ 357 billion

HOW AVERAGE FAMILY INCOME IS SPENT



In the graph below, we see how Americans are employed. In which category are most people employed? *clerical and sales*



1. If this graph represents 70 million employed people, how many people are engaged in farm work? *7.7 million*
2. What per cent of those employed are in services? *7 million*
3. Professional people make up what per cent of those employed? *10%*
4. Each category of work offers substantial differences in income. A person's salary may be paid by the month, by the week, by the hour, or on some other basis. For each category in the graph, what is the usual basis for payment? *See front.*
5. Which is a higher annual salary, $400 a month or $95 a week? *$ 95 a week*

SPECIAL REPORTS

1. Before deciding on a career, it is important to know whether there is a demand for workers in that occupation. Consult newspapers, local employment agencies, and your guidance office to determine what careers offer the most opportunity in your community. Skilled workmen, clerical workers, business, or professional people are in demand in many places. Investigate the many opportunities in Civil Service jobs.
2. Have each member of your class write his vocational choice on a sheet of paper. Make a table and a bar graph to show the per cent desiring to enter each category mentioned on the graph.

A *savings* account is opened by a person who wants to deposit his money, leave it there, and receive interest on it. The savings bank pays you interest for the use of your money. If the interest is added to your account every three months and the next three months' interest is calculated on the new and larger principal, we are receiving interest compounded quarterly.

The *principal* is the amount on which interest is calculated.
The *rate* is the per cent of interest earned.
The *time* is the period for which the interest is being calculated.
The formula for calculating interest is:

$$i = prt$$

where i is interest, p is principal, r is rate expressed as hundredths, and t is time expressed in years or fraction of a year. In using this formula, a year is counted as 360 days.

EXAMPLE

Find the interest on \$840 at 5% for 9 months compounded quarterly. Notice the interest is calculated on a new principal every three months.

$$i_1 = 840 \cdot \frac{5}{100} \cdot \frac{1}{4} = \frac{42.0}{4} = 10.50 \qquad p_1 = 850.50$$

$$i_2 = 850.50 \cdot \frac{5}{100} \cdot \frac{1}{4} = \frac{42.5250}{4} = 10.63 \qquad p_2 = 861.13$$

$$i_3 = 861.13 \cdot \frac{5}{100} \cdot \frac{1}{4} = \frac{43.0565}{4} = 10.76 \qquad p_3 = 871.89$$

The interest earned is 871.69 − 840.00 = \$31.69.

If this account receives *simple* interest, $i = prt$ would amount to $\frac{840}{1} \times \frac{5}{100} \times \frac{3}{4} = 31.50$. You can see how compounding interest increases the return on your savings.

EXAMPLE

Find the simple interest on \$600 at $6\frac{1}{2}\%$ for 72 days.

$$i = prt = 600 \times \frac{13}{200} \times \frac{72}{360} = 7.80$$

Find the simple interest in each of the following exercises and calculate the interest compounded quarterly for the first 3 exercises.

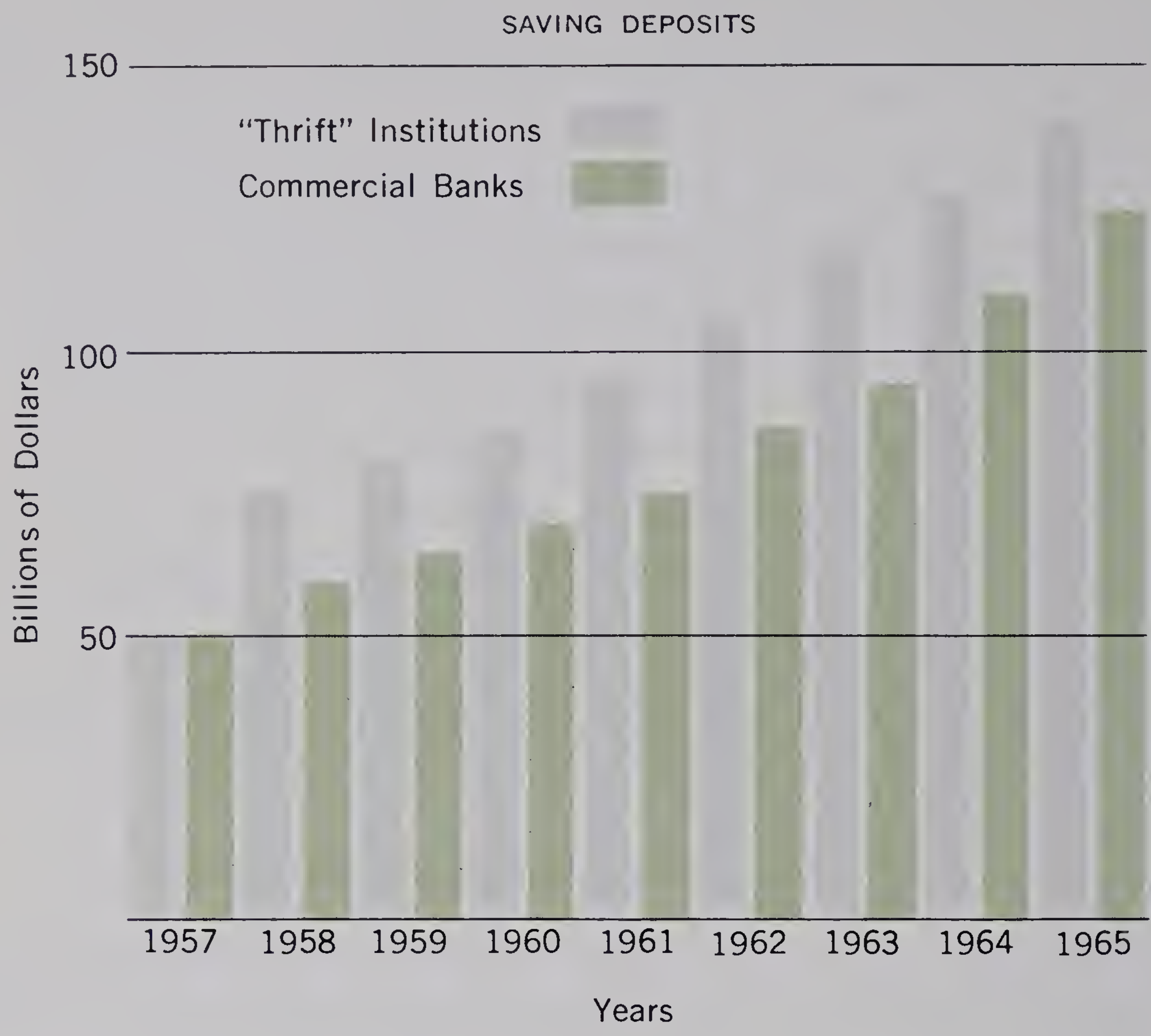
1. \$1200 at 6% for $1\frac{1}{2}$ years \$108 \$112.13
2. \$2400 at 5% for 1 year \$120
3. \$450 for 1 year at 4% \$122.28 \$18, \$18.28
4. \$1600 for 90 days at 7% \$28
5. \$240 for 80 days at 7% \$3.73
6. \$600 for 6 months at $5\frac{1}{2}$% \$16.50
7. \$500 for 66 days at 6% \$5.50
8. \$750 at 6% for 75 days \$9.38
9. \$400 at $5\frac{1}{2}$% for 90 days \$5.50
10. \$360 for 6 months at 8% \$14.40
11. \$750 for 72 days at $7\frac{1}{2}$% \$11.25
12. \$600 for 8 months at 4% \$16
13. \$8000 for 96 days at $7\frac{1}{2}$% \$160
14. \$750 at $4\frac{1}{2}$% for 180 days \$16.88
15. \$500 for 9 months at 6% \$22.50
16. \$600 for 120 days at 5% \$10
17. \$3500 for 2 months at 8% \$46.67
18. \$300 for 90 days at $6\frac{1}{2}$% \$4.88
19. \$750 at $4\frac{1}{2}$% for 72 days \$6.75
20. \$750 at $7\frac{1}{2}$% for 66 days \$10.31
21. \$450 at 6% for 90 days \$6.75
22. \$800 at 5% for 4 months \$13.33
23. \$500 for 120 days at $7\frac{1}{2}$% \$12.50
24. \$250 for 6 months at 6% \$7.50
25. \$720 for 60 days at 8% \$9.60
26. \$900 for 270 days at $6\frac{1}{2}$% \$43.88

QUESTIONS FOR RESEARCH AND REPORT

Savings banks have regulations for calculating interest designed to encourage deposits and discourage withdrawals. Find out some of the regulations from banks in your community.

1. What rate of interest is being paid? How often is interest calculated?
2. Suppose a deposit were made on the 15th of the month. When would it start drawing interest? When would it start drawing interest if it were deposited on the 9th of the month? Is any pattern apparent from your investigations?
3. What is the regulation regarding interest on amounts withdrawn between interest dates?
4. Is interest figured on the entire balance or only on whole dollars in the balance? How would this regulation affect a \$4000 balance over 3 three years?
5. Many periodicals contain current information on savings plans, interest rates, and trends. Find out what some of these sources of information are and see if they are available in the library.

About 7% of personal income each year in this country is directed into savings. For the most part savings are invested in savings banks, stocks and bonds, or real estate.



1. If the total personal annual income is $510 billion and 7% of this is saved, how much do the savings amount to? *$ 35.7 billion*

2. The graph shows how savings deposits have increased in two general types of savings institutions. The "thrift" institutions include savings and loan companies and mutual savings associations. How much did thrift deposits amount to in 1957? in 1965? (Answer to the nearest billion.) *$ 50 billion ; $ 140 billion*

3. How much did thrift deposits increase from 1957 to 1965? *$ 90 billion*

4. How much did deposits increase in commercial banks from 1957 to 1965? *$ 75 billion*

5. What did deposits total in both types of institutions in 1957? in 1965? What was the increase? *$ 100 billion ; $ 265 billion; $ 165 billion*

6. The rate of interest paid on savings in commercial banks from 1955 to 1966 was $4\frac{1}{2}\%$ per year. To the nearest $10 million, how much interest did commercial banks pay in 1957? in 1965? *$ 2.25 billion ; $ 5.625 billion*

Many people find it profitable to invest some of their savings in stocks or bonds, which often pay a higher return on their investments than do savings accounts. A bond is a written promise by a corporation or a government to pay a certain sum of money at a stated time to the holder of the bond, and in the meantime, to pay a fixed rate of interest at regular intervals. A bond is a promissory note, and like other promissory notes, it may be secured by collateral or merely by the signature of the officers of the corporation.

Interest on a bond is always a certain per cent of the face value. Thus a \$1000 bond at 6% pays \$60 interest annually. Payments are usually made semiannually. In this case \$30 would be paid every six months. Accrued interest on a bond is the interest that has been earned since the last interest payment.

1. On September 15 Mr. Brown purchased from Mr. Adams a \$1000 6% bond issued by the Northern Pacific Railway. Interest is paid January 1 and July 1. Mr. Adams had collected interest on July 1. On January 1 Mr. Brown will collect the interest earned since July 1. The interest up to September 15 belongs to Mr. Adams. What is the accrued interest that Mr. Brown should pay to Mr. Adams when he purchases the bond? *\$12.50*
2. Mr. Smith owns a \$1000 5% bond, paying interest March 15 and September 15. What is the annual income from the bond? How much is each payment? *\$50 ; \$25*
3. Mr. Jones has five \$1000 $5\frac{1}{2}$% bonds of the United States Steel Company, paying interest semiannually. How much is each payment? *\$137.50*
4. Mr. Edmonds owns three \$1000 $4\frac{1}{2}$% bonds of the Great Northern Railway, paying interest semiannually. How much is each payment? *\$67.50*
5. Mr. Johnson owns two \$1000 $4\frac{1}{4}$% United States bonds that pay interest on April 1 and October 1. What is the interest paid on each bond at each payment? *\$21.25*

Interest on each of the following bonds is paid semiannually. Calculate the amount of each payment.

6. A \$1000 4% bond *\$20*
7. A \$500 7% bond *\$17.50*
8. A \$5000 $6\frac{1}{2}$% bond *\$162.50*
9. A \$2000 $4\frac{1}{2}$% bond *\$45*
10. A \$6000 $3\frac{1}{2}$% bond *\$105*
11. A \$4000 2% bond *\$40*
12. A \$10,000 6% bond *\$300*
13. A \$3000 $2\frac{1}{2}$% bond *\$37.50*

When large amounts of money are needed in a short time, the United States Government issues bonds, just as any other corporation does. From the standpoint of the investor government bonds have the advantage of security, since they have the financial backing of the government. Also they can be converted to cash easily. Two types of bonds are designed especially to encourage family saving. One is Series H, which after two years of lower rates, pays 4.15% interest per year in semiannual payments. The other is Series E, which does not pay periodic interest. Instead these bonds are sold at a discount of 25% of their face value, which is paid in full when the bonds mature.

Here is a picture of a $75 bond, Series E, which may be purchased for $56.25.

1. The amount for which the bond can be cashed if it is held until maturity is its *face value.* What is the face value of the United States Savings Bond shown? $ 75

2. These bonds can be cashed for their face value after seven years from the date of issue. The time when they are paid up is called the *date of maturity.* On what date will a bond mature if it was issued on October 1, 1966? Oct. 1, 1973

3. The bond shown above can be purchased for $56.25. How much more than this is its value at maturity? $ 18.75

4. What is the face value of a bond purchased for $18.75? $ 25 ; For $37.50? $ 50

5. The difference between the purchase price and the value at maturity is the accumulated interest. What per cent of the purchase price is the accumulated interest? 33.3 %

6. At the same rate, what is the purchase price of a bond whose face value is $500? $ 375

7. A \$100 E bond costs \$75. It matures in 7 years when the holder will receive its face value. Each 6 months after its purchase the value for which a \$100 bond can be redeemed increases as follows:

First $\frac{1}{2}$ year	\$75.00	$2\frac{1}{2}$ to 3 years	82.08	5 to $5\frac{1}{2}$ years	91.44
$\frac{1}{2}$ to 1 year	75.84	3 to $3\frac{1}{2}$ years	83.84	$5\frac{1}{2}$ to 6 years	93.44
1 to $1\frac{1}{2}$ years	77.28	$3\frac{1}{2}$ to 4 years	85.68	6 to $6\frac{1}{2}$ years	95.52
$1\frac{1}{2}$ to 2 years	78.80	4 to $4\frac{1}{2}$ years	87.56	$6\frac{1}{2}$ to 7 years	97.68
2 to $2\frac{1}{2}$ years	80.40	$4\frac{1}{2}$ to 5 years	89.48	Maturity value	100.00

How much interest is earned at the end of one year? \$ 2.28

8. How much interest does a \$100 bond earn during the second year? during the seventh year? \$ 3.12 ; \$ 4.24

9. The value of a bond is established each year so as to encourage the owner to hold it until maturity. Explain how this provision helps the owner to see the advantage of holding his bonds to maturity.
The increase in value in later years is greater than in earlier years.

10. Another type of government bond is a *treasury bond*. These bonds may be purchased through a bank, and their market value changes as the interest rates change. There are several series of treasury bonds, each identified by its interest rate and date of maturity. The market values for several series recently were as follows:

	Series	*Market Value*
\$ 496.25	Series of 1970, 4% bond	99.8
\$ 462.19	Series of 1971, $2\frac{1}{2}$% bond	92.14
\$ 500.63	Series of 1974, $4\frac{1}{4}$% bond	100.4
\$ 500.94	Series of 1987, $4\frac{1}{4}$% bond	100.6
\$ 459.38	Series of 1980, $3\frac{1}{2}$% bond	91.28

The price quoted is for a \$100 bond. Note that the market price is stated in dollars and thirty-seconds, not dollars and cents. Thus a \$500 bond of series of 1970 would cost $5 \times \$99\frac{8}{32}$ or \$496.25. Find the price of a \$500 bond in each series. See above.

11. The interest on a treasury bond is paid each six months. Find how much the owner of a \$500 bond in each series would receive every interest date. \$ 10 ; \$ 6.25 ; \$ 10.63 ; \$ 10.63 ; \$ 8.75

12. At maturity a treasury bond is redeemed at its face value, regardless of what the holder paid for it. Thus if one buys a bond for less than the face value its value will increase if he holds it until it matures. Suppose you bought a \$1000 $2\frac{1}{2}$% series 1971 bond in 1966 at the market price quoted in Exercise 10. How much would its value increase by the time it matures in 1971? \$ 75.62

$ 1.88

13. Suppose you bought a series 1987 $4\frac{1}{4}\%$ \$1000 bond in 1966 at the stated market price. What is the decrease in value to maturity?

14. The gain or loss per year is taken into account in determining the *rate of yield* of a bond—that is, the per cent the annual income is of the investment in the bond. An annual gain is added to the interest. Why? The gain in value is the same as income.

15. Why should an annual loss be deducted from the interest?
Decrease in value is loss of income.

16. Suppose we wished to determine the rate of yield of a \$1000 bond, series 1970, purchased in 1966 and held to maturity. Thus we have:

Investment (cost of a \$1000 bond)			\$992.50
Annual income:			
Interest	\$40.00		
Annual gain in value	1.88	($\frac{1}{4}$ of \$7.50)	
Total annual income	\$41.88		

To determine the rate of yield, find n: 41.88 is $n\%$ of 992.50. What is the rate of yield, to one tenth of 1%. 4.2%

17. Calculate the rate of yield on a \$1000 bond in each of the other series if purchased at the market price in 1966 and held to maturity.
1971, 4.3%; 1974, 4.2%; 1987, 4.2%; 1980, 4.4%

18. As a treasury bond matures, its market value approaches 100. Explain why.

19. On the financial page of a daily paper you can probably find the current market value of various series of treasury bonds. If not, you can obtain the information from a bank. Find out if the market value of each series listed in Exercise 10 has increased or decreased from that of 1966.

20. Interest rates vary with business conditions. In times of active business the rate of interest paid on a savings account or charged on a loan is apt to be higher than when business is dull. In 1966 interest rates advanced sharply, and a savings bank that paid interest every three months at the rate of 5% annually published these figures to show what one has at the end of each year if he deposits \$1 weekly in his account:

At the end of:	*He has a total of:*
1 year	\$52.98
2 years	108.68
3 years	167.18
4 years	228.67
5 years	293.32

To find the amount you could accumulate with larger weekly deposits, multiply any of the amounts listed by the ratio of the larger amount to \$1. How much would you have at the end of 5 years with weekly deposits of \$5? $ 1466.60

Mr. Adams and Mr. Johnson were partners who owned and operated the Parkview Grocery Store. It was a well-managed and prosperous corner grocery. But plans were made to move to an attractive new shopping center. Since this move would call for additional funds and attendant risks, the partners decided to change their partnership to a corporation, in which they would sell stock.

The ownership of the corporation was divided into 2000 shares, each valued at $100. The former partners each retained 500 shares, representing the current value of the business. The remaining 1000 shares were sold to secure the funds needed to move and to set up the new store. Each share of stock entitles the holder to one vote at the annual stockholders' meeting, at which officers are elected, profits distributed, and other business is transacted. Since the former partners own half the stock, they still retain effective control of the business. Mr. Adams was elected president of the corporation, and Mr. Johnson, secretary and treasurer, as well as chairman of the board of directors.

1. During its first year the Parkview Grocery, Incorporated had profits of $5000, after taxes. It was decided to distribute $4000 as *dividends* to stockholders, and to retain $1000 in the business. How much did a person receive for each share? $ 2

2. Mr. Adams and Mr. Johnson each owned 500 shares of stock. How much did each receive in dividends? $ 1000

3. After the first year the directors decided to issue dividends quarterly (every three months). During the second year the profits (after taxes) were $9000. Of this, $3000 was retained to improve the business equipment and stock, and the remainder was distributed in four equal quarterly dividends. How much did the holder of one share of stock receive each quarter? 75 ¢

4. How much did the holder of one share of stock receive as dividends during the year? $ 3

5. If he paid $100 for his share of stock, the annual income in dividends is what per cent of his investment? (This is called the *rate of return* on the investment.) 3%

6. In the following year the profits (after taxes) were $12,000. Of this, $4500 was put back into the business and the rest was distributed in four quarterly dividends. How much did the owner of one share receive in dividends during the year? $ 3.75

7. How much did Mr. Adams receive? What was his rate of return?
$ 1875 ; $3\frac{3}{4}$ %

A person may have one or more reasons for purchasing stock in a corporation: to secure a voice in its management, to share in the growth of the company, and to receive dividends.

For most investors the second and third reasons are the important ones. Investment in stocks is becoming increasingly common as a savings plan. Each year about a million Americans are becoming corporation shareholders for the first time. More than 20 million individuals and families are investing in stocks as a means for saving. It is an interesting fact that more women than men are shareholders in corporations. Stock in corporations like American Telephone and Telegraph, and United States Steel is owned by shareholders all over the nation.

The selling price of a share of stock may vary with the business outlook and the rate of earnings. It is quoted in the financial pages of many newspapers. Stock is purchased through a licensed stock broker or a bank. Quotations listed below represent a typical newspaper report on stocks.

1. Under *Sales* is listed the number of shares sold on the previous day. What was the largest number of sales for a corporation? (Ampex) 112,600

2. Under *Close* is the price in dollars and fractions for one share at the end of the previous day. If all the shares had been sold at that price, what was the amount paid for the shares sold in Exercise 1? $ 3,138,725

3. The difference between the closing price and the opening price is listed under *Net Change* (N. Ch.). How can you tell whether the N. Ch. is above or below the opening price? +, above ; –, below

Sales	*Stocks*	*Close*	*N. Ch.*	*Sales*	*Stocks*	*Close*	*N. Ch.*
77,200	Am T T	$57\frac{5}{8}$	$+\frac{1}{8}$	14,700	Gen Dynam	$54\frac{3}{4}$	$+\frac{1}{8}$
112,600	Ampex	$27\frac{7}{8}$	$+\frac{3}{8}$	30,000	Holt R W	$59\frac{1}{2}$	$+1\frac{1}{2}$
19,900	Bulova	24	+1	80,000	Int Bus M	$389\frac{3}{4}$	$-1\frac{1}{4}$
51,200	Chrysler	$35\frac{1}{2}$	$-\frac{3}{8}$	15,000	Mattel	$12\frac{1}{8}$	...
2,000	Fedders	$14\frac{3}{4}$	$+\frac{5}{8}$	33,600	Minn M M	$79\frac{1}{4}$	$+\frac{5}{8}$
26,600	Ford Mot	$45\frac{3}{4}$	$-\frac{1}{4}$	27,700	US Steel	$43\frac{7}{8}$	$-\frac{1}{8}$

The total sales for Am T T amounted to $4,448,650. As you can see, a great deal of money is involved in these transactions. Determine the total sales for Bulova, Ampex, and Minn M M.
Bulova, $ 477,600 ; Minn MM, $ 2,662,800

4. Which stock increased most in value? Which stock declined the most? Holt RW ; IBM

$57 \frac{5}{8}$

5. What was the closing price of American Telephone and Telegraph? What was the closing price on the previous day? $57 \frac{1}{2}$

6. If United States Steel is paying dividends of $2.40 per share each year, what per cent of the market price reported above is the $2.40 dividend? Carry out your calculations to the nearest tenth of 1%. (This is the *rate of return,* or *yield.*) 5.5 %

7. American Telephone and Telegraph is paying dividends of $2.20 per year. Since dividends are commonly paid quarterly, Am T and T is paying 55¢ per share each 3 months. What is the rate of annual return at the market price listed above? 3.8 %

8. The dividends for the preceding year for the following stocks were:

Bulova 2.5 %	.60	Int Bus M 1.1%	4.40
Chrysler 5.6 %	2	Mattel 3.3 %	.40
Ford 5.2 %	2.40	Minn M M 1.5 %	1.20
Gen Dynam 1.8 %	1	Holt R W 1.5 %	.90

Select the stock which you estimate has the highest rate of return, and the one which you estimate has the lowest. Calculate the rate of return for each. *highest, Chrysler ; lowest, IBM*

9. How many of the corporations listed above have you heard of before? What kind of business is each engaged in? What economic factors do you think might increase or decrease the value of the stock in any of these companies?

Even though the small stockholder has little to say about operations of a company, he can certainly try to:

1. Select an industry or company that best fits his investment program.
2. Secure the best type of advice about the industry or company.
3. Read the financial information each company sends out to its shareholders, so that he knows what his company is doing and how the financial outlook is changing.
4. Recognize that stock prices may decline as well as rise.

Which stock should you buy? The desire for steady income or profit on your investment are two determining factors. Any broker can give you a list of common stocks that have paid dividends over a long period of time if you are looking for steady income.

The desire to obtain a larger return on your investment entails a greater risk. Both of these factors should enter into a person's thinking before he undertakes an investment program.

An important step in problem solving is to estimate a reasonable answer before you make your written computations. This helps prevent mistakes and provides a useful check on your answer. Before you work each of these exercises, complete each of the statements to show how you would find an approximate answer. Then find the exact number and see if it checks.

EXAMPLES

1. If 65% of Fremont's 800 boys took part in athletics last year, how many took part in athletics? Most of us would agree that 65% of 800 is not an easy mental problem. However, 62.5% of 800 would provide a reasonable estimate. We know 62.5% $= \frac{5}{8}$ and $\frac{5}{8}$ of 800 = 500. *About* 500 boys took part in athletics. Is the exact number more or less than 500?

2. Find 23% of 64. We know that 23% is close to 25% or $\frac{1}{4}$ and $\frac{1}{4}$ of 64 = 16. Since 23% is less than 25%, should the actual answer be more or less than 16?

Here is a list of some of the common equivalents.

TABLE OF EQUIVALENTS

Per Cent	*Fraction*	*Per Cent*	*Fraction*	*Per Cent*	*Fraction*
25%	$\frac{1}{4}$	60%	$\frac{3}{5}$	62.5%	$\frac{5}{8}$
75%	$\frac{3}{4}$	12.5%	$\frac{1}{8}$	70%	$\frac{7}{10}$
20%	$\frac{1}{5}$	37.5%	$\frac{3}{8}$	90%	$\frac{9}{10}$

1. Find 24% of 360. 24% is a little less than 25%, which is equal to $\frac{1}{4}$. $\frac{1}{4}$ of 360 is ? 86.4 ; 90

2. Find 51% of 60. 51% is a little more than ?, which is equal to ?. ? of 60 is ?. 51% of 60 should be a little more than ?. The exact answer is ?. 50% ; $\frac{1}{2}$; 30 ; 30 ; 30.6

3. Find 61% of 50. 61% is a little more than 60% which is equal to ?. ? of 50 is ?. 61% of 50 should be a little more than ?. The exact answer is ?. $\frac{3}{5}$; 30 ; 30 ; 30.5

Before doing any written computation on each of these problems, write your estimate of a reasonable answer. Then when you have completed your computation, see how close your estimate was.

4. Jane earned \$30 last week and saved 19% of what she earned. How much did she save? \$ 5.70

5. A tire priced at \$25 is sold at a reduction of 18%. What is the selling price? \$ 20.50

6. At an end-of-the-season sale Mary bought a pair of skates for \$6.15 that were regularly priced at \$8.00. What per cent of the regular price did she save? 23.125%

7. There were 650 students in Jefferson High School last year. The enrollment this year has increased by 8%. How many more pupils are there this year? 52

8. The Cleveland High School basketball team has played 14 games and won 9 of them. What per cent of its games has it won? 64.3%

9. Henry bought a motorcycle costing \$360. He paid 26% of the price in cash. How much was his payment? \$ 93.60

10. The Lakeside Band is earning money for new uniforms. The secretary says that so far they have earned \$304, which is 76% of what they need. How much do they need? \$ 400

11. Joe says his dog weighs 36 pounds. Joe weighs 140 pounds. The dog's weight is what per cent of Joe's weight? 25.7%

12. Last year the basketball team lost 35% of their games. They lost seven games. How many games did they play? 20

13. Helen saved \$4.80 by purchasing a camera for 12% off the regular price. What was the regular price of the camera? \$ 40

14. Jim earned \$13 last week and spent \$3 for lunches. What per cent of his earnings did he spend for lunches? 23.1%

15. The football team has played 11 games and won five of them. What per cent of its games has it won? 45.5%

16. A television set regularly priced at \$300 was sold for \$265. What was the per cent of reduction? 11.7%

17. Henry and George were driving from Centerville to Hazelton, a distance of 200 miles. After driving 123 miles, they stopped for lunch. What per cent of the distance to Hazelton had they driven? 61.5%

18. Mr. Adams earns \$6000 a year. He says that 26% of his salary goes for rent. How much is his annual rent? \$ 1560

19. There are 60 members in the camera club. Last week the secretary reported seven members absent. What per cent of the membership was present? 88.3%

Be sure to set up the conditional sentence for each problem.

1. Frank plans to save 40% of his earnings next summer for his college fund. If he earns \$350, how much should he save? *\$140*

2. According to recent statistics, the average earnings of college graduates is \$8000 per year. For those who did not attend high school the average is \$4500 per year. This is what per cent less than that of college graduates? *43.8%*

3. Jim Jones put \$200 in a savings bank that paid interest at the rate of 4% a year, payable every 3 months. He left his savings in the bank for a year without making deposits or withdrawals. Prepare a statement to show his balance at the end of each interval. *See front.*

4. Mr. Osen has a \$5000 $4\frac{1}{4}\%$ government bond that pays interest on November 15 and May 15. How much interest does he receive on each date? *\$106.25*

5. Stock in Pacific Gas and Electric Company was recently selling at \$34 a share. The company was paying dividends at the rate of \$1.40 per share annually. What was the rate of return? *4.1%*

6. Elmer sells newspapers on commission. Last week he earned \$12 by selling \$80 worth of newspapers. At that rate how much would he earn by selling \$250 worth of newspapers? *\$37.50*

7. Jane worked in a store last summer as an assistant cashier. She was paid $\frac{3}{5}$ as much as the cashier. Together they earned \$240 a week. What was Jane's weekly wage? *\$90*

8. Helen and Mabel sell subscriptions to a magazine in their spare time, receiving 25¢ per subscription. Together they sold 90 subscriptions last week, with Helen selling $\frac{4}{5}$ as many as Mabel. How much did each earn? *Mabel, \$12.50 ; Helen, \$10*

9. Eric works at a filling station after school and on Saturdays. When asked how much he earned last week he said, "If I had earned \$25 less than 4 times as much as I earned, it would have amounted to the same as if I had earned \$3 more than 3 times what I did earn." How much did he earn? *\$28*

10. There are 16 more girls than boys in Madison High School. The total enrollment is 540. How many girls are enrolled? *278*

11. A non-stop bus travels from Lakeview to Hillcrest, a distance of 120 miles, in $2\frac{1}{2}$ hours. At that rate how long will it take to travel from Hillcrest to Sunnyvale, a distance of 192 miles? *4 hr.*

Rearranging Formulas

Suppose you know that the area of a rectangle is 72 square rods and its width is 6 rods. You are to find its length. You can use the formula in the form: $A = lw$, substituting values for the known variables to obtain the conditional statement: $72 = 6l$.

A more direct method, especially if you are to use the formula more than once, is to rearrange the formula so as to isolate the variable, l, using axioms as they apply. Thus, $A = lw$. Multiply both sides by $\frac{1}{w}$ to obtain $\frac{A}{w} = l$. Substituting $A = 72$ and $l = 6$ you have $\frac{72}{6} = l = 12$.

EXAMPLE

What is the radius of a circle whose area is 154 square feet?

$$A = \pi r^2. \quad \text{Multiply both sides by } \frac{1}{\pi}.$$

$$\frac{A}{\pi} = r^2. \quad \text{Using } \pi \approx 3\frac{1}{7}, \frac{A}{\pi} = 154 \times \frac{7}{22} = 49$$

$$49 = r^2 \quad \text{then } r = 7$$

First rearrange the given formula to isolate the proper variable. Then substitute values and solve.

1. The perimeter of a rectangle is 360 rods. Its length is 120 rods. What is its width? ($P = 2l + 2w$.) *60 rd.*
2. The area of a triangle is 90 square feet. The base is 15 feet. What is the altitude? ($A = \frac{1}{2}ab$.) *12 ft.*
3. The area of a circle is 616 square feet. What is the measure of a diameter? ($A = \pi r^2$. $\pi \approx 3\frac{1}{7}$.) *28 ft.*
4. How long will it take to travel 720 miles at an average rate of 48 miles an hour? ($d = rt$.) *15 hr.*
5. The perimeter of a regular hexagon is 216 inches. How long is each side? ($P = 6s$.) *36 in.*
6. The volume of a cylinder 16 inches high is 2464 cubic inches. What is its diameter? ($V = \pi r^2 h$. $\pi \approx 3\frac{1}{7}$.) *14 in.*
7. The volume of a cone is 770 cubic inches. The radius of the base is 7 inches. What is the height? ($V = \frac{1}{3}\pi r^2 h$.) *15 in.*

Savings accounts and investments provide a reserve to meet unexpected expenses of sickness and accident. They may also prepare for major expenses such as a college education, a new home, or retirement.

There is always a risk that a wage earner's death or disablement will deprive his family of an income. Life insurance provides protection for the family should such an event occur. Protection consists of cash payments to a beneficiary. These may be either in a lump sum upon the death of a policy holder or in equal payments over a stated period of time following death. As we shall see, there are different kinds of life insurance which serve different purposes.

Many forms of insurance are available. Most old-line companies require a physical examination to certify that you meet specified health standards. Some companies waive this requirement when a person is covered in a group insurance. Group insurance is issued by insurance companies through employers. This type of insurance is written to cover a group of employees. Savings-bank insurance is becoming more and more available. Also, government life insurance has been made available to servicemen.

ORDINARY LIFE

Ordinary life insurance premiums are paid to the company during the life of the insured. Upon the policy holder's death, the insurance company pays the *beneficiary* the face amount of the policy. This type of policy is taken out mainly for immediate protection of a family.

LIMITED PAYMENT

A person who wishes to complete payment of his insurance premiums while his earning ability is highest may take out a limited payment policy. For a specified period, for instance 20 years, the insured pays premiums. At the end of this time his policy is paid up. Upon the death of the insured, whether it occurs before or after the policy is paid up, his beneficiary receives the face amount of the policy. Rates on this type of policy are higher than equivalent coverage under ordinary life.

ENDOWMENT INSURANCE

The endowment policy combines savings with limited payment insurance. The insured pays premiums for a specified period, as in the limited payment policy. If he dies during that period, his beneficiary receives the face amount of the policy. If he lives until the policy is paid up, the insured is entitled to receive the full face amount of the policy.

The cost of each kind of life insurance depends on how much saving is combined with the protection it affords. The following annual premium rates have been published by a large life insurance company for each $1000 coverage.

ANNUAL PREMIUM FOR $1000 POLICY

		Limited Payment		*Endowment*	
Issued at age:	*Ordinary Life*	*20-payment*	*30-payment*	*At age sixty*	*20-payment*
20	$15.49	$23.31	$18.72	$20.13	$43.90
25	17.42	25.51	20.52	23.68	44.21
30	19.91	28.16	22.77	28.60	44.58
35	23.24	31.51	25.73	35.82	45.63
40	27.71	35.76	29.68	47.05	47.05
45	33.69	41.18	35.00	65.82	49.81
50	41.70	48.18	42.27	103.09	54.31

1. Why is endowment insurance an expensive form of insurance? *It includes a considerable amount of saving.*

2. Why is an ordinary life insurance premium less than 20-payment life insurance for a given age? *The premiums will probably be paid over a longer term.*

3. By what per cent does the annual premium on ordinary life insurance increase between the ages of 25 and 35? 33.4%

4. What is the per cent of increase during the same period on 20-payment endowment insurance? 3.2%

5. The annual premium remains the same as it was in the year the policy was issued. How much cheaper is the total premium on a 20-payment life insurance policy if taken out at the age of 20 instead of the age of 40? $ 249

Find the annual premium on each of the following policies:

6. Ordinary life insurance issued at age 35 for $5000 $ 116.20

7. 20-payment life insurance issued at age 25 for $30,000 $ 765.30

8. Endowment at 60 issued at age 40 for $10,000 $ 470.50

9. 30-payment life insurance issued at age 20 for $7000 $ 131.04

10. Ordinary life insurance issued at age 45 for $7000 $ 235.83

11. 30-payment life insurance issued at age 30 for $8000 $ 182.16

12. 20-payment endowment issued at age 35 for $15,000 $ 684.45

13. 20-payment life insurance issued at age 35 for $20,000 $ 630.20

14. Endowment at age 60, issued at age 30 for $15,000 $ 429.00

A person buying life insurance should decide whether he is more interested in family income protection or in savings. Ordinary life insurance provides the greatest protection for a given premium. Endowment insurance provides a large measure of savings. Limited payment insurance provides a middle ground between the two.

1. What kind of insurance would you recommend for each of the following families? Give reasons for your judgment.
  a. A family of four, with no savings, with the wage earner in his early thirties.
  b. A family of three, with considerable savings, the wage earner in his late forties.
  c. A family of five, with no savings but a large salary, the wage earner in his early forties.

2. Mr. Jones is 35 years old and is able to save $40 a month out of his salary. He is thinking of four ways of using his savings: (a) buying corporation stock, (b) buying ordinary life insurance, (c) buying 20-payment life insurance, (d) buying 20-year endowment insurance. About how many thousand dollars of each kind of insurance could he get with his savings?

3. How would you advise him if he were a married man with five children for whom he wished the greatest protection?

4. Would you advise him differently if he had only one child? Explain.

5. The head of a family, in selecting life insurance, should plan to provide for his family in case of his death, and, if possible, to provide for income after he retires. Since endowment insurance is expensive, it takes a large income to accomplish both purposes through insurance.

Mr. Henderson is 30 years old, and has an income of $425 per month, of which he plans to save 15%. If he puts all his savings into life insurance, how large a policy (to the nearest $1000) can he buy of: $38,000 $27,000
a. Ordinary life insurance? $17,000 **b.** 20-payment life insurance?
c. 20-payment endowment insurance?

Calculate to the nearest $1000 how much insurance can be purchased by persons who can save the amounts listed:

	Amount	*Kind of Policy*		*Issued at Age*
6.	$75 per month	Ordinary life	$ 32,000	40
7.	$30 per week	20-payment life	$ 55,000	30

8. $250 per year	$5000	20-payment endowment	20
9. $25 per month	$10,000	Endowment at age 60	30

10. Mr. Arnold is 40 years old and is quite well-to-do. He has a wife and two children, and sufficient property to leave them well provided for. He is a contractor, and his income is quite large. He wishes to retire in twenty years. What kind of insurance would you recommend and why? How much can he buy with $100 a month? *See front.*

11. Mr. Smith is 35 years old. He has a wife and five children. He earns $750 a month, but up to the present has saved very little. He could save $100 a month from his salary. What kind of insurance do you think he should buy and how much? *See front.*

12. Mr. Adams has an income of $7500 a year. With careful budgeting, he could save 20%. How much will this amount to? *$ 1500*

13. Mr. Adams is 30 years old and has several dependents whom he wishes to protect while he accumulates his savings. How much 20-year endowment insurance can he buy with 20% of his salary? *$ 33,000*

14. How much would the same amount of ordinary life insurance cost him? *$ 657.03*

15. Mr. Adams believes that if he buys an amount of ordinary life insurance as found in Exercise 14 and invests the rest of his savings himself, he will have more money in twenty years than if he buys endowment insurance. How much will he have left to invest each year? *$ 842.97*

16. Mr. Adams figures that by buying stocks and bonds he can secure 5% interest on his savings. For each $10 saved per year at 5% he will have $347.20 at the end of twenty years. What will his total savings amount to in twenty years? *$ 29,268.96 (using 84.3 from problem 15)*

17. An insurance company offers a $1000 ordinary life policy taken out at age 35 for $24.83 annually. A person carrying this insurance for 8 years decided to cash in the policy for $116.80. How much did this protection cost him for the 8 years? *$ 81.84*

18. If the annual premium of a term insurance policy is $33.35 per $1000, how much would an $18,000 policy cost Mr. Banks in a 10-year period? *$ 6084.84*

19. A life insurance company invested its reserve funds in the following manner. Present this information using a circle graph.

Government bonds	40%	*144°*	Business securities	30%	*108°*
Loans	5%	*18°*	Real estate	5%	*18°*
Mortgages	20%	*72°*			

(Students should make their own graphs; the measures of the central angles are given.)

Part One

Solve and check each of these conditional statements.

A. **1.** $x + 14 = 27$ 13

2. $\frac{x}{2} + 3 = 13$ 20

3. $\frac{x}{12} = \frac{1}{3}$ 4

4. $2x + 5 = 9$ 2

5. $\frac{x}{6} = \frac{2}{3}$ 4

6. $\frac{2x}{3} + 14 = 20$ 9

7. $3x = 16 - x$ 4

8. $4x + 3 = 65$ 15.5

9. $5x - 7 = 2x - 1$ 2

10. $3x + 4 = 7 + 2x$ 3

B. **1.** $n\%$ of 25 is 6 24

2. 20% of n is 30 150

3. n is 50% of 28 14

4. 30 is $n\%$ of 15 200

5. n is 40% of 120 48

6. $n\%$ of 32 is 40 125

7. 80% of n is 28 35

8. 19 is $n\%$ of 95 20

9. 35% of 86 is n 30.1

10. 120% of n is 60 50

C. **1.** $\frac{x}{9} = \frac{2}{3}$ 6

2. $\frac{9}{12} = \frac{x}{48}$ 36

3. $\frac{3}{18} = \frac{x}{20}$ $3\frac{2}{3}$

4. $\frac{5}{18} = \frac{15}{x}$ 54

5. $\frac{5}{9} = \frac{20}{x}$ 36

6. $\frac{6}{16} = \frac{x}{21}$ $7\frac{7}{8}$

7. $\frac{x}{12} = \frac{12}{18}$ 8

8. $\frac{4}{9} = \frac{16}{x}$ 36

9. $\frac{5}{15} = \frac{x}{9}$ 3

10. $\frac{6}{x} = \frac{9}{12}$ 8

11. $\frac{15}{x} = \frac{5}{6}$ 18

12. $\frac{8}{x} = \frac{5}{18}$ $28\frac{4}{5}$

Find the interest on the following sums.

1. \$150, 2 years, 6% \$ 18

2. \$240, 3 years, 4% \$ 28.80

3. \$516, 2 years, 5% \$ 51.60

4. \$2600, 90 days, $4\frac{1}{2}\%$ \$ 29.25

5. \$640, 60 days, 5% \$ 5.33

6. \$480, 60 days, 4% \$ 3.20

7. \$350, 90 days, $5\frac{1}{2}\%$ \$ 4.81

8. \$5000, 72 days, 6% \$ 60

9. \$480, 120 days, $4\frac{1}{2}\%$ \$ 7.20

10. \$900, 60 days, 5% \$ 7.50

Part Two

List the numerals 1 through 13 on a sheet of paper. Then read each of the following statements carefully. After each numeral write the letter

indicating your choice as to the word or phrase that is defined in the statement having that number.

1. A share in the ownership of a corporation c
   a. bond **b.** dividends **c.** stock **d.** yield

2. A plan defining in advance allocations in the expenditure of income b
   a. taxes **b.** budget **c.** moratorium **d.** interest

3. The amount a share of stock will sell for on the stock exchange d
   a. income **b.** interest **c.** return **d.** market value

4. A share of profits in a corporation b
   a. yield **b.** dividends **c.** interest **d.** bond

5. The per cent the annual income is of the amount invested
   a. rate of return **b.** premium **c.** term **d.** income a

6. The amount to be repaid as stated on a bond d
   a. market value **b.** endowment **c.** dividend **d.** face value

7. A written promise by a corporation to repay a loan a
   a. bond **b.** stock **c.** premium **d.** interest

8. The length of time for which a loan is made d
   a. premium **b.** beneficiary **c.** rate **d.** term

9. The interest earned on a bond since the latest payment c
   a. cumulative amount **b.** yield **c.** accrued interest
   d. market value

10. The person you would see to purchase stock in a large corporation d
    a. the treasurer **b.** a stock exchange official
    c. beneficiary **d.** stock broker

11. The regular interest payment ordinarily made on a \$5000 $4\frac{1}{2}\%$ bond c
    a. \$225 annually **b.** \$225 semiannually **c.** \$112.50 semiannually
    d. \$112.50 quarterly

12. The stock, among the four listed below, that has the highest rate of return c

	Market value of stock	Annual dividend
a. 2.9 %	\$85	\$2.50
b. 3.2 %	\$34	\$1.10
c. 4.4 %	\$41	\$1.80
d. 3.6 %	\$55	\$2.00

13. Find the stock among the four listed above that has the lowest rate of return. a

Part Three

1. Mr. Bush has five \$1000 bonds issued by United States Steel which pay $5\frac{1}{2}\%$ interest per year. The interest is paid each six months. How much is paid on each interest date? \$ 137.50

2. Jim Brown works afternoons and evenings at the Acme Supermarket. He earns \$25 a week. He plans to save 40% of what he earns in order to buy a camera. How much does he plan to save each week? \$ 10

3. The camera Jim plans to buy will cost \$60. How many weeks will it take him to save this amount? 6 weeks

4. Mr. McConnell purchased 100 shares of American Can stock at \$47 a share. Last year his dividends were \$200. What was the rate of return on his investment? 4.3 %

5. The Jones Auto Sales Company was organized with 17,000 shares of stock selling for a total of \$850,000. What was the value of each share? \$ 50

6. During the first year of operation the Jones Auto Sales Corporation earned \$51,000. Of this amount \$17,000 was retained in the business to improve plant and equipment, and the rest was distributed as dividends. How much was distributed as dividends? \$ 34,000

7. How much was the dividend per share? \$ 2

8. What was the rate of return on the investment? 4 %

9. The General Motors Corporation has 283,545,000 shares of common stock. Recently the Corporation distributed \$708,862,500 as annual dividends on common stock. How much were the dividends per share? \$ 2.50

10. Mr. Bauer owned 200 shares of General Motors stock, which he had purchased at \$55 per share. What was the total amount he paid for his stock? \$ 11,000

11. How much were his annual dividends? \$ 500

12. What was the rate of return on his investment? 4.5 %

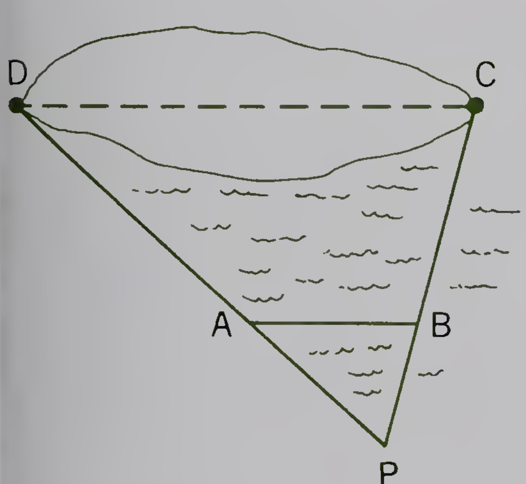
13. Mr. Anderson owns a \$5000 $4\frac{1}{2}\%$ bond issued by the Pacific Gas and Electric Company, that pays interest twice a year. How much does he receive on each interest date? \$ 112.50

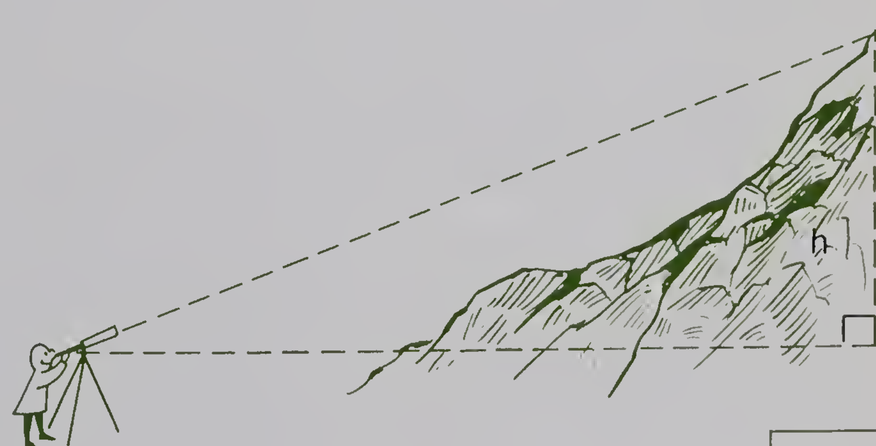
14. In 1960 there were 31,400 mathematicians in this country. 207% of that number will be needed by 1970. How many more mathematicians will be needed than we had in 1960? 33,598

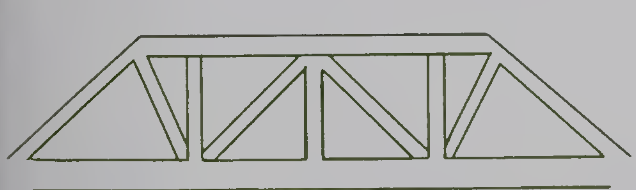
THE RIGHT TRIANGLE

WORDS TO WATCH FOR

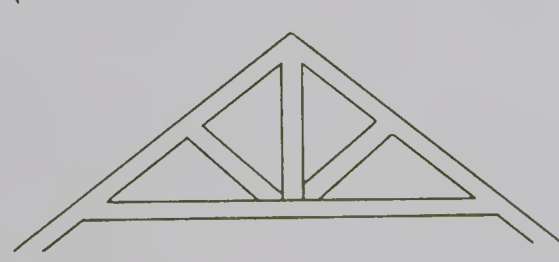
angle of depression
angle of elevation
corresponding angles
corresponding sides
cosine ratio
equilateral
hypotenuse
indirect measurement
legs of a triangle
perfect square
Pythagorean formula
Pythagorean triples
radical
scalene
sine ratio
square root
tangent ratio
trigonometric ratios



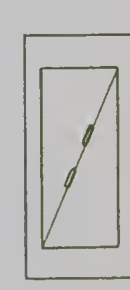




BRIDGE SPAN



ROOF TRUSS



DOOR BRACE

The triangle is the most widely used geometric figure. It is used by the designer and the architect to provide forms that are pleasing to the eye.

The craftsman and engineer use it to secure rigidity in construction. The surveyor uses triangles as a basis for measurement of distances over water or impassable terrain, and the astronomer uses them in measuring distances to the planets.

In the illustrations on the previous page you will find examples of each of the uses of triangles mentioned. Which of the uses discussed does each figure represent?

List another example of each of these three uses of the triangle:

a. Design **b.** Construction **c.** Measurement

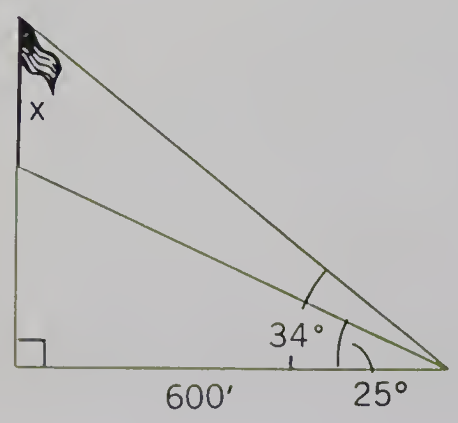
Bring to class other examples of these uses that you can find in periodicals, newspapers, or in your locale.

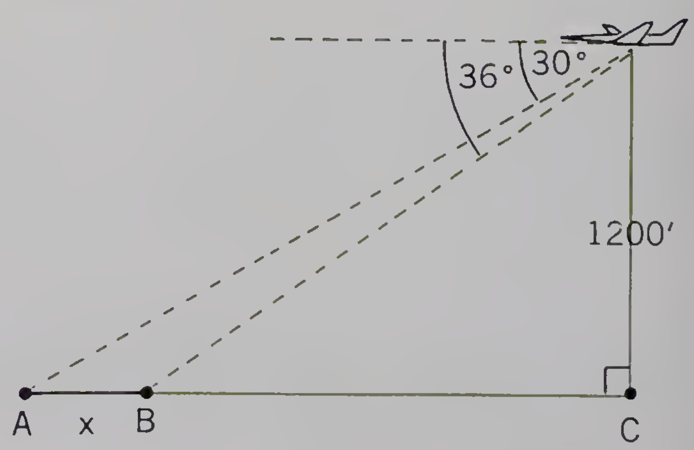
If you look around the room, you will find that there are more right angles than any other kind. The intersection of walls and ceiling, walls and floor, the door and window frames, the panes of glass in the window—all suggest perpendicular lines or planes intersecting to form right angles. What other examples can you find within and outside the room that suggest right angles?

The prevalence of right angles is not an accident. Where a vertical line (as determined by a plumb line) intersects a horizontal line (as determined by a spirit level) a right angle is formed. The right angle is a basic feature of any major construction. For this reason many instruments for construction feature the right angle. The carpenter's square, the cabinet-maker's try square, and the draftsman's T-square are a few examples.

In many instances a right angle occurs where construction is designed to support the weight of a horizontal load, as in bridge construction. In the beautifully designed Glen Canyon bridge, an arch supports the vertical members that carry the weight of the roadway from below.

Not so evident but equally important are the uses of the right triangle in problems where the height of an object is to be measured indirectly. The important relationships that exist between the parts of a right triangle provide the means for calculations of height and distance where it is impractical or impossible to measure directly.





It was known by the ancient Egyptians that a triangle whose sides have measures proportional to 3, 4, and 5 is a right triangle. In the process of laying out the boundaries of their fields after the periodic floods of the Nile river, surveyors used a rope with knots at intervals of 3, 4, and 5 units. When the rope was stretched out so that the segments were straight and the ends were joined, a right angle was formed at the knot between the 3 and 4 segments. Using this property, the surveyors could lay out rectangular plots.

Egyptian Surveying

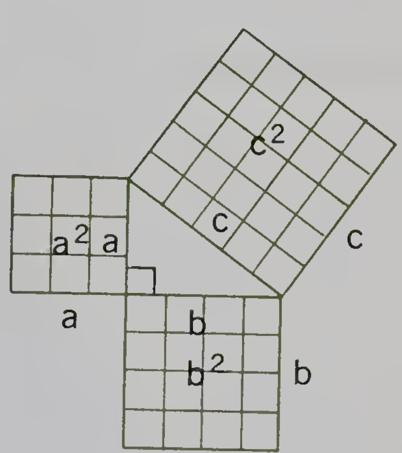
The Greek mathematicians were deeply interested in this property and analyzed it further. They found other proportions among the measures of the sides that produced right triangles. Some of these are:

5, 12, and 13; 7, 24, and 25; 8, 15, and 17; 9, 40, and 41

Later it was found that these are only special cases of a general property.

The square of the measure of the hypotenuse of a right triangle is equal to the sum of the squares of the measures of the other two sides.

You will recall that to *square* a number means to multiply it by itself, or in other words, to use the number twice as a factor. Thus $5^2 = 5 \times 5$. The longest side of a right triangle is called the hypotenuse. The other two sides are called the legs. In expressing this rule as a formula, we usually use a single lower-case letter to designate the measure of a side of a right triangle: a and b designate the measures of the legs of the right triangle, and c designates the measure of the hypotenuse. Thus the formula becomes: $c^2 = a^2 + b^2$.

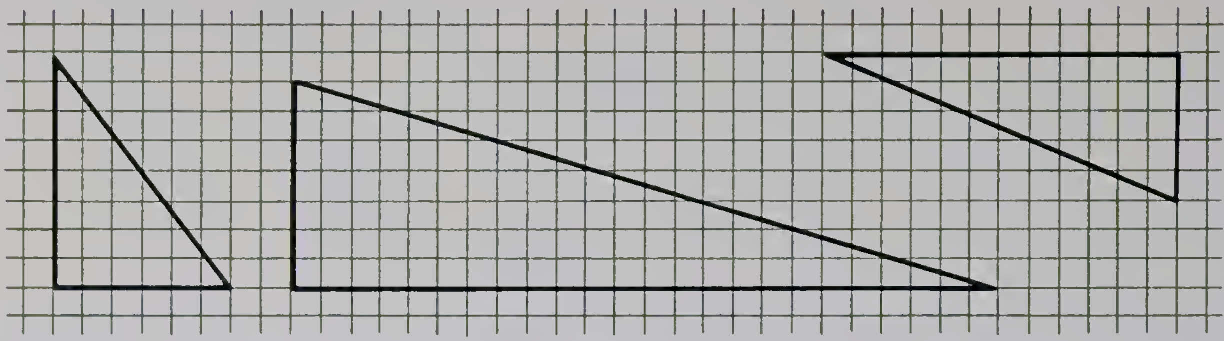


1. In the triangle above, $a = 3$, $b = 4$, and $c = 5$. Does

$a^2 + b^2 = c^2$? *yes* $c^2 - a^2 = b^2$? *yes* $c^2 - b^2 = a^2$? *yes*

Does the addend-sum relationship show that these equations are true? Use the numbers to see if they form true statements.

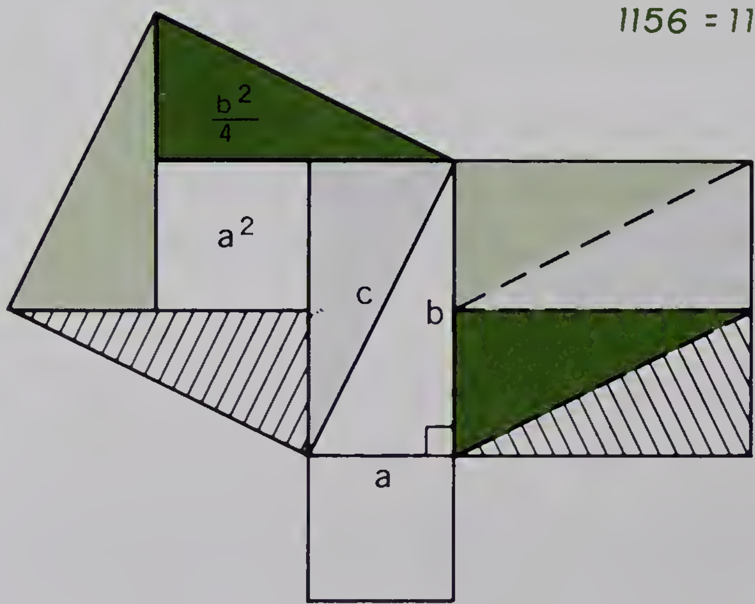
2. In each of these right triangles, determine the value of a, b, and c. Then show that in each triangle $a^2 + b^2 = c^2$. See front.



3. Using squares cut from graph paper, as shown in the triangles above, construct triangles $a = 5$, $b = 12$, $c = 13$ and $a = 8$, $b = 15$, $c = 17$.
4. Determine whether $a^2 + b^2 = c^2$, $c^2 - a^2 = b^2$, and $c^2 - b^2 = a^2$. yes
5. The relationship you have been using which is described by $a^2 + b^2 = c^2$ was taught by the Greek philosopher, Pythagoras, and is thus called the Pythagorean Rule or Pythagorean Theorem. The sets of numbers that can be used to construct a right triangle, such as 3, 4, and 5; 5, 12, and 13; and so forth, are called *Pythagorean triples*. The multiples of a set of triples are also Pythagorean triples: $a = 10$, $b = 24$, and $c = 26$ is a multiple of which triple? 5,12,13
6. Another set of Pythagorean triples is 13, 84, and 85. Show that the Pythagorean rule holds for this set. See front.
7. Is a triangle with sides 16′, 30′ and 34′ a right triangle? Explain.

Many ways of establishing the Pythagorean property have been presented. President Garfield was responsible for developing a proof of this theorem. You may be interested in finding this proof and seeing how he established that $a^2 + b^2 = c^2$. This right triangle property can be demonstrated by cutting the squares a^2 and b^2 in various ways and fitting them together to form a square c^2.

yes ; $16^2 + 30^2 = 34^2$
$256 + 900 = 1156$
$1156 = 1156$



The inverse operation to squaring a number is finding the *square root* of a number. The symbol for the operation is $\sqrt{\ }$. Thus $\sqrt{81}$ names one of the two *non-negative* equal factors of 81, which you recognize as 9.

1. If the square root of a number is a rational number, it is called a *perfect square*. These numbers are perfect squares:
a. 16 4 **b.** 25 5 **c.** 121 11 **d.** 169 13 **e.** 49 7 **f.** 64 8 **g.** $\frac{1}{4}$ $\frac{1}{2}$
What is the square root of each?

The table of squares (in the back of the book) is convenient for finding the square root of many perfect squares.

EXAMPLE

Find: $\sqrt{729}$
In the table of squares, under n^2, find 729.
Opposite 729, under n, find 27. Then: $\sqrt{729} = 27$

2. Find the square root of each of the following:

a. 841 29	**c.** 1681 41	**e.** 1225 35	**g.** 961 31
b. 1156 34	**d.** 784 28	**f.** 2601 51	**h.** 1936 44

3. By finding the square root of c^2, you can determine the length of the hypotenuse of a right triangle if you know the measures of its legs.

EXAMPLE

In the right triangle ABC, if $a = 10$, $b = 24$, $c = ?$
Then $c^2 = 100 + 576$ and $c = \sqrt{676}$. Why?
Under n^2 in the table of squares find 676. Opposite 676, under n, is _?_. What is the length of the hypotenuse? 26

Use the Pythagorean formula and the table of squares to find c.

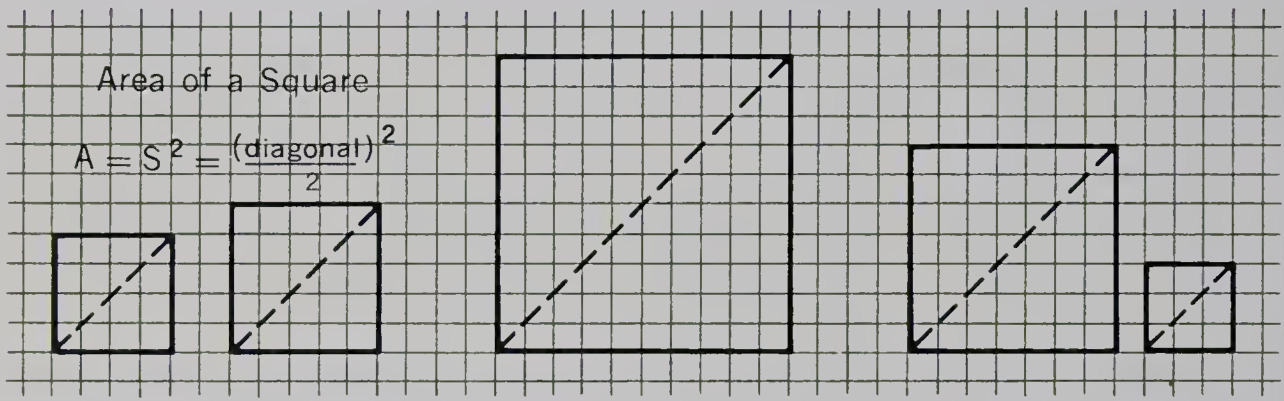
	a	b	c		a	b	c
4.	5	12	☐ 13	**9.**	14	48	☐ 50
5.	6	8	☐ 10	**10.**	24	32	☐ 40
6.	18	80	☐ 82	**11.**	25	60	☐ 65
7.	12	16	☐ 20	**12.**	21	72	☐ 75
8.	16	30	☐ 34	**13.**	9	12	☐ 15

In these exercises you will be using the squares of numbers between 1 and 25 extensively. You may have already memorized most of them. Make a list of those you do not readily remember, and refer to them when necessary.

g. 256 sq. rd.
h. 625 sq. yd.

1. You will recall that the formula for the area of a region enclosed by a square is $A = s^2$, where s represents the measure of a side of the square. Find the area of squares with sides measuring as follows:

a. 18 in. 324 sq. in. **c.** 19 rd. 361 sq. rd. **e.** 21 yd. 441 sq. yd. **g.** 16 rd.
b. 23 in. 529 sq. in. **d.** 24 in. 576 sq. in. **f.** 22 ft. 484 sq. ft. **h.** 25 yd.



2. Figure 1 below shows a rectangle whose length measures twice that of its width. The measure of its area is equal to the sum of the measures of the areas of the two square regions into which the rectangular region can be divided. Find the area of the rectangular region without written computation. 50 sq. units

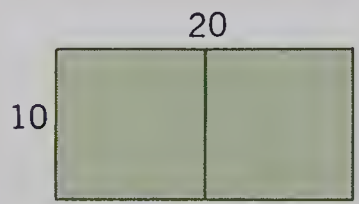


Figure 1

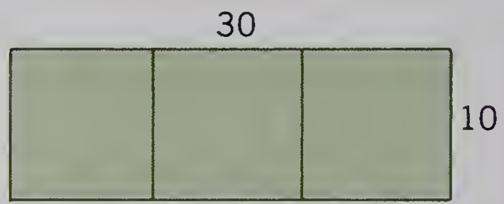


Figure 2

3. The rectangle in Figure 2 above can be divided into three square regions. Find the area of the rectangular region without written computation. 75 units

4. Before doing any calculation, make a sketch of each of the following rectangles. Then calculate the area of each rectangular region without written calculation.

	Length	*Width*			*Length*	*Width*
a.	16 in. 128 sq. in.	8 in.	800 sq. rd.	**d.**	40 rd.	20 rd.
b.	24 in. 288 sq. in.	12 in.	675 sq. yd.	**e.**	45 yd.	15 yd.
c.	28 ft. 196 sq. ft.	7 ft.	900 sq. in.	**f.**	60 in.	15 in.

We frequently need to find the square root of a number that is not a perfect square. Suppose we were to find the side of a square whose area is 30 square feet. We know that 30 is not a perfect square because $\sqrt{30}$ is not a rational number. In other words, there is no rational number when multiplied by itself will equal 30. We can find, however, an *approximate* square root to any desired number of decimal places by using the simple procedure that follows:

EXAMPLE

Find $\sqrt{30}$ to 3 decimal places.

1. You know that $\sqrt{25} = 5$ and $\sqrt{36} = 6$.
2. Then $\sqrt{30}$ is between 5 and 6. Try 5.5, the average of 5 and 6. $30 \div 5.5 = 5.45$
3. Since the two factors 5.5 and 5.45 are not equal, the square root of 30 lies between them. Try the average of 5.5 and 5.45: 5.475. $30 \div 5.475 = 5.4795$
4. The average of 5.475 and 5.4795 is 5.4773 so 5.477 is the square root of 30 correct to 3 decimal places, and the fourth place is a close approximation.

In general, the steps for finding an approximate square root are:

Step 1. Find the square roots of perfect squares next above and below the number whose square root you wish to find. For large numbers, use the table of squares.

Step 2. Divide the given number by the average of the numbers found in Step 1.

Step 3. Find the average of the divisor and quotient of Step 2. If you need a more exact square root, use this average as a new divisor, and find the quotient to an additional decimal place.

Step 4. Find the average of the divisor and quotient of Step 3 to find a more exact square root.

Find the square root of the following correct to three decimal places.

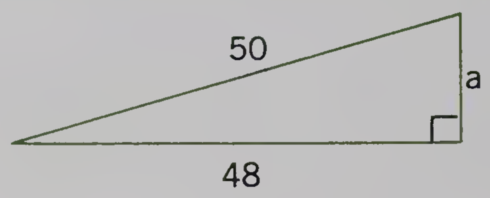
1. 40 6.325	**6.** 350 18.708	**11.** 700 26.458	**16.** 1550 39.370
2. 75 8.660	**7.** 215 14.663	**12.** 410 20.248	**17.** 3200 56.569
3. 28 5.292	**8.** 175 13.229	**13.** 340 18.439	**18.** 1235 35.143
4. 55 7.416	**9.** 650 25.495	**14.** 555 23.558	**19.** 4234 65.069
5. 406 20.149	**10.** 896 29.933	**15.** 2510 50.100	**20.** 1725 41.533

If you know the measures of the hypotenuse and one leg of a right triangle, and need to know the length of the other side, you apply the addend-addend-sum relationship to the Pythagorean formula. That is,

If	$a^2 + b^2 = c^2$	$\text{addend}_1 + \text{addend}_2 = \text{sum}$
then	$a^2 = c^2 - b^2$	$\text{addend}_1 = \text{sum} - \text{addend}_2$
or	$b^2 = c^2 - a^2$	$\text{addend}_2 = \text{sum} - \text{addend}_1$

Therefore, each of these equations is equivalent.

EXAMPLE



Find the length of a.

$a^2 = c^2 - b^2$

a. Replace c^2 by 50^2 and b^2 by 48^2. $\quad a^2 = 50^2 - 48^2$

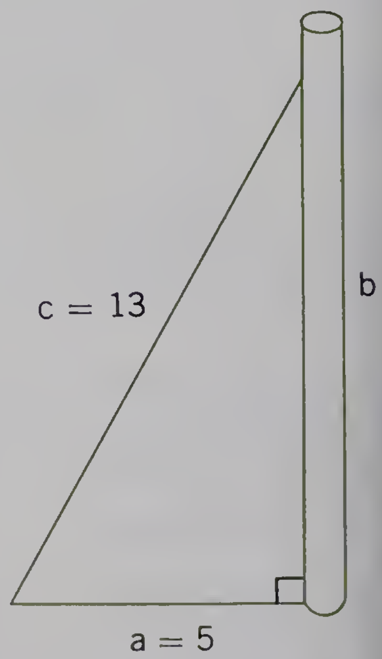
b. Write the square of 50 and 48 $\quad a^2 = 2500 - 2304 = 196$

c. Use the table of squares to find a. $\quad a = 14$

Find the length of the unknown side by using the appropriate form of the Pythagorean formula.

	a	b	c		a	b	c
1.	10	□	26 *24*	**4.**	15	36	□ *39*
2.	□	21	75 *72*	**5.**	18	□	82 *80*
3.	11	□	61 *60*	**6.**	□	15	25 *20*

Before trying to solve each of these problems, first make a sketch (as in Exercise 7) and label sides a, b, and c.



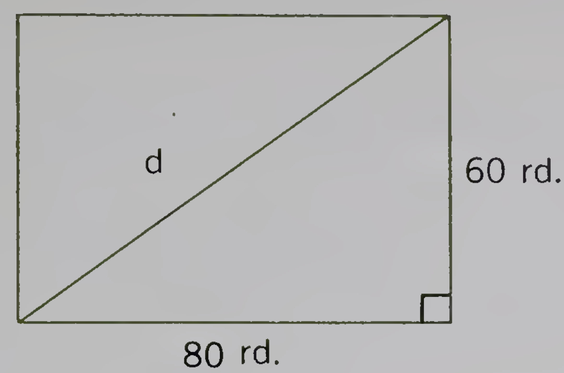
7. A telephone pole is to be braced by a prop 13 feet long, which is to be set in the ground 5 feet from the pole. How high up the pole will the prop reach? (See the figure at the right.) *12 ft.*

40 ft.

8. A 41-foot ladder leans against a wall with its base 9 feet from the wall. How high up the wall does the ladder reach? Draw a figure similar to the one for Exercise 7.

9. A 34-foot ladder rests against a wall with the top of the ladder resting against the sill 30 feet from the ground. How far is the foot of the ladder from the house? *16 ft.*

10. **a.** What is the length of the diagonal of a rectangular field that measures 60 rods wide and 80 rods long? **b.** How long is the diagonal of a rectangular field 60 feet long and 25 feet wide? *100 rd.; 65 ft.*



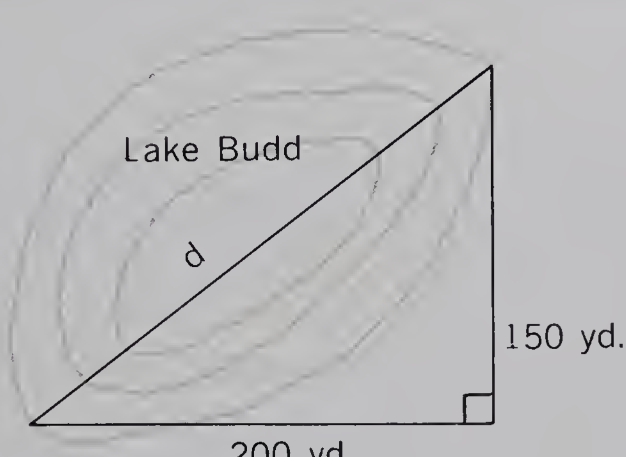
11. In order to get to school by following the sidewalk, Jim has to walk 120 yards north and 160 yards east. If he takes a shortcut across the diagonal of the field, how much distance can Jim save? *80 yd.*

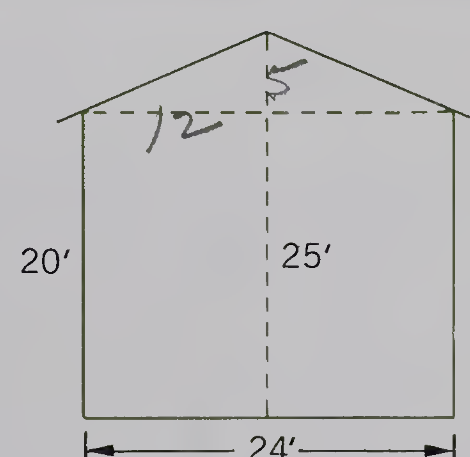
12. A baseball diamond is square and measures 90 feet on each side. What is the distance from first base to third base? *127.3 ft.*

13. A tree is supported by a cable fastened 24 feet above the ground. How long is the cable if it is fastened on the ground 10 feet from the base of the tree? *26 ft.*

14. While Mike is flying his kite, Henry is standing directly beneath it. When 200 feet of string are out, Mike is 56 feet from Henry. How high is the kite above Henry? *192 ft.*

15. To measure the distance across a lake, the Boy Scouts made the measurements shown below. How far is it across the lake? *250 ft.*





16. The ridge of a garage is 25 feet above the ground. The sides of the garage are 20 feet tall; the width is 24 feet. How long must each rafter be if the overhang is 2 feet? *15 ft.*

17. The rectangular swimming pool in the park is 60 feet long and 25 feet wide. Mary swam diagonally from one corner of the pool to the other and back 10 times. How many feet did she swim? *1300 ft.*

In Chapter 3 you examined the set of rational numbers that can be expressed as $\frac{a}{b}$, where a is an integer and b is a natural number. Thus, 5, $\frac{7}{4}$, and $\frac{3}{13}$ are all rational numbers. Expressed as decimals, they are 5.0, 1.75, and $0.\overline{230769}$. When a rational number is expressed as a decimal, it either repeats or terminates. (There are those who say that all rational numbers repeat when expressed as decimals. For example, 5 or $5.\overline{0}$ name the same number. Also, $\frac{24}{5} = 4.8\overline{0}$ and $\frac{17}{4} = 4.25\overline{0}$.)

A repeating decimal must be carefully written. In writing the decimal equivalent of $\frac{1}{3}$, if you write 0.3 you have written a terminating decimal, which is the equivalent of $\frac{3}{10}$, but which is only an approximation of $\frac{1}{3}$. If you write $0.3\cdots$, you have indicated that the decimal is endless, but you have given no clue to the following digits. The decimal equivalent of $\frac{1}{3}$ is $0.\overline{3}$ which shows that the 3 may be repeated endlessly.

The square roots of numbers that are not perfect squares neither terminate nor repeat. $\sqrt{5}$, for example, is $2.2361\cdots$. There is no repeating cycle of digits, no matter how far the calculation is carried out.

The square roots of 2, 3, 5, and so on are elements in the set of irrational numbers. They are, however, real numbers even though they cannot be expressed in the form $\frac{a}{b}$.

There are other irrational numbers besides the square roots of numbers that are not perfect squares. There are, for example, the cube roots of numbers that are not perfect cubes. There are the trigonometric ratios that you will become acquainted with later in this chapter. Another irrational number that you have frequently used is π, which is the ratio of the circumference of a circle to the measure of its diameter. This ratio, as you know, is expressed as a decimal that does not terminate, beginning: $3.141592\cdots$. You can readily calculate this value to as many places as you wish, using the formula:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \pm \frac{1}{2n - 1}$$

n is the number of terms, and $\pm$ indicates that the general term is plus or minus depending on whether it is an odd or even term of the series.

You can calculate the value to any desired precision, but the decimal will never terminate.

The symbol $\sqrt{\ }$ is called a *radical*. We can compute with square roots when they are expressed as radicals. Thus $\sqrt{5}$ is one of the two equal factors of 5. Then $\sqrt{5} \cdot \sqrt{5} = 5$. Or $(\sqrt{5})^2 = 5$.

1. Which of the following are rational numbers? b, d

a. $\sqrt{7}$ **b.** $\sqrt{9}$ **c.** $\sqrt{11}$ **d.** $\sqrt{16}$ **e.** $\sqrt{19}$

2. Find the value for each expression.

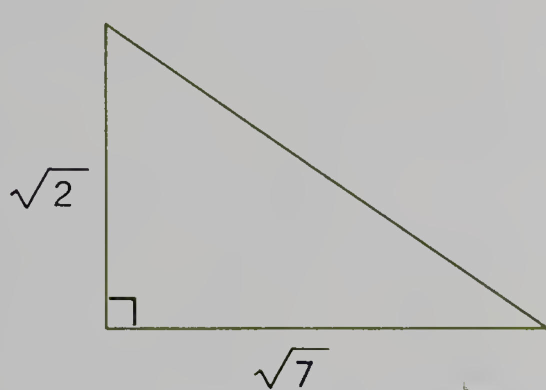
a. $\sqrt{2} \cdot \sqrt{2}$ 2 **c.** $\sqrt{3} \cdot \sqrt{3}$ 3 **e.** $\sqrt{11} \cdot \sqrt{11}$ 11

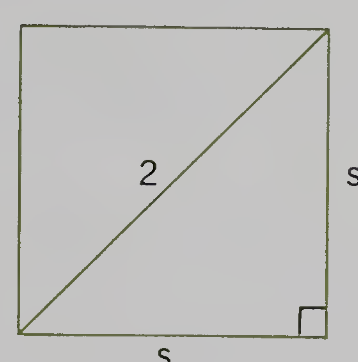
b. $\sqrt{7} \cdot \sqrt{7}$ 7 **d.** $(\sqrt{6})^2$ 6 **f.** $(\sqrt{13})^2$ 13

In the following exercises, if your answer is an irrational number, express it with a radical. That is, do not compute the square root.

3. One leg of a right triangle is $\sqrt{2}$, the other is $\sqrt{7}$. What is the measure of the hypotenuse?

HINT: $a = \sqrt{2}$ What is a^2? $b = \sqrt{7}$ What is b^2? What is c^2? What is c? 2 ; 7 ; 9 ; 3



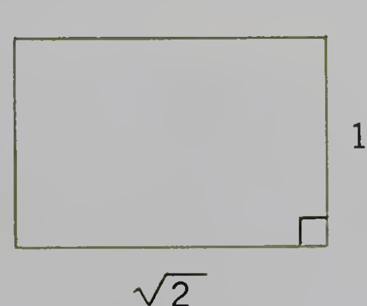


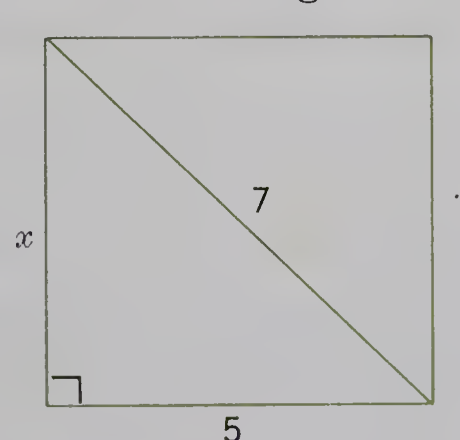
4. What is the measure of each side of the square above on the right whose diagonal is 2?

HINT: $c = 2$ $c^2 = \square$ $c^2 = 2s^2$ Why? Find s^2. 2 Find s. $\sqrt{2}$

4: $a^2 + b^2 \rightarrow s^2 + s^2 = 2s^2$

5. The measure of the length of a rectangle is $\sqrt{2}$. The measure of the width is 1. What is the measure of the diagonal? $\sqrt{3}$





6. The measure of a diagonal of a rectangle is 7. The measure of the longer side is 5. What is the measure of the shorter side? $\sqrt{24}$

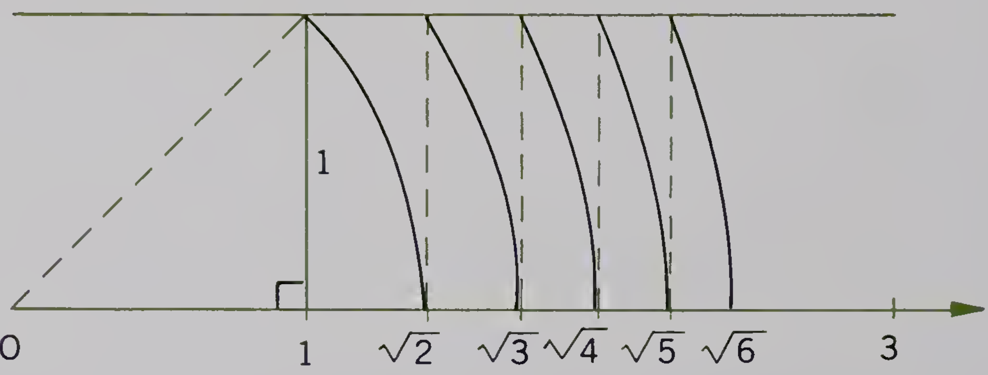
7. Find the measure of the unknown side in each of these right triangles. The variable c represents the hypotenuse.

	a	b	c		a	b	c
a.	3	5	? $\sqrt{34}$	**d.**	? 8	6	10
b.	2	? 2	$\sqrt{8}$	**e.**	4	? $\sqrt{20}$	6
c.	$\sqrt{2}$	$\sqrt{7}$	? 3	**f.**	$\sqrt{5}$	$\sqrt{11}$	? 4

8. Find the measure of the unknown side or diagonal in each of these rectangles:

	l	w	*Diagonal*		l	w	*Diagonal*
a.	8	5	? $\sqrt{89}$	**d.**	$\sqrt{5}$	? $\sqrt{95}$	10
b.	$\sqrt{15}$	1	? 4	**e.**	? $\sqrt{2}$	1	$\sqrt{3}$
c.	5	$\sqrt{11}$	? 6	**f.**	$\sqrt{6}$	? $\sqrt{2}$	$\sqrt{8}$

9. Each irrational number is associated with a point on the number line. The square roots of numbers that are not perfect squares can readily be located by construction, using a compass. Thus we can locate $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$:



Locating Irrational Numbers

Show the construction for finding points $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$. Using the same procedure, locate $\sqrt{6}$, $\sqrt{7}$, and $\sqrt{8}$.

10. Explain the difference in meaning among: 3.3, 3.3· · ·, and $3.\overline{3}$ See front.

11. Construct a line that measures $\sqrt{10}$ inches. Explain how you know it is $\sqrt{10}$ inches long. See front.

12. The square of an odd number is an odd number. This must be true, since 2 cannot be a factor of the square if it is not a factor of the number. On the other hand, the square of an even number is even. Complete the table to illustrate this fact:

n	1	2	3	4	5	6	7	8	9	10	11	12	13
n^2	1	4	9	16	25	36	49	64	81	100	121	144	169

It was demonstrated in the fifth century B. C. that the square roots of numbers that are not perfect squares are not rational numbers. Let's examine the way that the proof is developed. This is an indirect proof. Suppose that $\sqrt{2}$ *is* a rational number. Then $\sqrt{2} = \frac{a}{b}$, where $\frac{a}{b}$ is in lowest terms. Squaring both sides of the equation

$$2 = \frac{a^2}{b^2}$$

and

$$2b^2 = a^2$$

This means that a^2 has 2 as a factor.

This also means that a can be divided by 2 without a remainder. Thus, we can assume $a \div 2 = c$ (the quotient of $a \div 2$). So $a^2 = 2c \times 2c = 4c^2$. Substituting in the equation, $2b^2 = 4c^2$, and $b^2 = 2c^2$

Then b must be an even number.

BUT this is contrary to the assumption made at the start, that $\sqrt{2} = \frac{a}{b}$, where $\frac{a}{b}$ is in lowest terms. Since $\sqrt{2}$ cannot be expressed as a fraction $\frac{a}{b}$ in lowest terms, it is not a rational number. This is why we call $\sqrt{2}$ an irrational number.

1. Use the same line of reasoning to show that $\sqrt{3}$ is not a rational number. See above.

2. Use the same line of reasoning to test whether $\sqrt{4}$ is a rational number. See above.

1, 4, 9, 16, 25, 36, 49, 64, 81

You can find the value of $\sqrt{n}$ for values of n from 1 to 100 by using the table of squares. For what values of n is $\sqrt{n}$ a natural number rather than an irrational number? We call these values of n *perfect squares.* For example, $\sqrt{324} = 18$ and thus we would say that 324 is a perfect square because its square root is 18, a natural number. On the other hand, $\sqrt{325}$ is not a perfect square because there is no natural number which is the square root of 325. Is 361 a perfect square? Is 1369 a perfect square? Is 7089 a perfect square? If $27^2 = 729$, why can we say that 729 is a perfect square? yes; yes; no; $27 \cdot 27 = 729$

For most values of $\sqrt{n}$ it is simpler to leave $\sqrt{n}$ as an *indicated root,* writing it with the radical symbol $\sqrt{\ }$, unless the numerical value is required. In that case, the symbol $\sqrt{\ }$ is a symbol of operation, just as $+$, $-$, $\times$, and $\div$ are symbols of operation. It tells us that if there is a positive number r, such that $n = r^2$, then $\sqrt{n} = |r|$. This fact is frequently used in conditional equations.

Find the value for n: $8 - \sqrt{n} = 5$ $\qquad s - a_1 = a_2$

$\sqrt{n} = 8 - 5 \qquad a_1 = s - a_2$

$\sqrt{n} = 3 \qquad n = 3^2 = 9$

1. Find the values for n that make these statements true.

a. $\sqrt{n} = 7$ 49

b. $2\sqrt{n} = 12$ 36

c. $\sqrt{n} + 7 = 9$ 4

d. $\dfrac{\sqrt{n}}{3} = 1$ 9

e. $\sqrt{n} - 5 = 2$ 49

f. $4\sqrt{n} = 2$ $\frac{1}{4}$

j. $8 - \sqrt{n} = 5$ 9

k. $5\sqrt{n} = 15$ 9

l. $6\sqrt{n} = 3$ $\frac{1}{4}$

m. $12\sqrt{n} = 4$ $\frac{1}{9}$

n. $15\sqrt{n} = 5$ $\frac{1}{9}$

o. $16\sqrt{n} = 4$ $\frac{1}{16}$

NOTE: $\sqrt{n} = \frac{2}{4} = \frac{1}{2}$. Then $n = (\frac{1}{2})^2 = \frac{1}{2} \times \frac{1}{2}$. $n = \square$

g. $\sqrt{n} - 2 = 3$ 25

h. $\dfrac{\sqrt{n}}{4} = 2$ 64

i. $9\sqrt{n} = 27$ 9

p. $36\sqrt{n} = 6$ $\frac{1}{36}$

q. $5\sqrt{n} = 1$ $\frac{1}{25}$

r. $3\sqrt{n} = 5$ $\frac{25}{9}$

2. Is $\sqrt{2} \cdot \sqrt{5} = \sqrt{10}$? We know that $\sqrt{2} \cdot \sqrt{2} = \sqrt{2 \cdot 2} = \sqrt{4} = 2$. Let us try $\sqrt{2} \cdot \sqrt{3} = \sqrt{2 \cdot 3} = \sqrt{6}$.
In the Table of Squares we find (to 3 decimal places) that $\sqrt{2} = 1.414$ and $\sqrt{3} = 1.732$. Then, $(1.414) \cdot (1.732) = 2.449048$ which yields to 3 decimal places $\sqrt{6} = 2.449$ which agrees with the table value. Any difference probably will be due to rounding values of the square roots. Thus, we have shown that it appears true that $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$.

$1.414 \times 2.236 = 3.161704$
$= 3.162$

a. Using the values of $\sqrt{2}$ and $\sqrt{5}$ to three decimal places, test $\sqrt{2} \cdot \sqrt{5} = \sqrt{10}$ to see if it is a true statement.

Write the products under one radical sign:

$\sqrt{15}$ **b.** $\sqrt{3} \cdot \sqrt{5}$ $\sqrt{114}$ **d.** $\sqrt{6} \cdot \sqrt{19}$ $\sqrt{30}$ **f.** $\sqrt{3} \cdot \sqrt{10}$ $\sqrt{77}$ **h.** $\sqrt{11} \cdot \sqrt{7}$

$\sqrt{14}$ **c.** $\sqrt{7} \cdot \sqrt{2}$ $\sqrt{26}$ **e.** $\sqrt{2} \cdot \sqrt{13}$ $\sqrt{35}$ **g.** $\sqrt{5} \cdot \sqrt{7}$ $\sqrt{65}$ **i.** $\sqrt{13} \cdot \sqrt{5}$

From Exercise 2 it appears that we can find the products of radical numbers, using them as factors:

$$\sqrt{f_1} \cdot \sqrt{f_2} = \sqrt{p}$$

It seems only reasonable that the converse is true and that we should be able to write the product in terms of its factors. $\sqrt{p} = \sqrt{f_1} \cdot \sqrt{f_2}$

Is $\sqrt{6} = \sqrt{2} \cdot \sqrt{3}$? Yes! Note that we verified this in numerical form in Exercise 2. Similarly $\sqrt{10} = \sqrt{2} \cdot \sqrt{5}$

3. Write each of the following as the product of two radical factors:

a. $\sqrt{15}$	$\sqrt{5} \cdot \sqrt{3}$	**c.** $\sqrt{33}$	$\sqrt{11} \cdot \sqrt{3}$	**e.** $\sqrt{39}$	$\sqrt{3} \cdot \sqrt{13}$	**g.** $\sqrt{77}$	$\sqrt{11} \cdot \sqrt{7}$
b. $\sqrt{14}$	$\sqrt{7} \cdot \sqrt{2}$	**d.** $\sqrt{26}$	$\sqrt{2} \cdot \sqrt{13}$	**f.** $\sqrt{35}$	$\sqrt{7} \cdot \sqrt{5}$	**h.** $\sqrt{65}$	$\sqrt{13} \cdot \sqrt{5}$

4. Factoring the number under the $\sqrt{\ }$ may reveal one or more factor pairs that are perfect squares. The square root of these factors may be written as a coefficient of the $\sqrt{\ }$. If $\sqrt{98} = \sqrt{49} \cdot \sqrt{2}$, then $\sqrt{98} = 7\sqrt{2}$ or $7(1.414) = 9.898$, which is correct to 2 decimal places. Since the Table of Squares extends only to 100, this procedure may be useful when you are finding the square root of a number greater than a 100. If you can recognize the squares of numbers 1 to 12, many square roots of numbers can be found by factoring the number under the radical sign.

EXAMPLE

What is $\sqrt{125}$ to three decimal places?

$$\sqrt{125} = \sqrt{25 \cdot 5} = \sqrt{25} \cdot \sqrt{5} = 5\sqrt{5}$$

Finding $\sqrt{5}$ in the Table of Squares, we can multiply it by 5 to get 11.181. Thus $\sqrt{125} \approx 11.18$

If you cannot recognize the squares of numbers from 1 to 12, you can write out the prime factors of the number under the radical sign. Any pair of identical factors represents a perfect square.

EXAMPLE

Find $\sqrt{128}$ to three decimal places.

$$\sqrt{128} = \sqrt{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}$$

Combining the 3 pairs of identical factors, we have $1.414 \times 8 = 11.312$
$\sqrt{128} = \sqrt{64 \cdot 2} = \sqrt{64} \times \sqrt{2} = 8\sqrt{2}$. Complete the solution.

Simplify each of the following and find to three decimal places.

a. $\sqrt{12}$	3.464	**d.** $\sqrt{75}$	8.660	**g.** $\sqrt{45}$	6.708	**j.** $\sqrt{300}$	17.321
b. $\sqrt{480}$	21.909	**e.** $\sqrt{72}$	8.485	**h.** $\sqrt{112}$	10.583	**k.** $\sqrt{18}$	4.243
c. $\sqrt{200}$	14.142	**f.** $\sqrt{600}$	24.495	**i.** $\sqrt{44}$	6.633	**l.** $\sqrt{147}$	12.125

A. *Add:*

1. ${}^{+}8 + ({}^{-}3)$ +5
2. ${}^{+}11 + ({}^{+}7)$ +18
3. ${}^{-}11 + ({}^{-}13)$ -24
4. ${}^{-}2 + ({}^{+}21)$ +19
5. ${}^{+}28 + ({}^{-}15)$ +13
6. ${}^{-}4 + ({}^{+}4)$ 0

B. *Subtract:*

1. ${}^{+}15 - ({}^{+}3)$ +12
2. ${}^{-}18 - ({}^{+}16)$ -34
3. ${}^{+}17 - ({}^{-}19)$ +36
4. ${}^{-}18 - ({}^{-}18)$ 0
5. ${}^{-}13 - ({}^{-}19)$ +6
6. ${}^{+}15 - ({}^{+}15)$ 0

C. Find the value for n:

1. $\frac{5}{6} = \frac{n}{18}$ 15
2. $\frac{3}{8} = \frac{27}{n}$ 72
3. $\frac{15}{n} = \frac{25}{30}$ 18
4. $\frac{n}{8} = \frac{3}{4}$ 6
5. $\frac{19}{20} = \frac{n}{100}$ 95
6. $\frac{7}{9} = \frac{14}{n}$ 18

D. Without reference to tables, find the square root of each number to two decimal places.

1. $\sqrt{7}$ 2.65
2. $\sqrt{19}$ 4.36
3. $\sqrt{81}$ 9.00
4. $\sqrt{90}$ 9.49

E. Find the value of n that makes each a true statement.

1. $\sqrt{n} - 3 = 2$ 25
2. $\frac{\sqrt{n}}{5} = 2$ 100
3. $18 - \sqrt{n} = 15$ 9
4. $\sqrt{n} + 15 = 18$ 9
5. $\frac{\sqrt{n}}{2} = 4$ 64
6. $13\sqrt{n} = 39$ 9

F. Simplify each expression by removing factors that form perfect squares.

1. $\sqrt{48}$ $4\sqrt{3}$
2. $\sqrt{50}$ $5\sqrt{2}$
3. $\sqrt{300}$ $10\sqrt{3}$
4. $\sqrt{20}$ $2\sqrt{5}$
5. $\sqrt{72}$ $6\sqrt{2}$

G. Find the products:

1. 1.7×2.18 3.706
2. $4.76 \times .007$ .03332
3. 38.75×1.875 72.65625

H. Find the quotients:

1. $6.86 \div 4.9$ 1.4
2. $24.3 \div 2.7$ 9
3. $0.111 \div 3.7$ .03

If you need further practice, turn to the Practice Exercises on page 483. If not, you may work in the Experts' Corner.

Shortcut for Squaring Numbers

In the next few pages you will be dealing extensively with squares of numbers. There is an easy shortcut for mentally calculating the square of any number between 25 and 75. You will need to memorize the squares of numbers between 1 and 25 in order to use this shortcut. Knowing these squares will be useful for many calculations.

Let us list the squares of a few numbers to see a pattern.

$$26^2 = 676 = [24^2 = (576)] + 100$$
$$27^2 = 729 = [23^2 = (529)] + 200$$
$$28^2 = 784 = [22^2 = (484)] + 300$$

a decreases
b increases
50, 25

Do you see the pattern? Try setting up the statement for 29^2.

Each statement is in the form $n^2 = a^2 + 100b$. 24; 1; 25

When $n = 26$, what is a? what is b? what is $a + b$? As n increases, what is the relationship between n and a? n and b? a and b?

For each value of n, what is the value of $n + a$? of $n - b$? When you think you sense a pattern, try it by finding the squares of 31, 32, and 33. If your idea holds, write a description of the procedure for finding squares. Then, read the following discussion.

For numbers between 25 and 50 the steps are listed using 37 as an example:

Step 1.	Subtract 25 from the number . . . 12	
	Multiply the difference by 100	1200
Step 2.	Subtract the number from 50 . . . 13	
	Square the difference: $13^2 =$	169
Step 3.	Add the results of Step 1 and Step 2	1369

In the first five exercises, write down each of the three steps. In the rest, write only the result of Step 3.

Find the square of:

1. 27 729	**4.** 36 1296	**7.** 41 1681	**10.** 48 2304	**13.** 30 900
2. 31 961	**5.** 43 1849	**8.** 32 1024	**11.** 49 2401	**14.** 47 2209
3. 42 1764	**6.** 29 841	**9.** 28 784	**12.** 45 2025	**15.** 40 1600

The steps for squaring a number between 50 and 75 are the same as those listed above, except that instead of subtracting the number from 50 in Step 2, you subtract 50 from the number. 3025; 3136; 3249; 3364; 3481; 3600; 3721; 3844; 3969; 4096

16. Write the squares of numbers from 55 to 64 inclusive.

17. What is the area of the surface of a table 68″ square?
4624 sq. in.

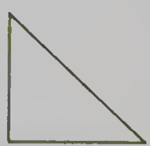
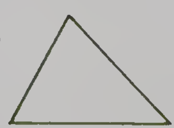
A. On a sheet of paper list the numerals 1 through 10. Next to each numeral, write the word (or words) from the right-hand column that fits the description given.

1. A triangle with three sides having equal measures *Equilateral*
2. A triangle with two sides having equal measures *Isosceles*
3. The longest side of a right triangle *Hypotenuse*
4. A triangle having one angle whose measurement is 90° *Right*
5. A triangle having no two sides with equal measures *Scalene*
6. The side on which a triangle appears to rest *Base*
7. The sides of a right triangle that determine the right angle *Legs*
8. A triangle each of whose angles measures less than 90° *Acute*
9. A triangle with one angle more than 90° *Obtuse*
10. The shortest distance between a base of a triangle and its opposite vertex *Altitude*

Acute
Altitude
Area
Base
Diagonal
Equilateral
Horizontal
Hypotenuse
Isosceles
Legs
Oblique
Obtuse
Right
Scalene

B. On the left below is a list of different kinds of triangles. On the right are illustrations of seven triangles. Write the numerals 1 through 7 on a sheet of paper. After each write the letter to indicate which of the triangles is named by the word having that numeral. There may be more than one triangle that is identified by one word.

1. Acute *b,d,f,g*
2. Equilateral *f,g*
3. Isosceles *d,f,g*
4. Obtuse *c*
5. Right *a,e*
6. Scalene *a,b c,e*
7. Equiangular *f,g*

a. 
b. 
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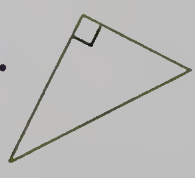
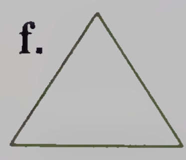
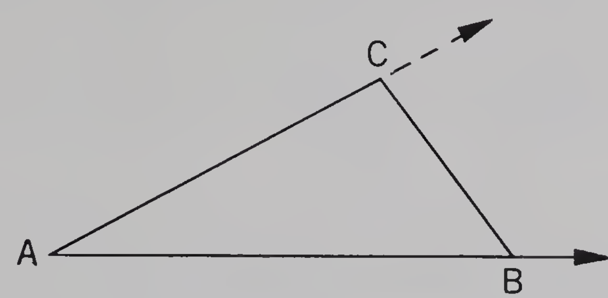
d. 
e. 
f. 
g. 

Angles, as you know, are formed by two rays that have the same endpoint. Since the sides of a triangle are segments, rather than rays, a triangle, strictly speaking, has no angles. However, it is possible to imagine each side extended as a ray, thus forming an angle at each vertex of the triangle. The measure of each angle is determined by the sides of the triangle. Therefore, these angles are referred to as the angles of the triangle. (See Exercise 14 on page 291.)



In the figure above, what angle is formed by $\overrightarrow{AB}$ and $\overrightarrow{AC}$? Show how you can draw rays to form angles with vertices at B and C. $\angle CAB$

An important relationship in the triangle is that the sum of the measures of the angles in any one triangle is the same as the sum of the measures in any other triangle. Using your ruler and protractor, draw four triangles given the following conditions:

a. All angles of the triangle are acute.
b. One angle of the triangle is obtuse.
c. One angle of the triangle is a right angle; the other two are equal in measure.
d. One angle of the triangle is a right angle; the other two angles are not equal in measure.

In each triangle, label the vertices A, B, and C.

In discussing the measurement of an angle, it would *not* be correct to write: $\angle A = 45°$. The symbol for equality is used only to state that two expressions name the same number or the same set. Since $\angle A$ is not a number but a set of points, it is not equal to a number that denotes measurement. What we wish to say is: *the degree measure of* $\angle A$ *is 45.* In symbolic terms, the statement is: $m\,\angle A = 45°$.

From discussions in Chapter 1 you will recall that the same consideration applies to statements about measurement of line segments. Suppose we measure a segment, say $\overline{AB}$, and find that it is 18 inches long. To record this measurement, we write $m\,\overline{AB} = 18$ inches. The same idea may also be expressed as $AB = 18$ inches if, as now, the meaning is specified. As pointed out in Chapter 1, we will use this latter expression in denoting measure of line segments.

Such insistence on precision in expression may seem unnecessarily technical at times. However, mathematics is a language of precision, and precision in thought is impossible without precision in vocabulary. Here it is important to be consistent when using the symbol for equality.

Now use your protractor to measure each of the angles in each of the four triangles. Record your measurements in a table like this:

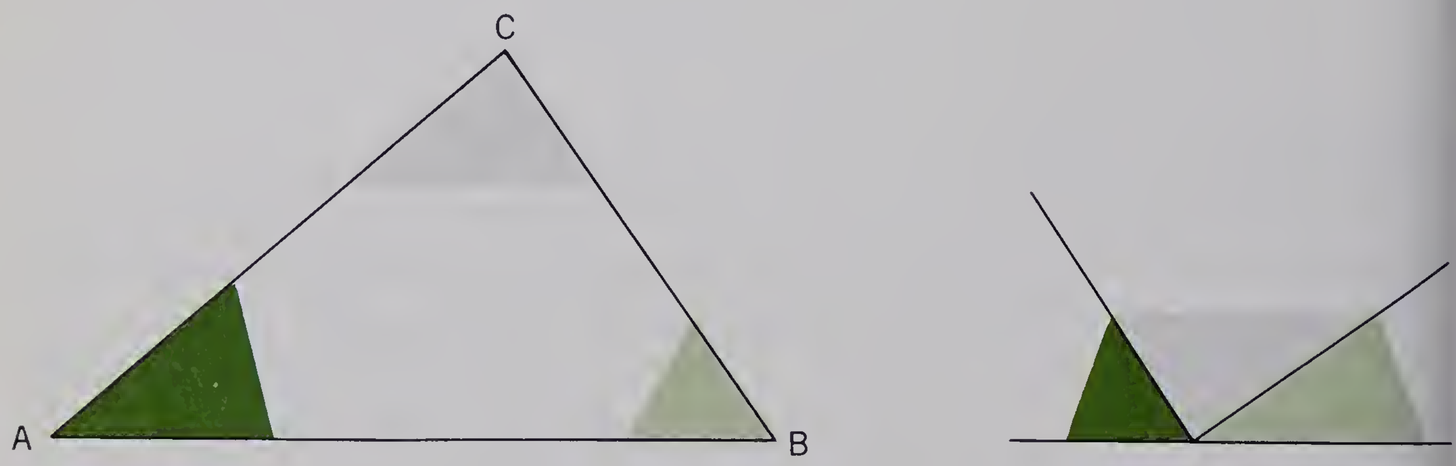
Answers will vary.

Triangle	$m\angle A$	$m\angle B$	$m\angle C$	*Sum of measures*
a.	☐	☐	☐	☐
b.	☐	☐	☐	☐
c.	☐	☐	☐	☐
d.	☐	☐	☐	☐

1. Find the average of the four sums in the right-hand column. Measure the angles with your protractor again to check the accuracy of the measurements. Can you measure more precisely than to the nearest degree?

2. It can be shown that the sum of the measures of the angles in a triangle is 180°. Do your findings agree with this?

3. Here is another way to show that the sum of the measures of the angles of a triangle is 180°. Cut out any one of the triangles you have drawn and "cut off" the angles. When you fit them together their two "outer" sides will form a straight line. How many degrees does the straight line represent? *180°*



4. Explain why a triangle can have no more than one right angle.
The measure of the third angle would be zero.

5. Explain why a triangle can have only one obtuse angle.
The measure of the third angle would be less than zero.

6. Draw a triangle that contains an obtuse angle and two angles of equal measure. *Drawings will vary.*

7. Draw a triangle that contains a 30° and a 75° angle. Can you classify this triangle in any manner? *isosceles*

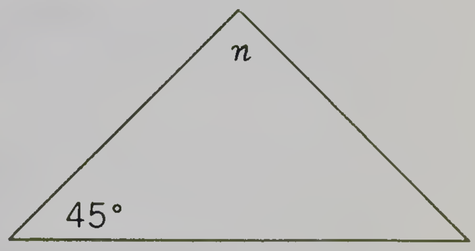
Knowing that the sum of the measures of the angles of a triangle is 180°, it is possible to determine the measure of any angle if the measures of the other two are given. If you have information about the relative measures of the three angles you can find the measure of each angle. It is always helpful to make and label a sketch before you set up an equation.

1. Measures of two of the angles of a triangle are 48° and 50°. What is the measure of the third angle?

   HINT: Let n represent the number of degrees in the third angle. Then: $n + 48 + 50 = 180$. Explain why this is a true statement. Find n. 82°

2. What is the measure of each angle in an equilateral triangle? 60°

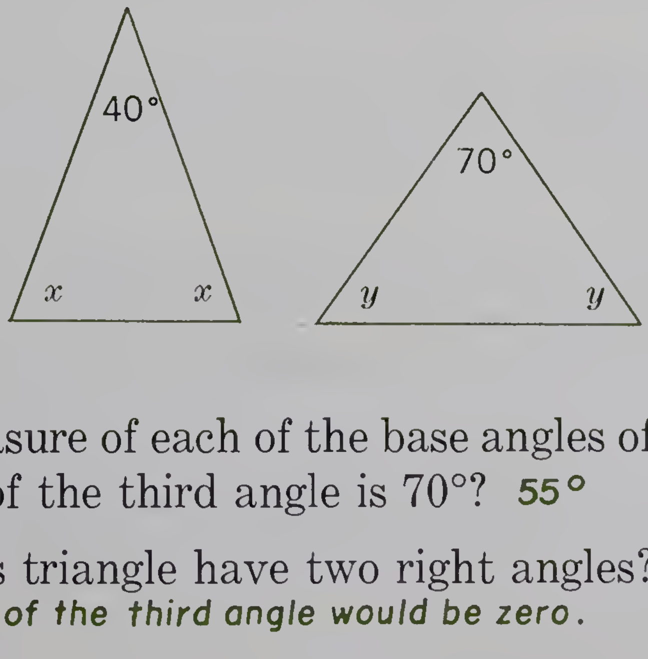
3. If one of the angles of equal measure in an isosceles triangle is 45°, what is the measure of each of the other two angles? 45°, 90°

   HINT: Since the measure of one of the two angles of equal measure is 45°, what is the measure of the other? Let n represent the number of degrees in the third angle. Now set up the equation.

4. One of the angles of equal measure in an isosceles triangle is 30°. What is the measure of each of the other angles? Hereafter we will call the angles of equal measure of an isosceles triangle the *base* angles. 30°, 120°

5. The measure of the base angles of an isosceles triangle is unknown, but the measure of the third angle is 40°. What is the measure of each of the base angles? 70°



6. What is the measure of each of the base angles of an isosceles triangle if the measure of the third angle is 70°? 55°

7. Can an isosceles triangle have two right angles? Explain.
   No, the measure of the third angle would be zero.

8. The measure of an angle of an isosceles triangle is twice that of either of the base angles. What is the measure of each angle? 45°, 45°, 90°

9. The measure of an angle in an isosceles triangle is 30° greater than that of either of the base angles. What is the measure of each base angle? 80°, 50°, 50°

10. The measure of the greatest angle in a scalene triangle is 100°; that of the least is 30°. What is the measure of the third angle? 50°

11. The measure of the greatest angle in a scalene triangle is three times that of the least angle. The measure of the third angle is twice that of the least angle. Find the measure of each angle. 90°, 30°, 60°

12. The measure of the greatest angle in a scalene triangle is twice that of the least angle. The measure of the third angle is 1.5 times that of the least angle. Find each measure. 80°, 40°, 60°

HINT: If n represents the number of degrees in the smallest angle, what is the number of degrees in the greatest angle? in the third angle? What is the sum of the measures?

13. The measure of one of the angles in a scalene triangle is 60°. The measure of one of the other angles is four times that of the third. What is the measure of each angle? 24°, 96°

14. The measure of one of the acute angles in a right triangle is 10° greater than that of the other. What is the measure of each angle? 40°, 50°, 90°

15. What is the measure of each of the acute angles in an isosceles right triangle? 45°

16. Can a right triangle be equilateral? Explain. No, the hypotenuse is always the largest side.

17. The measure of one of the acute angles in a right triangle is 1.5 times the measure of the other. Find the measure of each. 36°, 54°

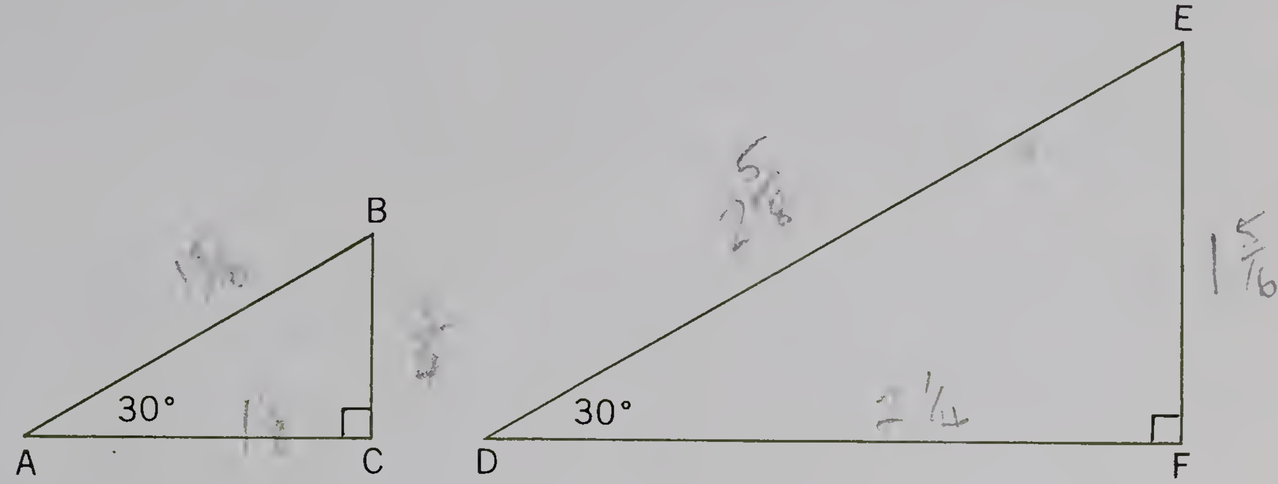
18. The measure of one of the acute angles in a right triangle is 30° more than twice that of the other. Find each measure. 20°, 70°

19. The measure of one of the acute angles of a right triangle is 10° more than three times the measure of the other. Find the measure of each. 20°, 70°

20. The measure of the greatest angle in a scalene triangle is 50° more than that of the least angle. The measure of the third angle is 40° more than the measure of the least angle. Find the measure of each. 30°, 70°, 80°

21. The measure of one of the angles in a scalene triangle is 30°. The measure of one of the other angles is twice that of the third angle. What is the measure of each? 30°, 50°, 100°

Two plane figures are *similar* if they are of the same shape but not necessarily the same size. The two triangles illustrated below are similar. Although triangle DEF is not the same size as triangle ABC, it has the same shape. Angles C and F are right angles.



This is indicated by the small squares inscribed in the angles. Note that $m\angle B = m\angle E$, and $m\angle A = m\angle D$. These are *corresponding angles,* and they illustrate the following fact.

In similar figures, measures of corresponding angles are equal.

The sides opposite corresponding angles in similar figures are called *corresponding sides.* It is important to remember that

---

In similar figures, the measures of corresponding sides are in the same ratio.

---

1. Which angle in the triangle on the right corresponds to $\angle C$ in the triangle on the left? Which angle corresponds to $\angle B$? Which angle corresponds to $\angle A$? $\angle F$; $\angle E$; $\angle D$

2. Which side in the triangle on the right corresponds to $\overline{AB}$ in the triangle on the left? to $\overline{AC}$? to $\overline{BC}$? $\overline{DE}$; $\overline{DF}$; $\overline{EF}$

3. Write the value of these ratios: $\dfrac{EF}{BC}, \dfrac{ED}{BA}$. $\dfrac{1\frac{5}{16}}{\frac{3}{4}} = \dfrac{7}{4}$; $\dfrac{2\frac{5}{8}}{1\frac{1}{2}} = \dfrac{7}{4}$

4. Since measures of corresponding sides are in the same ratio, what is the measure of $\overline{FD}$? of $\overline{ED}$? $2\frac{1}{4}$" ; $2\frac{5}{8}$"

If a proportion equating ratios of sides is true, then the ratios must list the sides in the same order.

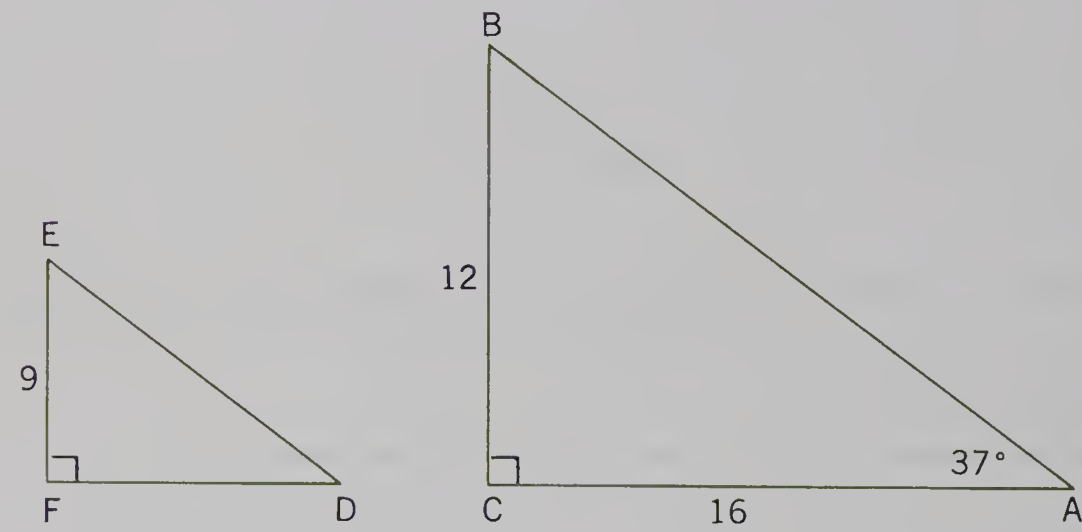
5. Using this test, state which of the following proportions are true.

a. $\frac{BC}{BA} = \frac{EF}{ED}$ T c. $\frac{CA}{BA} = \frac{FD}{ED}$ T e. $\frac{CB}{CA} = \frac{EF}{FD}$ T

b. $\frac{CB}{BA} = \frac{ED}{FE}$ F d. $\frac{BA}{CA} = \frac{FD}{ED}$ F f. $\frac{AC}{DF} = \frac{AB}{DE}$ T

6. Write three other true proportions, not listed above, that exist between the sides of the triangles illustrated on the previous page. See front.

7. In triangle BCA below, $m \angle A$ is 37°. Find $m \angle B$. Give a reason for your answer. 53°; $37° + m\angle B + 90° = 180°$



8. Find $m \angle E$. 53° Give two reasons why you know that your answer is correct. In similar figures, measures of corresponding angles are equal; $180° = 90° + m\angle E + 37°$.

9. What is the value of the ratio $\frac{FE}{CB}$? $\frac{3}{4}$

10. If the two triangles are similar, what is the value of $\frac{FD}{CA}$? $\frac{3}{4}$ of $\frac{ED}{BA}$? $\frac{3}{4}$

11. What is the value of the ratio $\frac{BC}{BA}$? $\frac{3}{5}$

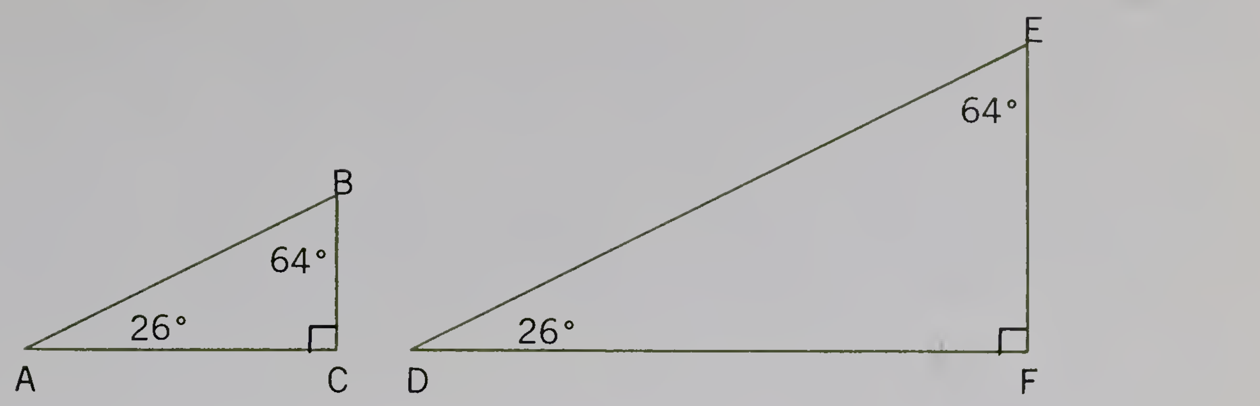
12. What is the ratio between the two corresponding sides in triangle EFD? $\frac{3}{4}$

13. Write two proportions equating the ratios between two sides of triangle BCA with two corresponding sides of triangle EFD.

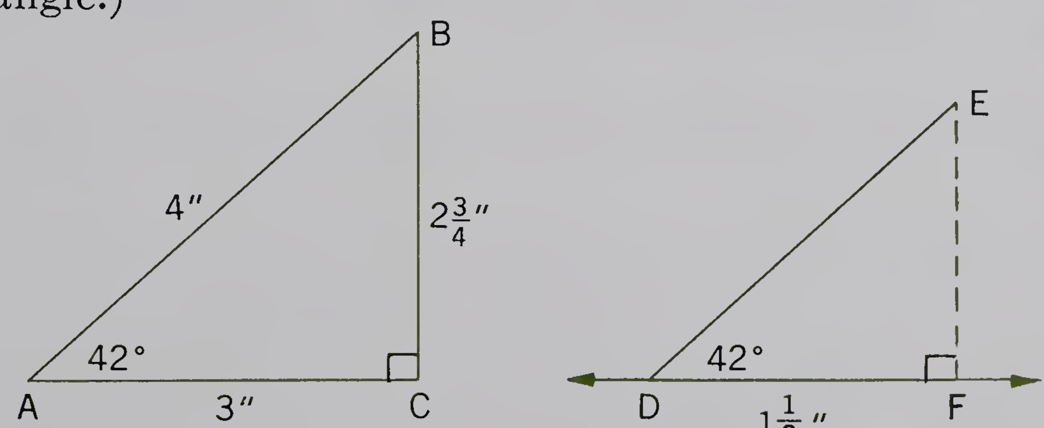
$\frac{BC}{CA} = \frac{EF}{FD}$; $\frac{CA}{BA} = \frac{FD}{ED}$; $\frac{AB}{BC} = \frac{DE}{EF}$

Conditions for Similarity

Two triangles are similar if the measures of their corresponding angles are equal or if the measures of their sides are proportional. Thus the two right triangles illustrated at the top of the next page are similar, since their respective angles have the same measure.



1. If the measures of angles B and E had not been given, would you be able to calculate what they were? Explain. yes, $m\angle B + 26° = 90°$, $m\angle E + 26° = 90°$, $m\angle B = m\angle E = 64°$
2. If the measure of one acute angle of a right triangle is 30°, what is the measure of the other acute angle? 60°
3. Explain why this statement is true: If the measure of an acute angle in each of two right triangles is equal, the two right triangles are similar. See front.
4. It is a simple matter to construct a right triangle similar to a given right triangle, with the measures of corresponding sides having a stated ratio. Suppose we are given $\triangle BCA$ below and are required to construct $\triangle EFD$, the measure of each of whose sides is $\frac{1}{2}$ of the corresponding side of $\triangle BCA$. (We will hereafter use the symbol $\triangle$ for triangle.)



   a. On a line measuring somewhat greater than $1\frac{1}{2}''$ long, locate points F and D such that they are $1\frac{1}{2}''$ apart.
   b. Using your protractor, construct an angle measuring 90° at F and an angle measuring 42° at D.
   c. Extend the sides of the two angles until they intersect. Label this point of intersection E. What is $m\angle E$? 48°

5. What is the measure of $\overline{FE}$? of $\overline{ED}$? $1\frac{3}{8}''$; 2″
6. What is the value of the ratio $\frac{FE}{CB}$? of $\frac{ED}{BA}$? $\frac{1}{2}$; $\frac{1}{2}$

7. Test this proportion and state if it is a true relationship.

$$\frac{CB}{CA} = \frac{FE}{FD} \quad \frac{2\frac{3}{4}}{3} = \frac{1\frac{3}{8}}{1\frac{1}{2}}; \frac{33}{8} = \frac{33}{8}$$

8. Write two other proportions equating ratios between the measure of the sides of $\triangle BCA$ and $\triangle EFD$. $\frac{FE}{CB} = \frac{ED}{BA}$; $\frac{DF}{AC} = \frac{DE}{AB}$

9. Using the steps outlined above, construct right $\triangle ABC$ with $AC = 2\frac{3}{16}''$ and $m\angle A = 30°$. Construct right $\triangle DEF$ with $DF = 4\frac{3}{8}''$ and $m\angle D = 30°$. What are the measures of angles B and E? 60°; 60°

10. Write the value for each of the following ratios in the triangles you drew.

a. $\frac{FD}{CA}$ $\frac{2}{1}$ b. $\frac{CB}{FE}$ $\frac{1}{2}$ c. $\frac{BA}{ED}$ $\frac{1}{2}$

11. Write the value for each of the following ratios as a decimal rounded to the nearest hundredth.

a. $\frac{CA}{BA}$ .87 b. $\frac{CB}{FE}$ .50 c. $\frac{CB}{CA}$ .57 d. $\frac{FD}{ED}$ .87 e. $\frac{FE}{FD}$ .57 f. $\frac{FE}{ED}$ .44 g. $\frac{BC}{BA}$ .50

Problems in Review

1. A square field measures 72 rods on each side. What is its area? 5184 sq. rd.

2. What is the area of a square garden each of whose sides measures 36 feet? 1296 sq. ft.

3. The measure of the length of a rectangular field is 84 rods. The measure of its width is 42 rods. What is its area? 3528 sq. rd.

4. The measure of the length of a rectangular field is 102 rods. The measure of its width is 34 rods. What is its area? 3468 sq. rd.

5. The measure of each of the base angles in an isosceles triangle is 15° more than that of the third angle. Find the measure of each angle. 65°, 65°, 50°

6. The measure of the least angle in a scalene triangle is $\frac{1}{4}$ the measure of the greatest angle and $\frac{1}{3}$ the measure of the third angle. What is the measure of each angle? $22\frac{1}{2}°$, $67\frac{1}{2}°$, 90°

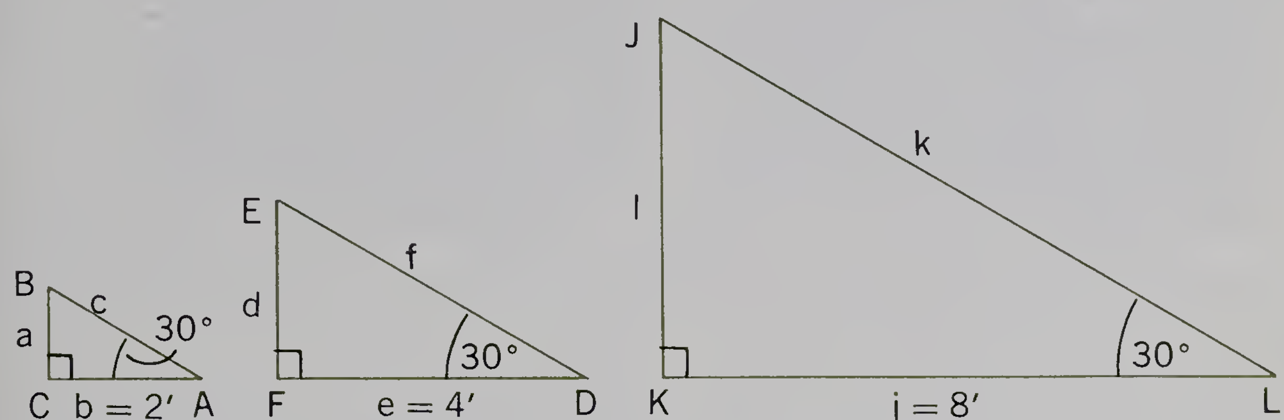
7. The measure of the least angle in a scalene triangle is $\frac{1}{5}$ the measure of the greatest angle and $\frac{1}{4}$ the measure of the third angle. What is the measure of each angle? 18°, 72°, 90°

8. The bisectors of the base angles of an isosceles triangle meet to determine an angle. If the vertex angle is 20°, what is the measure of the angle formed by the bisectors? 100°

9. In a right triangle the measure of one acute angle is twice the measure of the other acute angle. What is the measure of each angle? 30°, 60°, 90°

To simplify our discussion of ratio, we will hereafter use a single letter to designate the measure of a segment. The measure of the side opposite $\angle A$ in a triangle will be a, the measure of the side opposite $\angle B$ will be b, and the measure of the side opposite $\angle C$ is c.

1. Each of the right triangles illustrated below has an acute angle measuring 30°. What is the measure of the other acute angle in each triangle? 60°



2. Are the triangles similar? Explain your answer. Yes, the measures of all three angles are equal.

3. If the triangles are similar, the ratios between measures of corresponding sides are equal. Complete this statement:

$$\frac{a}{b} = \frac{\square\ d}{e} = \frac{l}{\square\ j}$$

4. In any right triangle in which $m\angle A = 30°$, the ratio $\frac{a}{b} = \frac{58}{100}$. Then if $b = 2$, you have this proportion: $\frac{a}{2} = \frac{58}{100}$. Solve the proportion to find a. 1.16

5. Since $\frac{a}{b} = \frac{d}{e}$ in triangles BCA and EFD above, then $\frac{d}{e} = \frac{58}{100}$ must also be a true statement. Solve this proportion to find d if $e = 4$. 2.32

The tangent ratio for an acute angle of a *right* triangle is defined as:

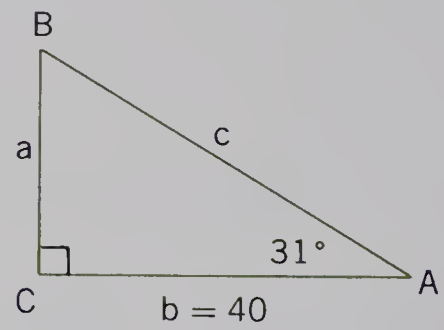
$$\frac{\text{the measure of the side opposite the angle}}{\text{the measure of the side adjacent to the angle}}$$

In the triangles illustrated above, we also have the following tangent ratios

$$tan(m\angle B) = \frac{b}{a} \qquad tan(m\angle E) = \frac{e}{d} \qquad tan(m\angle J) = \frac{j}{l}$$

6. Reexamine page 203. Notice the ratio $\frac{a}{b}$ is called the *tangent ratio* for $m\angle A$ in $\triangle BCA$. This is expressed as: $tan(m\angle A) = \frac{a}{b}$ (tan is the abbreviation for tangent). Similarly, $tan(m\angle D) = \frac{d}{e}$ in $\triangle EFD$, and $tan(m\angle L) = \frac{l}{j}$ in $\triangle JKL$.

Usually the ratio is expressed as a decimal. Using the Tables, the tangent of an angle measuring 31° (see triangle ABC below) is 0.601. Writing this as $\frac{601}{1000}$, write and solve the proportion to find a. 24.04

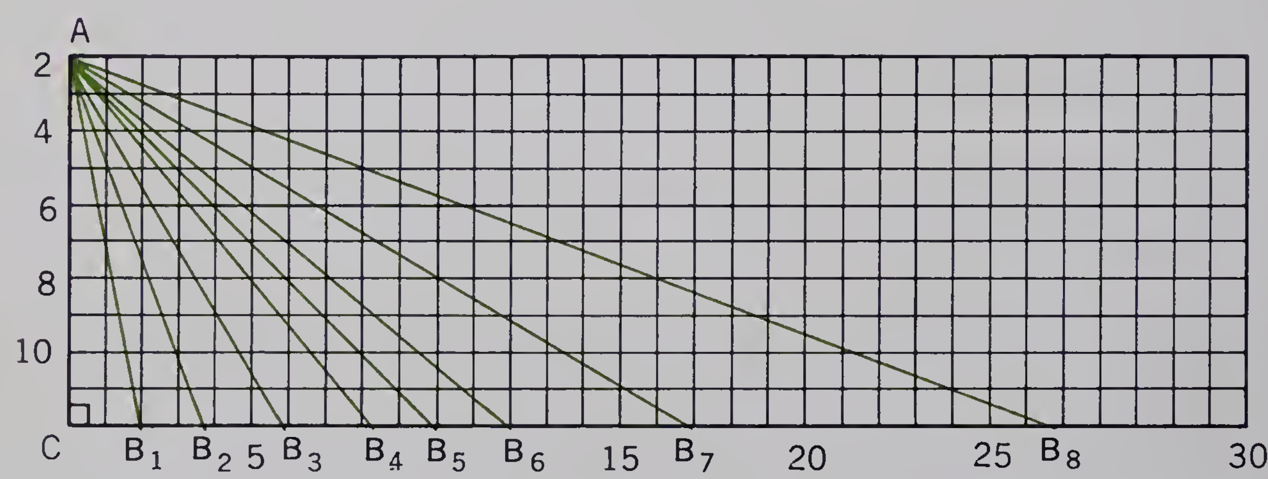


7. Suppose $b = 60$. What is a? 36.06

8. What is a if $b = 7.5$? 4.5075

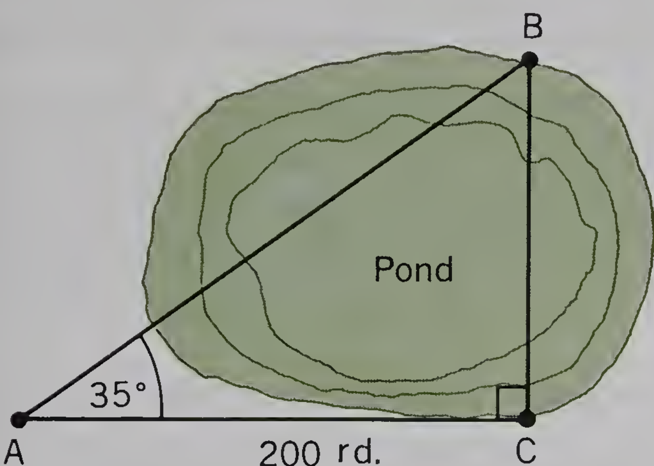
9. In the figure below, note that side $\overline{AC}$ has the same measure in all eight triangles. As $m\angle A$ increases, the measure of the side opposite $\angle A$ increases. Find the tangent ratio of each of the following angles to one decimal place.

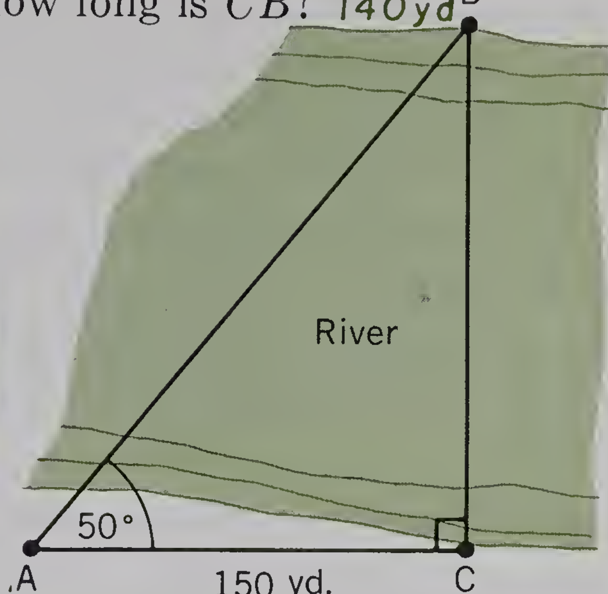
.2 **a.** 10° ($\angle CAB_1$) .6 **c.** 30° ($\angle CAB_3$) 1.2 **e.** 50° ($\angle CAB_6$)
.4 **b.** 20° ($\angle CAB_2$) 1.0 **d.** 45° ($\angle CAB_5$) 1.7 **f.** 60° ($\angle CAB_7$)



10. Examine your answers in Exercise 9 and answer these questions:
   a. Does the value of the tangent ratio increase as the measure of the angle doubles? (Compare, for example, *tan* 20° and *tan* 10°; *tan* 60° and *tan* 30°.) yes
   b. The tangent ratio is less than 1 for all angles less than what measure? 45°
   c. When $\angle A$ measures 45°, what can you say of the measures of the side opposite $\angle A$ and the side adjacent to $\angle A$? They are equal.
   d. Would it be possible to find the measure of the side opposite $\angle A$ if $m\angle A = 45°$? Explain your answer. yes, if you know the other side

1. A surveyor wished to measure the distance across the pond illustrated below. Notice *tan* 35° = 0.7. Set up the proportion to determine the length of $\overline{CB}$. How long is $\overline{CB}$? 140yd





2. Eric and John wanted to measure the distance across the river BC. Notice *tan* 50° = 1.19, or $\frac{119}{100}$. Set up and solve the proportion needed to find the distance across the river. 179 yd.

3. A surveyor wished to measure the distance across a pond. Using a diagram similar to the one for Exercise 1, find BC if AC equals 190′ and $m\angle A = 68°$. 471.2 ft.

4. a. Draw a sketch similar to the one above on the right, find the distance across the river if angle A equals 57° and the measure of AC is 187 yd. 287.98 yd.

b. If you knew a river was 880 feet wide and angle A was 60°, how would you determine AC? Using tan 60° = $\frac{880}{AC}$, it would be necessary to divide to find your answer. Is there a way to solve this problem that will avoid long division? Use $\angle B$.

5. Find the length of the side opposite $\angle A$ in each of these triangles.

	$m\angle A$	b		$m\angle A$	b	
57.7 yd. **a.**	30°	100 yd.	**d.**	60°	150 ft.	259.5 ft.
80 rd. **b.**	45°	80 rd.	**e.**	25°	30 yd.	13.98 ft.
165 ft. **c.**	70°	60 ft.	**f.**	14°	100 rd.	24.9 rd.

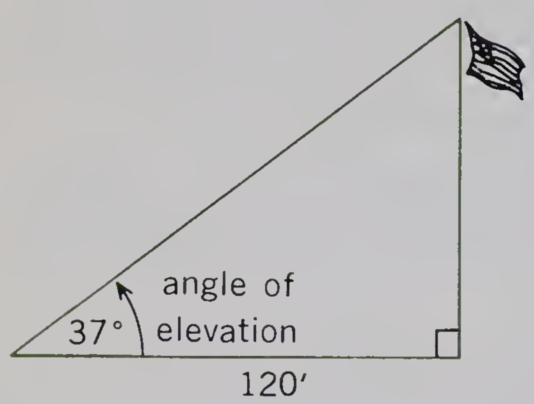
6. Construct a right triangle in which $m\angle A = 15°$, and the side adjacent to it measures 4 inches. Here is an easy way to construct it.

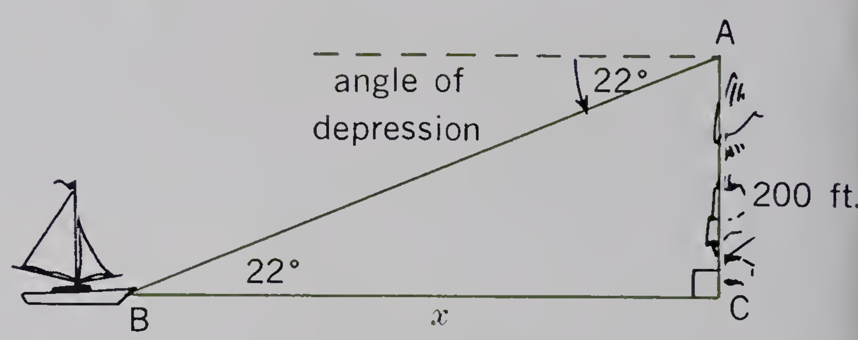
In the table of tangents find 15°. Under *tangent* opposite 15° you will find .268. This is the tangent of 15°. Set up and solve the proportion needed to find the length of the leg opposite $\angle A$.

$\frac{268}{1000} = \frac{x}{4}$; 1.072 in.

When one is taking a sight on a point above him, the angle between the horizontal and a ray extending along the line of sight to the point is the *angle of elevation.* If he is taking a sight on a point below him, the angle between the horizontal and a ray extending along the line of sight is the *angle of depression.* These expressions are very useful to surveyors and others who make indirect measurements. They will be illustrated in the following exercises. As with all problems dealing with geometric figures, you will find it useful to make a sketch of the figure, inserting the measures and indicating the measures you are to find before undertaking the computations. Use the table of tangents on page 000 as required.

1. The measure of the angle of elevation to the top of a flagpole from a distance of 80 feet is 37°. What is the height of the flagpole? 60.3ft.





2. Sighting a boat from the top of a 200′ cliff, we find the measure of the angle of depression is 22°. The measure of the height of the cliff is what fraction of the distance to the boat? How far is it from the cliff to the boat? .404 ; 496ft.

3. A bridge over a canyon with vertical walls is 483 feet in length. An observer at one end of the bridge finds the angle of depression to the bottom of the other side of the canyon measures 44°. How high is the bridge from the canyon floor? 466.8ft.

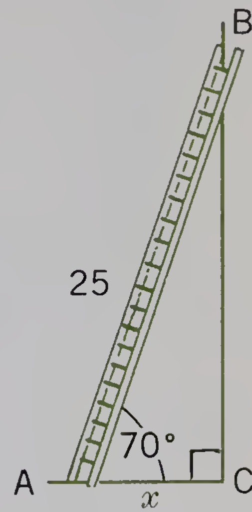
4. An observer sights a helicopter directly over a building 1200 feet from him. The angle of elevation measured by the observer is 30°. How high is the helicopter? 692.4 ft.

5. An observer in a balloon is 6000 feet from the ground. The angle of depression to a point A measures 30°; to point B, 40°; and to point C, 10°. Which point is closer to the observer? How far away is it from a point directly under the observer? B; 7140ft.

6. An observer finds that the angle of elevation to the top of a chimney 400 feet away measures 42°. How high is the chimney? 360ft.

A ladder 25 feet long is placed against the side of a house so that the angle determined by the foot of the ladder and the ground is measured as 70°. How far is the foot of the ladder from the house?

Can you use the tangent ratio to find b? If you examine the figure, you will see that you need a ratio that includes the known side c. Since a is not known, the tan $(m\angle A) = \frac{a}{b}$ cannot lead us to the solution. Since c (the length of the ladder) is known, we need a ratio that includes b and c. This requires the introduction of a new trigonometric ratio, the *cosine*.



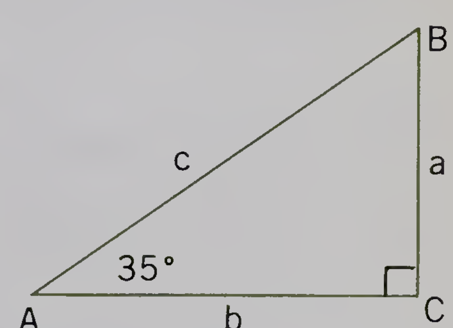
The *cosine* ratio for an acute angle of a right triangle is defined as:

$$\frac{\text{measure of the adjacent side}}{\text{measure of the hypotenuse}}$$

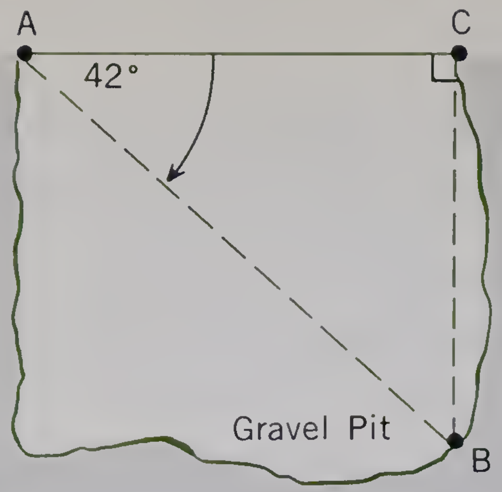
This can be abbreviated as $\cos (m\angle A) = \frac{b}{c}$.

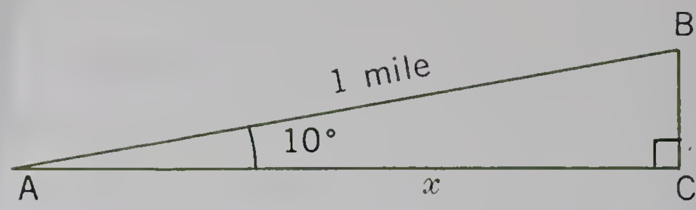
Using the table of cosine ratios, we find that $\cos (m\angle A) = .342$. Then $\frac{b}{25} = \frac{342}{1000}$. Solve the proportion for b. 8.55 ft.

1. In the triangle at the right $m\angle A$ is 35°. Measure sides b and c and calculate the ratio $\frac{b}{c}$ to three decimal places. In the table this ratio is given as 0.819. Is this approximately the value you found?

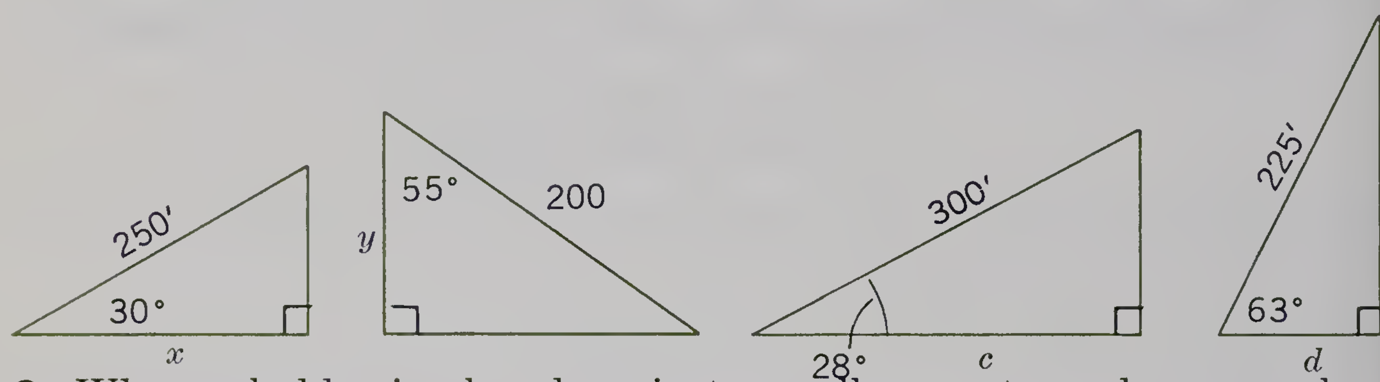


2. If c is 40, b is what fraction of 40? $\frac{819}{1000}$
3. Set up and solve the proportion to find b. 32.76 ft.
4. To measure the distance from A to C across a gravel pit, a surveyor used a transit and a tape line to make the measurements. AB is 800 feet, and $m\angle A$ is 42°. What is the length of $\overline{AC}$? See page 208. 594.4 ft.
5. A mountain road has a slope of 10°. What is the horizontal component AC as a car travels a mile along the road (AB)? See page 208. .985 mi. or 5200.8 ft.
6. A telephone pole is braced by a piece of timber 20 feet long. An angle measuring 25° is determined by the intersection of the telephone pole and the timber. How high up the pole does it reach? 18.12 ft.



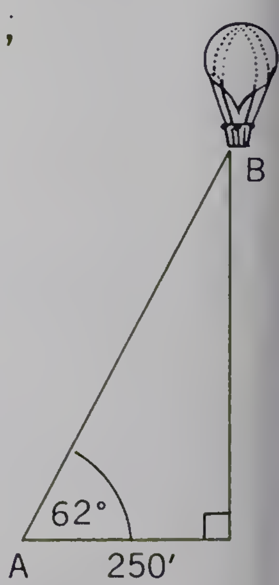


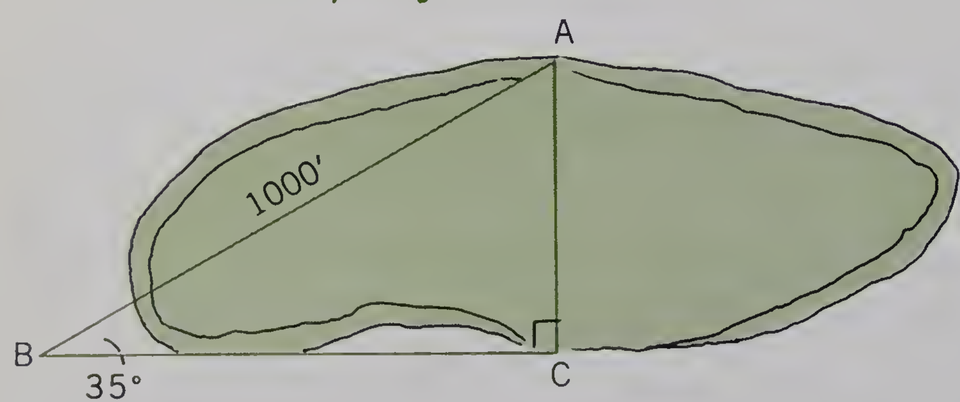
7. Solve for the indicated variable carrying your calculations to one decimal place. The variable represents the measure of one side.
216.5 ft. 114.8 ft. 264.9 ft. 102.15 ft.



8. When a ladder is placed against a wall so as to make an angle of 75° with the ground, it is said to be in its safest position for the worker. How far from the bottom of the wall should the base of a 32-foot ladder be placed? 8.288 ft.

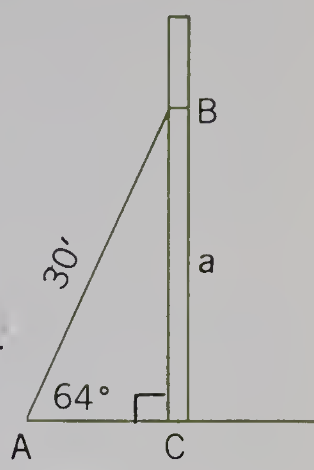
9. A balloon is anchored to the ground by a wire cable.
   a. How long is the wire (AB) that keeps the balloon from drifting? (Which ratio did you use?) 533 ft. ; cosine
   b. How high is the balloon from the ground? (Which ratio did you use?)
   470 ft. ; tangent





10. The Boy Scouts wanted to find the distance AC across the lake. The distance AB is 1000 feet. What is the distance from A to C?
231 ft.

A telephone pole is braced by a 30-foot wire that determines an angle measuring 64° with the ground. How high up the pole is the wire fastened?



What ratio should you use in the solution?

Tan $(m\angle A) = \frac{a}{b}$. But both a and b are unknown.

Cos $(m\angle A) = \frac{b}{c}$. But we need a ratio that includes a, which we are to find, and c, which is known. This requires a third trigonometric ratio, the *sine.*

The *sine* ratio for an acute angle of a right triangle is defined as:

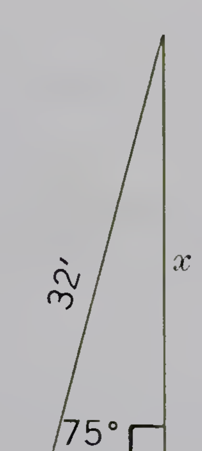
$$\frac{\text{measure of side opposite}}{\text{measure of the hypotenuse}}$$

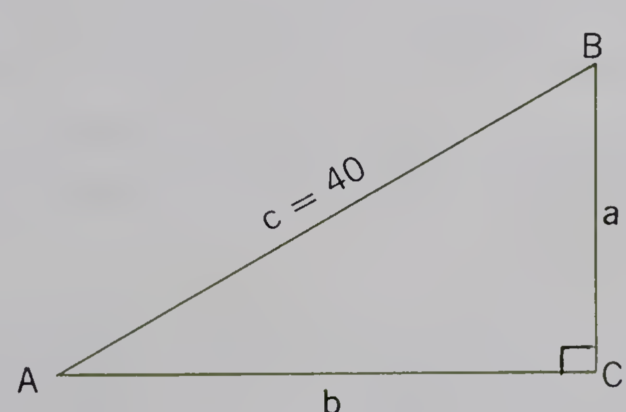
This is abbreviated as $\sin\,(m\angle A) = \frac{a}{c}$.

In this example, $\sin\,(m\angle A) = \frac{a}{30}$. Using the trigonometric tables we find that $\sin 64° = 0.899$. Therefore: $\frac{a}{30} = \frac{899}{1000}$. Solve for a. 26.97ft.

Make a sketch showing given measurements and measurements to be found before attempting the solution of each of these problems.

1. A ladder 32 feet long is placed against a building so that the angle with the ground measures 75°, as shown here. How far up the wall does the ladder extend? 30.912 ft.





2. Using the table of sines, find a when c is 40 and $\angle A$ has each of these measures.

a. 18°	**b.** 42°	**c.** 36°	**d.** 22°	**e.** 30°
12.36	26.76	23.52	15.00	20.00

Trigonometry originally developed as a science for measuring triangles, as the word indicates: *trigonon* (triangle) *metry* (measurement). This branch of mathematics has far outstripped its original usefulness. Today, trigonometry deals with the relationships between the trigonometric functions and their inverses as well as with the solution of triangles and other polygons.

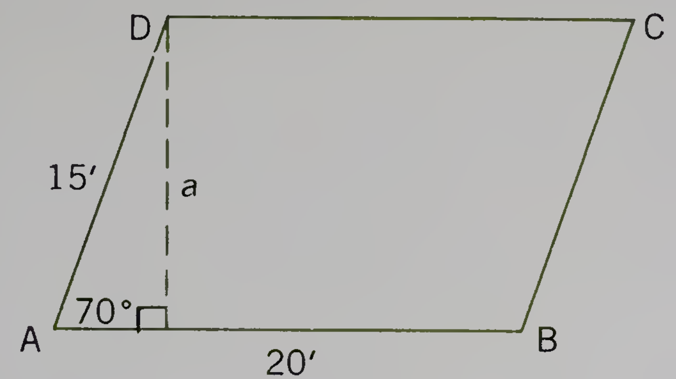
In solving a triangle problem, it is important to determine which trigonometric ratio will lead to a solution. The first step is to make a sketch that represents the given situation. Then determine what trigonometric ratio must be used to find the measure that is required. Write the trigonometric proportion and solve it for the missing term.

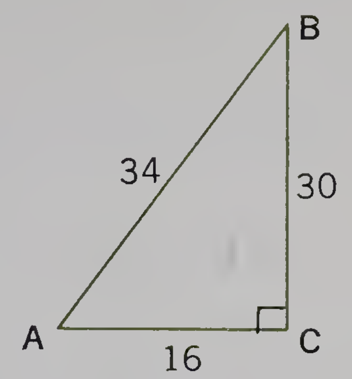
These problems will provide practice in solving right triangles. Assume that there is level ground for all problems unless otherwise stated.

1. The grade of a road measures 3° from the horizontal. As a car moves 2 miles along the road, how has the altitude changed? .1046 mi.; 552.3 ft.
2. What is the measure of the altitude of an equilateral triangle each of whose sides measures 12 inches? 10.392 ft.
3. A radar operator reported that an airplane was approaching at an angle of elevation of 20°. He reported the direct distance between the radar and airplane as 4 miles. What is the land distance from operator to a point directly under the airplane? 3.76 mi.
4. The pilot of an airplane 4500 feet above the ground sees an airfield. He measures the angle of depression from the plane to the field as 12°. What is the land distance from a point directly under the plane to the airfield? 21,195 ft.
5. The hypotenuse of a right triangle measures 380 feet. The measure of one of the acute angles is 65°. What is the measure of the other acute angle? What is the length of each leg? 25° ; 344.28 ft. ; 160.74 ft.
6. The slope of an open pit mine makes an angle measuring 30° with the horizontal. The slope measures 600 feet. How deep is the mine? 300 ft.
7. A ladder 41 feet long is placed against a wall so that the foot of the ladder is 9 feet from the wall. How far up the wall is the top of the ladder? (Remember: $a^2 + b^2 = c^2$.) 40 ft.
8. A wire 30 feet long is used to brace a telephone pole. The wire is fastened to a stake 15 feet from the pole. How high up the pole is the wire fastened? 25.98 ft.

9. Jim and Henry are flying a kite. Jim is holding the string which is out 300 feet. Henry is 150 feet from Jim and the kite is directly overhead. How high is the kite above Henry? 259.8 ft.

10. What is the altitude a of the parallelogram $ABCD$? 14.1 ft.





11. In $\triangle ABC$, $\sin (m \angle A) = \frac{a}{c}$. What is $\cos (m \angle A)$? $\frac{b}{c}$

12. In the same triangle, what is $\cos (m \angle B)$? $\frac{a}{c}$

13. In any given right triangle if $\angle A$ and $\angle B$ are the acute angles, is it always true that $\cos (m \angle B) = \sin (m \angle A)$? yes
Sketch a 3–4–5, 8–15–17, and 5–12–13 triangle, and compare the sine and cosines of $\angle A$ in each triangle if $\angle A$ is opposite the shorter leg. $\cos (m \angle A) > \sin (m \angle A)$

14. Look in the table of sines and cosines. Does it appear to be true that $\sin (m \angle A) = \cos (90° - m \angle A)$? yes

15. In a table of sines and cosines it is only necessary to give the ratios up to angles of 45°. Explain why. You can get the others by subtracting the measures of the angles from 90°.

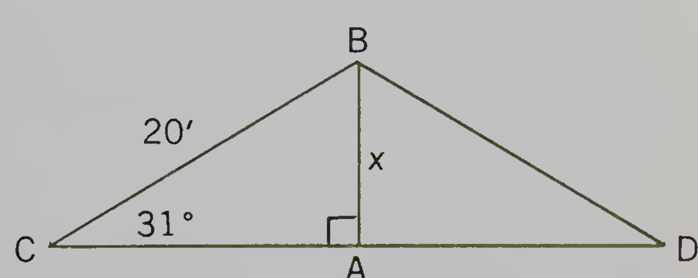
16. Can you find the sine of each of the following angles without using the table beyond 45°? For example, $\sin 88° = \cos 2° = .999$.

a. 55° $\cos 35° = .819$ **b.** 70° $\cos 20° = .940$ **c.** 60° $\cos 30° = .866$ **d.** 80° $\cos 10° = .985$ **e.** 75° $\cos 15° = .966$

17. A ship at sea sights a lighthouse with its base at sea level. The captain knows that the lighthouse is 120 feet in height. The angle of elevation to the top of the lighthouse measures 20°. How far is the boat from the lighthouse? 330 ft.

18. A wire running from the top of a pole to a stake that is 50 feet from the foot of the pole forms an angle with the ground measuring 70°. How high is the pole? 137.5 ft.

19. How long is the support AB in a roof truss if the length of the joist from B to C is 20 feet and the roof makes an angle of 31° with the horizontal? 10.3 ft.



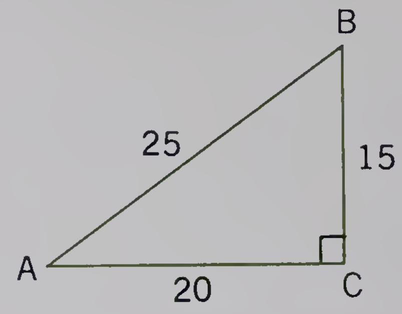
20. Find the length of the beam CD of Exercise 19.
34.28 ft.

If you know the measure of 2 or 3 sides of a right triangle, it is possible to find the trigonometric ratios of each acute angle. We can then find the measure of the angle in degrees. Thus, if you examine the right triangle ABC, you will see that

$$\tan(m\angle A) = \frac{a}{b} = .75$$

$$\sin(m\angle A) = \frac{a}{c} = .6$$

$$\cos(m\angle A) = \frac{b}{c} = .8$$



To find $m\angle A$ look in the table of tangents to find the measure of the angle whose tangent is .75. Here we find that $\tan 37° = .754$. To the nearest degree, then, $m\angle A = 37°$. What is $m\angle B$? 53°

1. What is the sine, cosine, and tangent of $m\angle B$? .8; .6; 1.3
2. Instead of using the tangent function to find $m\angle A$, could you use the sine or cosine? See if they identify the same angle. yes
3. In $\triangle ABC$, find the trigonometric functions to the nearest hundredth and the measure of the angles to the nearest degree if $a = 8$, $b = 15$, and $c = 17$.
   a. $\sin(m\angle A)$ .47 **c.** $\cos(m\angle A)$ .88 **e.** $m\angle A$ 28°
   b. $\sin(m\angle B)$ .88 **d.** $\cos(m\angle B)$ .47 **f.** $m\angle B$ 62°
4. Construct a right triangle with $a = 2\frac{1}{2}''$; $b = 4\frac{11}{16}''$; $c = 5\frac{5}{16}''$. Find $\sin(m\angle A)$; $\cos(m\angle A)$; $\tan(m\angle A)$. $\frac{8}{17} = .47$; $\frac{15}{17} = .88$; $\frac{8}{15} = .53$
5. What is the $m\angle A$ to the nearest degree? What is the $m\angle B$? Use a protractor to check your results on the triangle constructed in Exercise 4. 28°; 62°
6. Without using your protractor, construct an angle that measures 42°. In the table of tangents you will find that $\tan 42° = .9$ to the nearest tenth. Then, if you construct a right triangle with $a = 4.5$, $b = 5$, and $m\angle C = 90°$, you can check $m\angle A$ with your protractor. Does $\angle A$ measure 42 degrees? yes
7. Using the method outlined in Exercise 6, construct angles with each of these measures. Check your results with a protractor. See front.
   a. 37° **b.** 35° **c.** 58° **d.** 70°
   In Exercise 7a $\tan 37° = .8$ to the nearest tenth. If you selected $a = 4$ and $b = 5$ and $m\angle C = 90$, you will find that $m\angle A$ may be 37°, 38°, or 39°. Try $a = 3$ and $b = 4$ and see if $m\angle A$ is closer to 37°. Explain. $\tan 37° = .754$; $m\angle A$ is closer to $\frac{3}{4}$

8. What is the measure of the angle of elevation of the sun when a 14-foot post casts a 20-foot shadow? (Determine angles to the nearest degree.)
HINT: Sketch a right triangle and represent the post as a and the shadow as b. Then $\tan(m\angle A) = .70$. What is the $m\angle A$ to the nearest degree? 35°

9. A mountain path rises 150 feet for each 1200 feet measured along the path. What is the path's degree of slope? 7°
HINT: What trigonometric ratio is $\frac{150}{1200}$? tangent

10. In hiking a mile along a mountain path, John found the increase in altitude to be 1000 feet. What is the degree of slope of the path? 11°

11. What is the measure of the angle of elevation of the sun when a 60-foot flagpole casts a 15-foot shadow? Where would the angle be measured? 76°; at the end of the shadow

12. What is the angle of elevation of the sun when a boy 5 feet tall casts a 10-foot shadow? 27°

13. A searchlight is 200 feet above the ground. What is the angle of depression when the light shines directly on an object on the ground 1800 feet from the bottom of the tower? 6°

14. A ladder 30 feet long is placed against a wall with its foot placed 8 feet from the wall. What angle does the ladder make with the ground? 75°

15. Mike and Eric are flying a kite. Eric is holding the string, which is out 500 feet. If the kite is directly above Mike, who is 200 feet from Eric, what is the angle of elevation of the kite from Eric? 66°

16. A cable that supports a mast is 25 feet long. It is fastened to the mast at a point 20 feet above the deck. What angle does it make with the mast? (The cosine of the angle is 0.80. How do you know this?) What angle has this cosine, or the closest approximation? 37°

17. A ladder 30 feet long touches the wall at a point 27 feet above the ground. At what angle does it meet the wall? 26°

18. The string of John's kite is 240 feet long. When the string is stretched tight and determines an angle with the ground that measures 38°, how high is the kite? 147.84 ft.

19. A wire 26 feet long is fastened to the top of a pole and to a stake in the ground. It makes an angle of 67° with the ground. How tall is the pole? 23.946 ft.

Some Special Angles

If you examine the table of sines, cosines, and tangents closely, you will see that certain angles have some interesting properties.

1. First construct a right triangle ABC in which $m\angle A = 45°$ and $AC = CB = 1''$. Then $\tan(m\angle A) = 1$. This means that $\frac{a}{b} = 1$, and $a = b$. If $a = 1$ and $b = 1$, what is the value of c^2? What is the value of c? 2; 1.414
2. Is c the length of the diagonal in a square whose side is 1 unit in length? yes
3. What is the sin 45° and cos 45° in the triangle of Exercise 1? .707
4. Find $\sqrt{2}$ to 3 decimal places and calculate sin 45° with the values given in Exercise 3. Does your answer agree with that in the table? .707; yes
5. Another interesting angle is 30°. Construct a right triangle with $m\angle A = 30°$, $a = 2''$. What is c? 4 in.
6. Which trigonometric function can you use to find c? sin Does $c = 4''$? yes
7. Use a trigonometric function to find b. Measure b on your construction. Do your calculation and measurement agree when compared to two decimal places? 3.464, yes
8. As $m\angle A$ becomes very small, a number of interesting relationships develop. If $c = 1$ unit (as 1 foot, 1 meter, and so forth), $\sin(m\angle A)$ states the length of a, since then $\sin(m\angle A) = \frac{a}{1}$. If $m\angle A = 5°$ and $c = 1$ ft., $a = 0.087$ ft. (or approximately 1 inch). How can you tell this from the table? sin 5° = .0872
9. If $c = 1$ foot and $m\angle A = 2°$, $a = 0.035$ ft. (or approximately 0.4 in.). How can you tell this from the table? sin 2° = .0349
10. What is a in feet and in inches, when $m\angle A = 1°$ and $c = 1$ yard. .0525 ft. or .63 in.
11. Using $c = 1$ foot and remembering that $a^2 + b^2 = c^2$, find b when $m\angle A = 1°$. slightly less than 1 ft.
12. How could you use cos 1° to save computation with the Pythagorean Theorem in Exercise 11? $b = c\cos(m\angle 1°)$
13. As $m\angle A$ becomes smaller, a becomes smaller, and the value of b becomes more nearly equal to that of c. How does the value of $\cos(m\angle A)$ tell you this? cos 1° = 1.000; cos 0° = 1.000

14. Since $\sin(m\angle A) = \frac{a}{c}$ and $\tan(m\angle A) = \frac{a}{b}$, what will be true of their values as b and c become more nearly equal? Examine the tables and see if they confirm your answer. $\sin(m\angle A) \approx \tan(m\angle A$; $\sin 1° \approx \tan 1°$

15. As $m\angle A$ becomes less and b and c become more nearly equal, what must be true of the value of $\cos(m\angle A) = \frac{b}{c}$? Does the table confirm your answer? $\cos(m\angle A) \approx 1$

16. If $m\angle A = 0°$, then $a = 0$. Why? $\frac{a}{c} = 0$, $a = 0 \cdot c = 0$

17. If $c = 1$ and $a = 0$, what is the value of b? Remember, $a^2 + b^2 = c^2$. 1
Then what is the value of: **a.** $\sin 0°$ 0 **b.** $\cos 0°$ 1 **c.** $\tan 0°$ 0

18. Does the table confirm your answers to Exercise 17? yes

19. The ratios also are very interesting as $m\angle A$ approaches 90°. If $c = 1$ unit, $\cos(m\angle A)$ states the length of b, since then $\cos(m\angle A) = \frac{b}{1}$. How long is b if $c = 1$ foot, and $m\angle A = 89°$. .0175 ft.

20. As $m\angle A$ approaches 90°, a becomes more nearly equal to c. What ratio for 89° shows this to be true? $\sin 89° = 1.000$

21. As $m\angle A$ approaches 90° and b becomes less, the value of $\tan(m\angle A) = \frac{a}{b}$ increases rapidly. Explain why. See front.

22. Does the table confirm the fact that the tangent increases rapidly as the measure of the angle approaches 90°. yes

23. If $m\angle A = 90°$, $b = 0$. Why? See front.

24. If $m\angle A = 90°$, $a = c$. Why? See front.

25. Why is the $\tan(m\angle A) = \frac{a}{b}$ undefined at 90° when $b = 0$? See front.

26. Summarize briefly what you have learned about this topic.
$\sin 0° = \cos 90° = 0$ $\cos 0° = \sin 90° = 1$ $\tan 0° = 0$; $\tan 90°$ is undefined

STEPS FOR SOLVING MATHEMATICAL PROBLEMS

1. Understand the problem.
2. Analyze the data.
3. Discover new facts.
4. Follow up and verify promising leads.
5. Review your solution.

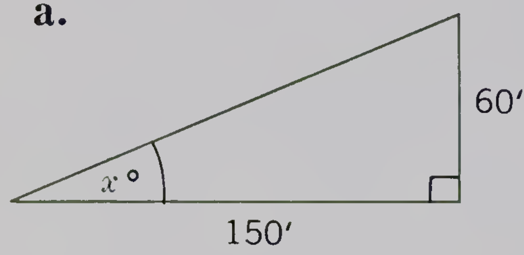
Part One

a. 22° b. 187.44 ft. c. 133.2 ft. d. 74.2 e. 10.152 ft. f. 61.728 ft.

Using one of the trigonometric ratios or the theorem of Pythagoras, solve for the indicated variable on each of these sketches.

g. 15 ft. h. 120 ft.

a.



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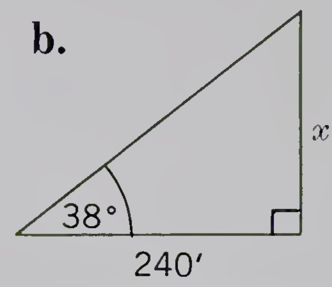

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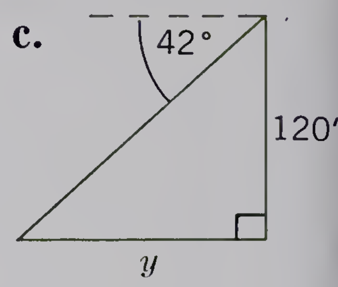

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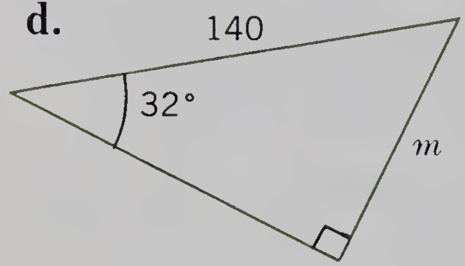

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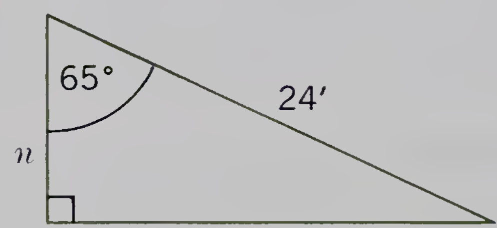
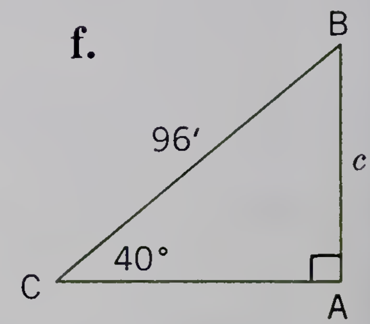
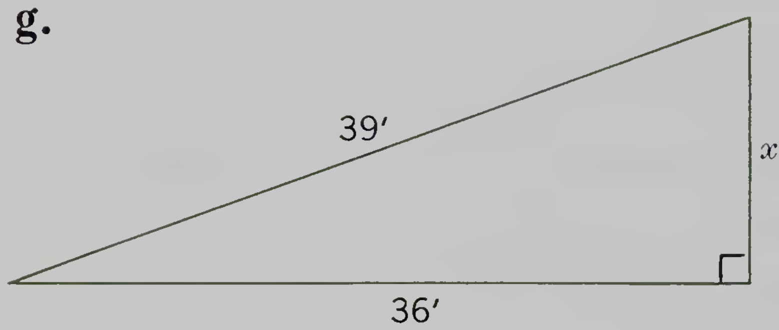


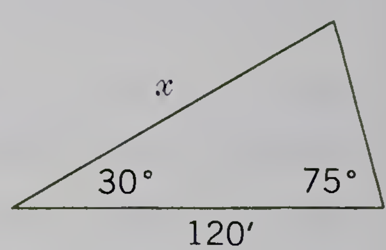
f.



g.

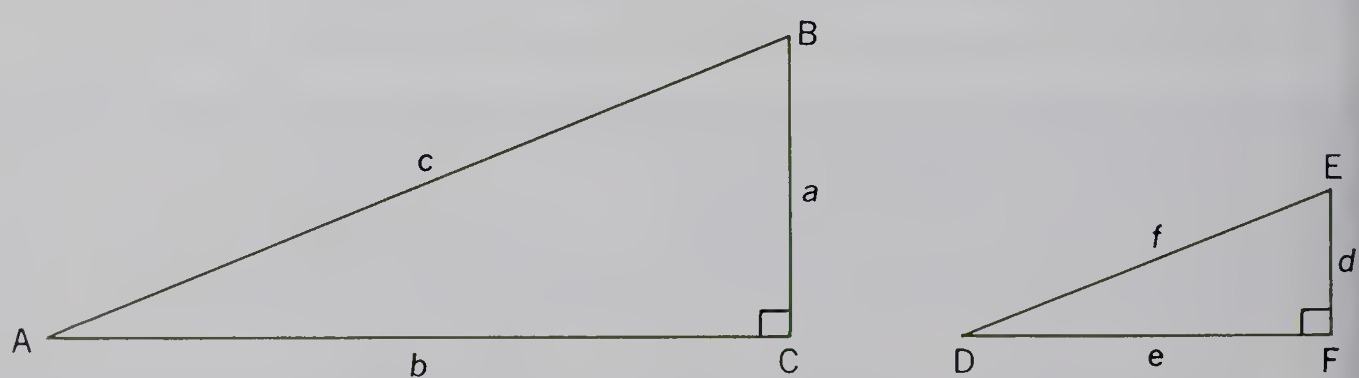


h.



Part Two

1. In the following illustration the two triangles are similar. Determine if each proportion is true or false.



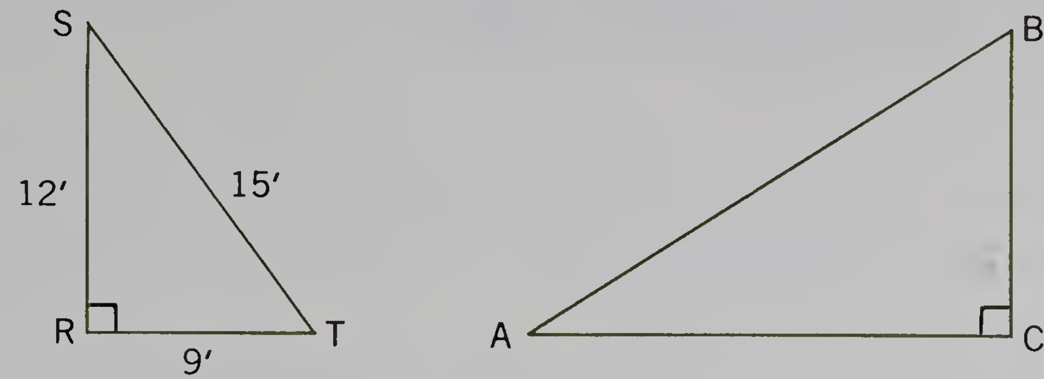
a. $\frac{a}{c} = \frac{b}{f}$ F **c.** $\frac{e}{f} = \frac{b}{c}$ T **e.** $\frac{d}{a} = \frac{e}{b}$ T **g.** $\frac{f}{d} = \frac{c}{a}$ T

b. $\frac{f}{c} = \frac{d}{a}$ T **d.** $\frac{a}{b} = \frac{e}{d}$ F **f.** $\frac{b}{e} = \frac{c}{d}$ F **h.** $\frac{b}{e} = \frac{a}{d}$ T

2. The $\triangle RST$ is a right triangle with the right angle at R. Write to two decimal places the value of:

a. tangent of angle T 1.33 b. cosine of angle T .60 c. sine of angle T .80

3. Triangle ABC is a right triangle with right angle at C.



Find the required measure in each of the following right triangles.

a. $m\angle A = 35°$, $m\angle B = ?$ 55°
b. $m\angle B = 30°$, $m\angle A = ?$ 60°
c. $m\angle A = 25°$, $m\angle B = ?$ 65°
d. $a = 12$, $b = 5$, $c = ?$ 13
e. $a = 9$, $c = 15$, $b = ?$ 12
f. $b = 6$, $c = 10$, $a = ?$ 8

STEPS FOR SOLVING APPLIED PROBLEMS

1. Understand the problem.
2. Note what the problem asks for.
3. Look for hidden questions.
4. Estimate a reasonable answer.
5. Set up and solve the conditional sentence(s).
6. Check your answer.

Part Three

1. If he follows the road to school, George has to walk 90 rods north and 120 rods east. What distance can he save by taking a shortcut directly from home to school? 60 rd.

2. A support cable on a telephone pole is 50 feet long and is attached to the pole 40 feet above the ground. What is the distance from the foot of the pole to the point at which the cable is fastened in the ground? 30 ft.

3. In Exercise 2, what is the measure of the angle formed by the cable and the pole? *approximately 37°*

4. A searchlight is on a tower 240 feet above the ground. How many degrees below the horizontal must it be depressed to shine directly on a spot 600 feet from the foot of the tower? *approximately 22°*

5. What is the angle of elevation of the sun when a tree casts a shadow one-half its length? *approximately 63°*

6. A ladder is most stable if the measure of the angle formed by the foot of the ladder with the ground is 75°. To form this angle, how far should the foot of a 30-foot ladder be placed from a wall against which it rests? *7.77 ft.*

$\frac{1}{2}\sqrt{194}$ ft. or 6 ft. 11.6 in.

7. A screen door 6 feet 6 inches high and 2 feet 6 inches wide is to be braced by a wire stretched diagonally from an upper corner to the lower corner on the opposite side. How long will the wire be?

8. When an observer stands 100 yards from a redwood tree, he finds that the angle of elevation to the top of the tree is 50°. How tall is the tree? *119 yd.*

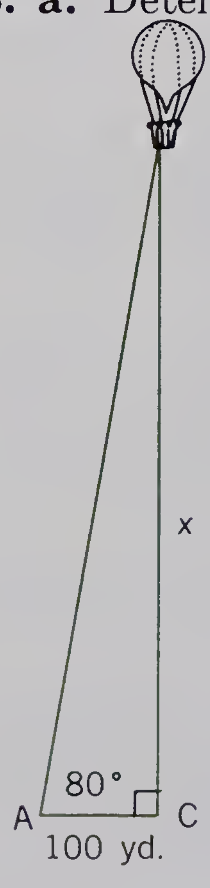
9. A rectangular field is 70 rods long and 50 rods wide. What angle does the diagonal make with the longer side? *approximately 36°*

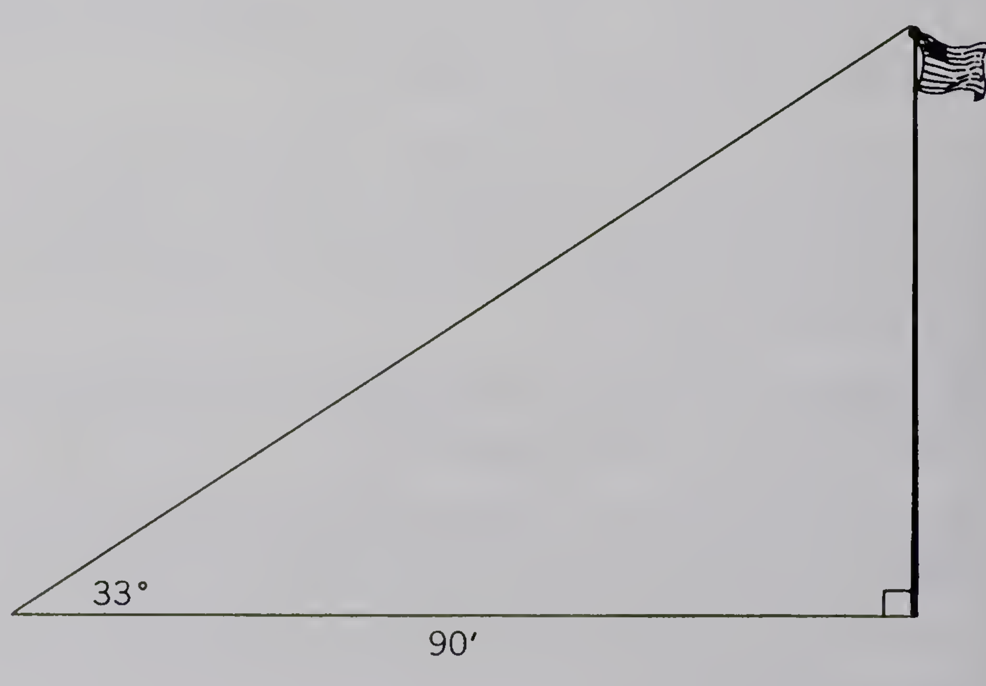
10. Two planes leave an airport at the same time. One travels south at 450 miles per hour, and the other travels west at 500 miles an hour. How far apart will they be after half an hour? *$25\sqrt{181}$ mi. or 336.3 mi.*

11. While Stan is flying his kite, Jim, who is 105 feet away, is directly beneath it. Stan has let out 250 feet of string. What is the angle of elevation of the kite from Stan? *approximately 65°*

12. A mountain road slopes at an angle of 9° with the horizontal. What is the change in altitude after a person has reached a point a mile up the road? *.156 mi. or approximately 824 ft.*

13. **a.** Determine the height of the balloon. *567 yd.* **b.** of the flagpole. *58.41 ft.*





CHAPTER SIX

THE TOOLS OF ALGEBRA

WORDS TO WATCH FOR

absolute value	*ellipse*	*ordered pair*
algebraic	*hyperbola*	*parabola*
circle	*identity element*	*positive number*
commutative property	*irrational number*	*rational number*
conditional statement	*mathematical phrase*	*real number*
conic section	*natural number*	*signed number*
coordinates	*negative number*	*solution set*
distributive property	*open phrase*	*variable*

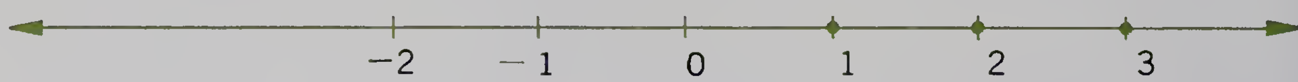
Algebra is a study of properties and relations of sets of numbers and their combinations according to assigned rules. One of the main differences between arithmetic and algebra is that arithmetic deals with specific quantities, while algebra deals with the principles and properties of numbers and relationships between quantities in general. For this purpose algebra makes use of variables that may have different values in different situations. You have used this form of algebra when you used a formula or a conditional sentence.

The basis of algebra, and indeed of all of mathematics, is the use of numbers. From earliest times man has found it necessary to use numbers to solve his problems, to advance his knowledge, and to communicate ideas about quantity. While the history of mathematics goes back to the earliest civilizations, our mathematical knowledge is now advancing more rapidly than ever. More advanced mathematics has been discovered and invented since the beginning of the present century than in all previous history. Much of this rapid advance is due to the improvement and understanding of our system of numeration and its properties. We will begin by reviewing some sets of numbers.

Basic knowledge of sets of numbers and their properties and an understanding of how to apply these properties in solving problems is important to everyone. Since algebra is concerned with sets of numbers and their combinations, we need to examine the most widely used sets in an organized fashion.

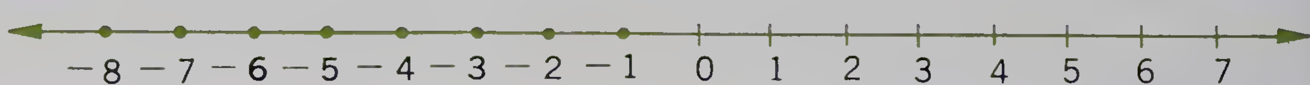
By further study of the number line, we can organize and extend our knowledge of numbers. We use 0 as a point of reference on the line. Using a convenient unit of length, we indicate points to the right

and left of 0 that we associate with 1 and $^{-}1$. Proceeding to the right and adding $^{+}1$ to the previous number, we identify the *positive integers*. Each new integer is called the *successor* of the one on its left. Any integer is less than the one to its right and greater than any to its left. Thus $^{+}5 < {}^{+}8$. Write several other statements, using both $<$ and $>$ to illustrate this fact.



Since we can always add $^{+}1$ to any integer to obtain its successor, there is no greatest positive integer. The set of positive integers is therefore said to have an infinite number of elements. Using set notation, we can represent the positive integers as $\{1, 2, 3, 4, 5 \cdots\}$.

Now let us examine the points associated with integers to the left of 0. Proceeding as we did on the right, but adding $^{-}1$ to each integer, we find that the set of negative integers is also infinite since we can always add $^{-}1$ to a negative integer to find the next integer to the left. Thus we can indicate the set of negative integers: $\{^{-}1, {}^{-}2, {}^{-}3, {}^{-}4, \cdots\}$.



An examination of the number line we now have reveals several facts:

1. We have identified three sets of numbers.
   a. The set of positive integers, with an infinite number of elements;
   b. The set of negative integers, with an infinite number of elements;
   c. Zero, a set with one element.
2. The number line is symmetrical with zero at the point of symmetry. Each point associated with a positive integer has its mate an equal distance to the left of zero. The distance of a point from zero determines its *absolute value*. The direction of a point from zero determines its sign. Thus, the absolute value of either $^{+}7$ or $^{-}7$ is $|7|$. The vertical bars are read as "the absolute value of". Both points are 7 units from zero and $^{+}7$ is to the right and $^{-}7$ is to the left of zero.
3. The "mate" of each integer is its additive inverse.

1. Draw a number line and locate the point associated with each of these integers:

a. 0 d. $^-3$ g. $^-5$ j. $^-6$ m. $^+4$ p. $^+5$ s. $^+2$ v. $^-4$
b. $^-2$ e. $^+1$ h. $^+3$ k. $^+7$ n. $^-1$ q. $^-7$ t. $^+6$ w. $^-9$
c. $^+9$ f. $^-8$ i. $^-10$ l. $^+8$ o. $^+10$ r. $^+12$ u. $^-11$ x. $^+11$

2. Copy the following pairs of integers and insert $<$ or $>$ to make a true statement regarding the pair.

a. $^-2, ^+7$ $<$ e. $^-1, 0$ $<$ i. $^-100, ^+3$ $<$ m. $0, ^-700$ $>$
b. $^+5, ^+3$ $>$ f. $^+14, ^-20$ $>$ j. $^+5, 0$ $>$ n. $^+1, ^-500$ $>$
c. $^-6, ^-2$ $<$ g. $^-3, ^-7$ $>$ k. $^+6, ^-8$ $>$ o. $^+6, ^-1$ $>$
d. $^-6, ^+1$ $<$ h. $^+4, ^+2$ $>$ l. $0, ^-7$ $>$ p. $^-1, ^-300$ $>$

3. Write a signed number that you would use to express the following numerical quantities.

a. 28 feet below sea level $^-28$
b. A loss of \$58 $^-58$
c. A gain of 35 yards $^+35$
d. An increase of 12 pounds $^+12$
e. A drop of 26° Celsius $^-26$
f. 3000 feet above sea level $^+3000$
g. A debt of \$75 $^-75$
h. A drop of 2000 in population $^-2000$

So far our discussion of the development of the number system has been confined to whole numbers and integers. We know that the set of integers is closed under the operations of addition, subtraction, and multiplication. It is also true that the set of integers is *not* closed under division. As an example, if we divide the integer 8 by the integer 11, we do not obtain a quotient that is an integer. Therefore, as we saw in Chapter 3, we must enlarge our number system so that, with the exception of dividing by zero, our new system is also closed under division. For this reason we added fractions and mixed numbers to our set of integers. The resulting set is called the set of *rational numbers*.

Any number which can be written as $\frac{a}{b}$ with a and b both integers, and b not zero, is called a *rational number*.

---

1. In the above definition of a rational number, why is $b \neq 0$?
Dividing by zero is undefined.
2. What have the numbers $\frac{a}{b}$ been called when $b = 1$? integers Are integers rational numbers? Why? Is the set of rational numbers closed under addition? subtraction? multiplication? division? See front.
3. The set of natural numbers is called a subset of the integers. Why?
Each natural number is an integer.
4. The set of integers is called a subset of the rationals. Why?
Each integer is a rational number.

Not all numbers can be written in the form $\frac{a}{b}$ when a and b are integers. Hence they are called *irrational numbers*. Some examples of irrational numbers are $\sqrt{2}$, $^{-}\sqrt{7}$, π, cos 45° and so on. We can compute the values of these numbers as accurately as necessary. For example, $\sqrt{2}$ is equal to 1.4142 to the nearest ten-thousandth. Nevertheless, it can be proved that none of them can ever be expressed exactly as $\frac{a}{b}$ with $b \neq 0$.

Irrational numbers are numbers of the number line that are not rational. They cannot be expressed exactly in the form $\frac{a}{b}$ with both a and b integers and b not equal to zero.

The radical sign $\sqrt{\ }$ does not necessarily indicate an irrational number.

EXAMPLES

1. $\sqrt{36} + 7 = 6 + 7 = 13$
   Therefore, $\sqrt{36} + 7$ is rational.

2. $\frac{\sqrt{25}}{8} = \frac{5}{8}$
   Therefore, $\frac{\sqrt{25}}{8}$ is rational.

3. $\sqrt{17}$ is irrational since 17 is not a perfect square.

4. $7 - \sqrt{54}$ is irrational since 54 is not a perfect square.

As you examine these arithmetic expressions, state whether each is irrational or rational and simplify all the rational numbers.

1. $\frac{\sqrt{49}}{14}$ R ; $\frac{1}{2}$

2. $3 - \sqrt{3}$ I

3. $\frac{6\sqrt{25}}{10}$ R;3

4. $\frac{8\sqrt{36}}{\sqrt{37}}$ I

5. $7 - 3\sqrt{4}$ R ; 1

6. $\frac{\sqrt{7}}{7}$ I

7. $8(\sqrt{5})^2$ R ;40

8. $3\sqrt{2} - 2\sqrt{2}$ I

9. $\sqrt{64} - \sqrt{25}$ R ;3

10. $16 - \sqrt{256}$ R ; 0

11. $(3\sqrt{2})^2$ R ; 18

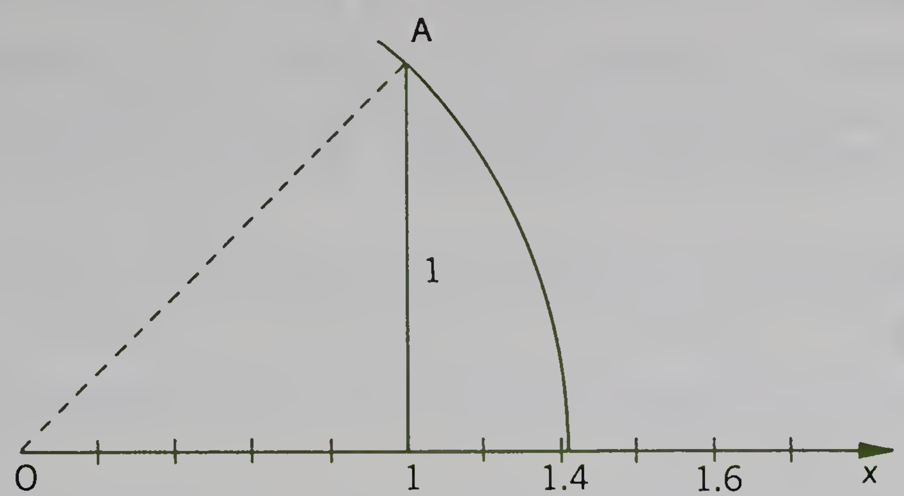
12. $\sqrt{48}$ I

13. $2(13 + \sqrt{7})$ I

14. $-\frac{\sqrt{5}}{3}$ I

15. $8 + \sqrt{36}$ R ; 14

While irrational numbers cannot be expressed exactly in decimal form, they can be located on the number line. To locate $\sqrt{2}$, for example:



The distance from 1 to A is equal to the distance from 0 to 1. Then, according to the Pythagorean Theorem, the distance from 0 to A is $\sqrt{2}$. Why? $1^2 + 1^2 = OA^2$ or $OA = \sqrt{2}$

With a compass this distance can be laid off on the number line. As we should expect, it falls between 1.41 and 1.42.

1. Using the Theorem of Pythagoras as illustrated, draw a number line and locate points corresponding to these irrational numbers: See front.

a. $\sqrt{5}$ **b.** $\sqrt{10}$ **c.** $\sqrt{13}$

2. Look up the approximate decimal value of each of these irrational numbers and locate them on a number line. Use two decimal places.

a. $\sqrt{3}$ 1.73 **b.** Tan 20° .36 **c.** Sin 45° .71 **d.** π 3.14 **e.** Cos 30° .87

3. Which of the tangents is a rational number? Which of the sines? Which of the cosines? tan 0°, tan 45°; sin 0°, sin 30°, sin 90°; cos 0°, cos 60°, cos 90°

4. Is the set of irrational numbers closed under addition? subtraction? multiplication? division? no in each case

5. Under what operations is the set of real numbers closed? Addition, subtraction, multiplication (cannot divide by zero)

6. Are negative numbers less than positive numbers? Do we have more positive numbers than negative numbers? yes; no

7. Are there more rational numbers than irrational numbers? no

8. Is the set of irrational numbers a subset of the rational numbers? no

9. Is the set of negative integers a subset of the rational numbers? yes

10. The set of rational numbers is not closed under division because of one exception. What is the exception? dividing by zero

You have been multiplying, dividing, adding, and subtracting numbers for several years now. Many of the results you accept without question because the answers are consistent and plausible. These results, however, are consistent and plausible only because the arithmetic processes obey certain basic properties. Here are some of those basic properties.

Commutative Property of Addition: If a and b are real numbers, then $a + b = b + a$.

Commutative Property of Multiplication: If a and b are real numbers, then $a \cdot b = b \cdot a$ or $ab = ba$.

Distributive Property of Multiplication over Addition: If a, b, and c are real numbers, then $a(b + c) = a \cdot b + a \cdot c$ or $ab + ac$.

EXAMPLES

Commutative $7 \cdot 3 = 3 \cdot 7 = 21 \qquad 8 + 7 = 7 + 8 = 15$

Distributive $4(3 + 2) = 4 \cdot 3 + 4 \cdot 2 = 20$.

State which property is illustrated. *Solve.*

1. $3 + 2 = 2 + 3 = \square$ commutative property of addition ; 5
2. $5 \cdot 7 = 7 \cdot 5 = \square$ commutative property of multiplication ;35
3. $3 + (^-7) = (^-7) + 3 = \square$ commutative property of addition; $^-4$
4. $4(3 + 5) = 4 \cdot 3 + 4 \cdot 5 = \square$ distributive ; 32
5. $7 + (^-13) = (^-13) + 7 = \square$ commutative property of addition; $^-6$
6. $9(7 + 3) = (9 \cdot 7) + (9 \cdot 3) = \square$ distributive ; 90
7. $\frac{3}{4} \cdot \frac{2}{3} = \frac{2}{3} \cdot \frac{3}{4} = \square$ commutative property of multiplication ; $\frac{1}{2}$
8. $\frac{5}{6} + \frac{3}{4} = \frac{3}{4} + \frac{5}{6} = \square$ commutative property of addition ; $1\frac{7}{12}$
9. $5(\frac{1}{5} + \frac{3}{10}) = 5 \cdot \frac{1}{5} + 5 \cdot \frac{3}{10} = \square$ distributive ; $2\frac{1}{2}$
10. $(^-\frac{7}{8}) + \frac{3}{4} = \frac{3}{4} + (^-\frac{7}{8}) = \square$ commutative property of addition; $\frac{-1}{8}$
11. $5 + (^-5) = (^-5) + 5 = \square$ commutative property of addition ; 0
12. $6 \cdot \frac{2}{3} = \frac{2}{3} \cdot 6 = \square$ commutative property of multiplication ; 4

Let us see if you can discover how these basic properties of arithmetic are used to prove the rules of multiplication and division of signed numbers. Review these properties before you proceed.

Multiplication and Division with Signed Numbers

In performing an exercise in multiplication or division in arithmetic, we know that the sign of the result will be positive. We are using only positive numbers, and the set of positive numbers is closed under multiplication and division.

When we are using signed numbers, however, it becomes important to determine the sign of the product or quotient. Fortunately, there are some simple rules that with a little investigation we can develop and understand.

In multiplication there are three possible situations with respect to the signs of the factors: $f_1 f_2 = p$.

I. Both may be positive: $(^+f_1) \cdot (^+f_2)$.
II. The signs may be unlike: $(^+f_1) \cdot (^-f_2)$ or $(^-f_1) \cdot (^+f_2)$.
III. Both signs may be negative: $(^-f_1) \cdot (^-f_2)$.

Remember! If $f_1 f_2 = p$, then $p \div f_1 = f_2$ or $p \div f_2 = f_1$.

Similar situations exist in division with respect to the signs of the product and the known factor: $p \div f_1 = f_2$.

I. Both may be positive: $^+p \div (^+f_1)$.
II. They may be unlike: $^+p \div (^-f_1)$, or $^-p \div (^+f_1)$.
III. Both signs may be negative: $^-p \div (^-f_1)$.

In this discussion we agree that a quotient or product must exist, and it will be positive or negative but not both. Situation I, for both multiplication and division, where both signs are positive, is the one we are familiar with in arithmetic. The product of positive factors is positive. When the product and one factor is positive, the other factor is positive. Thus:

I. *MULTIPLICATION* The product of two positive factors is positive.

I. *DIVISION* If both product and one factor are positive, the other factor is also positive.

Let us turn to Situation II in multiplication, where the signs of the two factors are different. Multiplication is the process of combining several sets, each having the same number of elements, into one set. In the formula $f_1 f_2 = p$, f_1 is the number of sets, and f_2 is the number of elements in each set. Thus, $(^+4) \cdot (^-3) = (^-3) + (^-3) + (^-3) + (^-3)$. This is the same as adding four $(^-3)$'s, and the sum must be negative, $^-12$.

Suppose the exercise were $(^-4) \cdot (^+3)$. What property of multiplication allows us to write this as $(^+3) \cdot (^-4)$? What is the sign of the product? In other words $(^+4) \cdot (^-3) = (^-3) \cdot (^+4)$, and $(^-4) \cdot (^+3) = (^+3) \cdot (^-4)$ Thus: *commutative ; —*

II. *MULTIPLICATION* If two factors have unlike signs, their product is negative.

---

1. Write the products (Don't forget the sign!)

a. $^+9 \cdot (^-7)$ *$^-63$*	**g.** $^+8 \cdot (^-16)$ *$^-128$*
b. $^-8 \cdot (^+6)$ *$^-48$*	**h.** $^+7 \cdot (^+18)$ *$+126$*
c. $^+9 \cdot (^+8)$ *$+72$*	**i.** $^+9 \cdot (^-15)$ *$^-135$*
d. $^+7 \cdot (^-8)$ *$^-56$*	**j.** $^-5 \cdot (^+25)$ *$^-125$*
e. $^-6 \cdot (^+16)$ *$^-96$*	**k.** $^+8 \cdot (^+14)$ *$+112$*
f. $^+8 \cdot (^+15)$ *$+120$*	**l.** $^+7 \cdot (^+22)$ *$+154$*

Let us now consider Situation II in division, when the product and known factor have unlike signs. For example:

$$^-24 \div (^+6) = n \qquad p \div f_1 = f_2$$

This is equivalent to: $\quad ^+6 \cdot n = {}^-24 \qquad f_1 f_2 = p$

We can see that the absolute value of n must be 4. But what is its sign? Since the sign of the product is negative, from what we have just said about situation II in multiplication, the signs of the factors must be unlike. Then n is $^-4$. Let us look at another example:

$$^+35 \div (^-7) = n \qquad p \div f_1 = f_2$$

This is equivalent to: $\quad ^-7 \cdot n = {}^+35 \qquad f_1 f_2 = p$

It is clear that the absolute value of n is 5. What is its sign? If it were positive, the sign of the product would be negative. Therefore, it cannot be positive, so $n = {}^-5$. Thus:

II. *DIVISION* If the signs of the product and one factor are unlike, the sign of the other factor is negative.

---

2. *Divide* (Don't forget the sign!)

$^-9$	**a.** $^+36 \div (^-4)$	*$^-8$*	**e.** $^+24 \div (^-3)$	*$^-9$*	**i.** $^+63 \div (^-7)$	*$+8$*	**m.** $+48 \div (^+6)$
$^-18$	**b.** $^-54 \div (^+3)$	*$^+16$*	**f.** $^+64 \div (^+4)$	*$+4$*	**j.** $^+28 \div (^+7)$	*$^-6$*	**n.** $^-42 \div (^+7)$
$+4$	**c.** $^+32 \div (^+8)$	*$^-3$*	**g.** $^-81 \div (^+27)$	*$^-3$*	**k.** $^-51 \div (^+17)$	*$^-7$*	**o.** $^-84 \div (^+12)$
$^-8$	**d.** $^+72 \div (^-9)$	*$+8$*	**h.** $^+56 \div (^+7)$	*$^-16$*	**l.** $^-96 \div (^+6)$	*$+17$*	**p.** $^+68 \div (^+4)$

In Situation III in division, both product and known factor are negative. Thus, we have, for example:

$$^-91 \div (^-7) = n \qquad\qquad p \div f_1 = f_2$$

This is equivalent to: $\quad (^-7) \cdot n = {}^-91 \qquad\qquad f_1 f_2 = p$

The absolute value of n is 13. What is its sign? Since the sign of the product is negative, the sign of n cannot be negative. Why? Therefore, the unknown factor must be $^+13$. *The product of two negative numbers is positive.*

III. *DIVISION* If the signs of the product and known factor are both negative, the sign of the unknown factor is positive.

---

3. *Divide* (Don't forget the sign!)

+9 **a.** $^-81 \div (^-9)$ +8 **e.** $^-64 \div (^-8)$ +3 **i.** $^-87 \div (^-29)$ +3 **m.** $^-78 \div (^-26)$
+6 **b.** $^-72 \div (^-12)$ +8 **f.** $^-56 \div (^-7)$ $^-5$ **j.** $^-35 \div (^+7)$ $^-3$ **n.** $^-57 \div (^+19)$
-9 **c.** $^-36 \div (^+4)$ +7 **g.** $^-91 \div (^-13)$ +3 **k.** $^-51 \div (^-17)$ +3 **o.** $^-69 \div (^-23)$
$^-3$ **d.** $^-48 \div (^+16)$ +6 **h.** $^-96 \div (^-16)$ +6 **l.** $^-66 \div (^-11)$ -11 **p.** $^-55 \div (^+5)$

What is the sign of the product in situation III when both factors are negative? For example:

$$^-9 \cdot (^-7) = n \qquad\qquad f_1 f_2 = p$$

This can be written: $\quad n \div (^-9) = {}^-7 \qquad\qquad p \div f_1 = f_2$

We know that the absolute value of n is 63. What is its sign? From Rule II for division we know that when the signs of the known and unknown factors are both negative, the sign of the product is positive. Therefore $n = {}^+63$. Thus:

III. *MULTIPLICATION* If both factors are negative, the sign of the product is positive.

---

4. Find the products (Don't forget the sign!)

+30 **a.** $^-5 \cdot (^-6)$ $^-63$ **e.** $^-7 \cdot (^+9)$ $^-88$ **i.** $^-11 \cdot (^+8)$ +96 **m.** $^-8 \cdot (^-12)$
+63 **b.** $^-9 \cdot (^-7)$ +48 **f.** $^-6 \cdot (^-8)$ +91 **j.** $^+13 \cdot (^+7)$ +341 **n.** $^-31 \cdot (^-11)$
$^-54$ **c.** $^+6 \cdot (^-9)$ +108 **g.** $^-12 \cdot (^-9)$ $^-72$ **k.** $^+12 \cdot (^-6)$ +132 **o.** $^-12 \cdot (^-11)$
$^-60$ **d.** $^-4 \cdot (^+15)$ +90 **h.** $^+9 \cdot (^+10)$ +84 **l.** $^-6 \cdot (^-14)$ +105 **p.** $^+7 \cdot (^+15)$

The sign variation in multiplication can conveniently be summarized as follows:

> If the factors have like signs, the sign of the product is positive. If the signs of the factors are unlike, the sign of the product is negative.

The sign variation in division can similarly be summarized:

> If the signs of the dividend and divisor are like, the sign of the quotient is positive. If they are unlike, the sign of the quotient is negative.

5. Find n (Don't forget the signs!)

a. $(^{+}8)(^{+}9) = n$ $^{+}72$
b. $(^{-}7)(^{-}6) = n$ $^{+}42$
c. $(^{-}3)(^{+}15) = n$ $^{-}45$
d. $(^{+}57) \div (^{-}3) = n$ $^{-}19$
e. $n(^{+}6) = {}^{+}42$ $^{+}7$
f. $n(^{-}9) = {}^{-}63$ $^{+}7$
g. $(^{-}7)(^{-}18) = n$ $^{+}126$
h. $(^{-}8)(^{+}9) = n$ $^{-}72$
i. $(^{+}13)(^{+}7) = n$ $^{+}91$
j. $(^{-}9)(^{+}6) = n$ $^{-}54$
k. $(^{-}5)(^{-}19) = n$ $^{+}95$
l. $(^{+}81) \div (^{-}9) = n$ $^{-}9$
m. $(^{+}15)(^{+}8) = n$ $^{+}120$
n. $n(^{-}8) = {}^{+}40$ $^{-}5$
o. $(^{-}108) \div (^{+}9) = n$ $^{-}12$
p. $(^{+}126) \div (^{-}18) = n$ $^{-}7$
q. $n(^{+}9) = {}^{+}54$ $^{+}6$
r. $(^{-}72) \div (^{-}6) = n$ $^{+}12$
s. $n \div (^{-}13) = {}^{+}7$ $^{-}91$
t. $n(^{-}14) = {}^{+}70$ $^{-}5$
u. $(^{+}16)(^{+}6) = n$ $^{+}96$
v. $(^{+}135) \div (^{-}15) = n$ $^{-}9$
w. $(^{+}136) \div (^{+}17) = n$ $^{+}8$
x. $(^{-}144) \div (^{-}9) = n$ $^{+}16$

6. If the product of two numbers is negative and one factor is positive, the other factor is ? negative

7. If the product of two numbers is positive and one factor is negative, the other factor is ? negative

8. If the quotient of two numbers is negative and the dividend is positive, the divisor must be ? negative

9. If the quotient of two numbers is positive and the divisor is positive, the dividend must be ? positive

10. If the quotient of two numbers is negative and the divisor is positive, the dividend must be ? negative

11. Find n. (Do not forget the signs!)

a. $(^{-}\frac{2}{3}) \cdot (^{-}\frac{3}{4}) = n$ $+\frac{1}{2}$
b. $(^{+}\frac{5}{8}) \cdot (^{-}\frac{4}{5}) = n$ $-\frac{1}{2}$
c. $(^{+}\frac{3}{7}) \cdot (^{-}\frac{7}{8}) = n$ $-\frac{3}{8}$
d. $(^{+}\frac{5}{6}) \cdot n = (^{-}\frac{5}{8})$ $-\frac{5}{8}$
e. $(^{-}\frac{2}{3}) \cdot n = (^{-}\frac{6}{7})$ $+1\frac{2}{7}$
f. $(^{-}\frac{4}{5})n = (^{-}\frac{3}{5})$ $+\frac{3}{4}$
g. $(^{-}\frac{5}{7}) \div n = (^{+}\frac{5}{7})$ -1
h. $(^{-}\frac{3}{5}) \div n = (^{-}\frac{3}{5})$ $+1$
i. $(^{+}\frac{4}{9}) \div n = (^{-}\frac{9}{10})$ $-\frac{40}{81}$
j. $n \div (^{-}\frac{2}{5}) = (^{-}\frac{5}{8})$ $+\frac{1}{4}$
k. $n \div (^{+}\frac{7}{8}) = (^{-}\frac{8}{9})$ $-\frac{7}{9}$
l. $n \div (^{-}\frac{5}{6}) = (^{+}\frac{6}{9})$ $-\frac{5}{9}$

Signed Numbers

A. *Add:*

1. $(^+3) + (^-7) + (^-5)$ $^-9$

2. $(^-5) + (^-7) + (^-11)$ $^-23$

3. $(^-7) + (^-3) + (^+10)$ 0

4. $(^-9) + (^-12) + (^+15)$ $^-6$

5. $\begin{array}{r} ^-3 \\ ^+8 \\ ^-5 \\ \underline{^+14} \end{array}$ $^+14$

6. $\begin{array}{r} ^-3\frac{1}{2} \\ ^+2\frac{1}{2} \\ \underline{^-1\frac{3}{4}} \end{array}$ $^-2\frac{3}{4}$

7. $\begin{array}{r} ^-1.78 \\ ^-2.63 \\ ^-4.77 \\ \underline{^-1.04} \end{array}$ $^-10.22$

8. $\begin{array}{r} ^-3\frac{1}{2} \\ ^+5\frac{5}{6} \\ \underline{^+7\frac{3}{4}} \end{array}$ $^+10\frac{1}{12}$

9. $\begin{array}{r} ^-11.38 \\ ^+2.66 \\ ^-5.01 \\ \underline{^+8.77} \end{array}$ $^-4.96$

10. $\begin{array}{r} ^-2\frac{2}{3} \\ ^-3\frac{5}{8} \\ \underline{^-6\frac{1}{2}} \end{array}$ $^-12\frac{19}{24}$

B. *Subtract:*

1. $(^+4) - (^+2)$ $^+2$ **2.** $(^-5) - (^-7)$ $^+2$ **3.** $(^+17) - (^+13)$ $^+4$ **4.** $(^-11) - (^-11)$ 0

5. Subtract $^-8$ from $^-5$. $^+3$

6. Subtract $^+8$ from $^+5$. $^-3$

7. Subtract $^+5$ from $^-11$. $^-16$

8. Subtract $^-2\frac{3}{4}$ from $^+3\frac{3}{8}$. $^+6\frac{1}{8}$

9. Subtract $^+.583$ from $^-.677$ $^-1.260$

10. Subtract $^+3.889$ from $^+7.66$. $^+3.771$

C. *Multiply:*

1. $(^+3) \cdot (^-7)$ $^-21$

2. $(^-5) \cdot (^-17)$ $^+85$

3. $(^-3) \cdot (^-6) \cdot (^+3)$ $^+54$

4. $(^-\frac{1}{2}) \cdot (^-\frac{3}{4}) \cdot (^+\frac{5}{6})$ $^+\frac{5}{16}$

5. $(^-3.67) \cdot (^-2.17)$ $^+7.9639$

6. $(^-\frac{25}{8}) \cdot (^+5\frac{2}{3})$ $^-17\frac{17}{24}$

7. $(^-3) \cdot (^-3) \cdot (^-3)$ $^-27$

8. $(^+18) \cdot (^+11) \cdot (^-17)$ $^-3366$

9. $(^-3\frac{1}{2}) \cdot (^+2) \cdot (^-2\frac{1}{2})$ $^+17\frac{1}{2}$

D. *Divide:*

1. $(^-84) \div (^-28)$ $^+3$

2. $(^-78.4) \div (^+14)$ $^-5.6$

3. $(^-25.74) \div (^-14.3)$ $^+1.8$

4. $(^+1728) \div (^-12)$ $^-144$

5. $(^-3.67) \div (^-.1835)$ $^+20$

6. $(^-2\frac{1}{2}) \cdot (^-1\frac{3}{4}) \div (^-5\frac{2}{3})$ $\frac{^-105}{136}$

7. $(^-14) \cdot (^-15) \div (^-35)$ $^-6$

8. $(^-32.6) \cdot (^+4.2) \div (^-7)$ $^+19.56$

9. $(^-3\frac{5}{8}) \cdot (^-2\frac{3}{4}) \div (^-2)$ $^-4\frac{63}{64}$

10. $(^-3.2) \cdot (^-4.0) \div (^-6.4)$ $^-2$

If you need practice, turn to the exercises on page 500. If not, you may work the Experts' Corner which follows.

Archimedes—Value of Pi

A variety of approximations for the value of π—the ratio of the circumference of a circle to the measure of a diameter—were used in ancient times. The best of these approximations was developed by Archimedes, a brilliant mathematician and engineer who lived in Syracuse, Sicily during the third century before Christ. His method of determining this ratio relied upon geometric constructions, such as you used in Chapter 1. He constructed inscribed and circumscribed polygons and found that as he increased the number of sides of the polygons, the ratios of the perimeter of the inscribed and circumscribed polygons compared to the diameter of the circle tended to approach a certain value. The ratio that he determined by this method was close to the value of π that we now use.

His procedure was one that you can readily use. As you see in the illustrations, it is simpler if you use separate equal circles of the same diameter for the inscribed and circumscribed figures. To inscribe the square, first construct the circle and then draw two diameters perpendicular to each other. Connect the points where the diameters intersect the circle and you have the inscribed square. For the circumscribed square, construct perpendiculars to each diameter at their endpoints. Remember, you will need to extend the diameter in order to construct the perpendiculars.

1. You can use the Pythagorean property to find the length of a side of an inscribed square. If the diameter of the circle is 4″, and the side is represented by s, explain why $2s^2 = 16$. $s^2 + s^2 = 16$

2. What is the ratio of the perimeter of the inscribed square to the measure of the diameter of the circle? $\frac{2.8284}{1}$

3. What is the perimeter of the circumscribed square? What is its ratio to the diameter of the circle? 16 in.; $\frac{4}{1}$

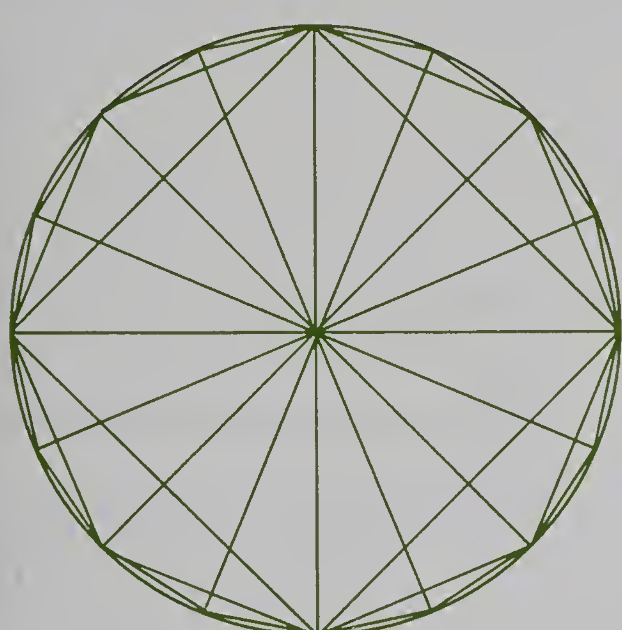
4. Complete this table for the squares:

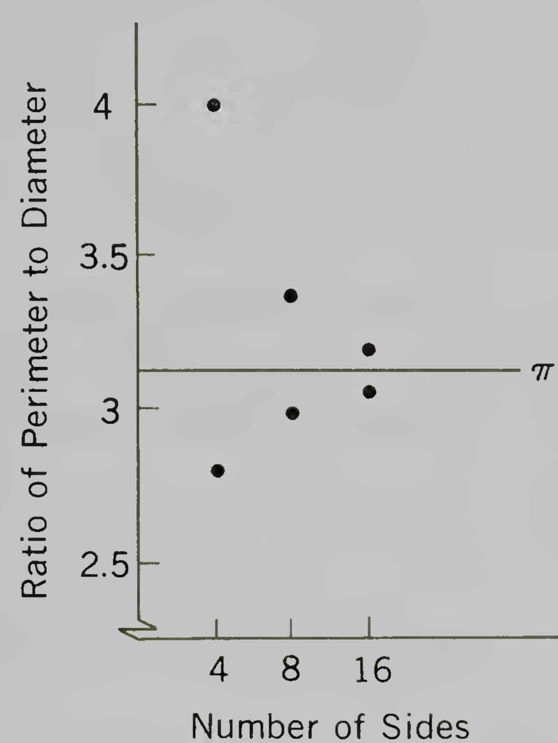
(ex. 5)	(ex. 6)		
8	16	Number of sides in polygon	4
13.3″	12.7″	Perimeter of circumscribed polygon	□ 16″
12.3″	12.5″	Perimeter of inscribed polygon	□ 11.3″
$\frac{3.325}{1}$	$\frac{3.175}{1}$	$\frac{\text{perimeter of circumscribed polygon}}{\text{measure of diameter}}$	□ $\frac{4}{1}$
$\frac{3.075}{1}$	$\frac{3.125}{1}$	$\frac{\text{perimeter of inscribed polygon}}{\text{measure of diameter}}$	□ $\frac{2.8284}{1}$

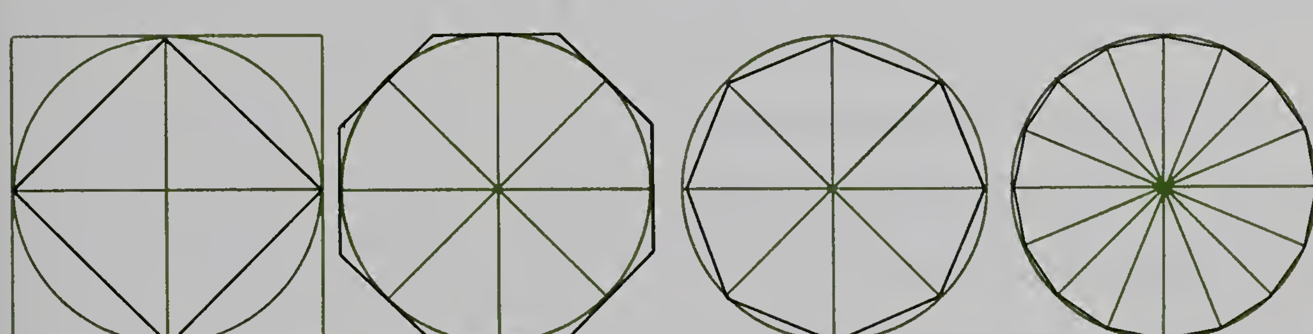
5. By bisecting the angles formed by the diameters, you can construct an inscribed and a circumscribed octagon as illustrated. Measure a side of each octagon and calculate its perimeter. Then prepare a table as we did in Exercise 4. *See exercise4, page 230.*

6. Again bisecting the angles formed by the diameters, you can construct sixteen-sided polygons, one inscribed and one circumscribed. Measure a side of each and calculate its perimeter. Prepare a table as you did in Exercise 4. *See exercise 4, page 230.*

7. From the values on the tables, prepare a line graph, as illustrated. Archimedes assumed that as the number of sides increased, the ratios would approach a constant value. Does it appear that this is the case? *yes*







8. It is apparent that this method is limited in precision by the accuracy of the constructions. Can you come within $\pm$ 0.01 to a value of 3.14 for π?

9. The history of man's efforts to determine the value of π can be found in many reference books. Investigate some of these sources and prepare a report of your findings.

We can solve many problems with arithmetic. We know how to find the sum of two numbers, their difference, their product, and their quotient. We have learned to solve many different kinds of problems which involve these processes. There is a limit, however, to what we can do with arithmetic alone. In order to extend our problem-solving ability, we need to study algebraic techniques.

One main difference between arithmetic and algebra is the use of variables. The most convenient symbols available for variables are the letters of the alphabet and therefore, they are the most commonly used. By the use of the signs of operation of arithmetic ($+$, $-$, $\times$, $\div$), we can translate English phrases into mathematical phrases.

EXAMPLES

1. English Phrase: A number increased by 7
   Mathematical Phrase: $n + 7$
2. English Phrase: A number decreased by 11
   Mathematical Phrase: $n - 11$
3. English Phrase: Three times a number increased by 3
   Mathematical Phrase: $3n + 3$

Translate the following English phrases into mathematical phrases.

1. A number increased by 2 $n+2$

2. 5 more than a certain number $n+5$

3. Twice a number decreased by 17 $2n-17$

4. Three times a number increased by 5 $3n+5$

5. 7 less than a certain number $n-7$

6. 4 more than 3 times a certain number $3n+4$

7. A number divided by 5 $\frac{n}{5}$

8. 11 less than 5 times a number $5n-11$

9. 10 more than the product of 4 and a certain number $4n+10$

10. Twice a number increased by the product of 5 and a second number $2n+5m$

11. 5 times a certain number decreased by 3 times the same number $5n-3n$

12. 7 decreased by 5 times a certain number $7-5n$

13. Twice a certain number increased by the product of 2 and 3 $2n+2\cdot 3$

TRANSLATING ALGEBRAIC PHRASES

We can also translate algebraic phrases into English phrases. Study these examples.

EXAMPLES

1. Algebraic Phrase: $3n + 7$
   English Phrase: (a) Three times a number increased by 7
   or (b) 7 more than 3 times a number.
2. Algebraic Phrase: $2n - 5$
   English Phrase: (a) Twice a number decreased by 5
   or (b) 5 less than twice a number.

Translate the following algebraic phrases into English phrases. *See front.*

1. $2n + 11$
2. $3r - 2$
3. $7 + 5x$
4. $11 - 3a$
5. $3a + 2b$
6. $8y + 5$
7. $\frac{1}{2}n - 3$
8. $14 - \frac{1}{3}a$
9. $3n - 2m$
10. $8y + 3r$
11. $5m - 17$
12. $\frac{1}{3}a + \frac{1}{2}b$
13. $8r + \frac{1}{2}s$
14. $3n - 18$
15. $18 - 3n$

It is quite obvious that the above phrases have different numerical values when different values are assigned to the letters.

What is the numerical value of $3r + 5$ if r is assigned the value 2? The answer is $3 \times 2 + 5 = 6 + 5 = 11$.

What is the value of the same phrase $3r + 5$ if r is assigned the value 4? The answer is $3 \times 4 + 5 = 12 + 5 = 17$.

Since each phrase has an unlimited number of values, we call it an *open phrase*. Since the value of the letter in the phrase can vary, the letter is called the *variable*. Evaluate the following open phrases if we assign n the value 4, and to m the value 3.

1. $3n - 2m$ 6
2. $5n + 7m$ 41
3. $5n + 2m$ 26
4. $\frac{1}{2}n - m$ $^-1$
5. $3m - 2n$ 1
6. $\frac{1}{2}n + \frac{1}{5}m$ $2\frac{3}{5}$
7. $7 + 5m$ 22
8. $8 - m$ 5
9. $5n + 8m$ 44
10. $\frac{1}{4}n - m$ $^-2$
11. $5n - 2m$ 14
12. $2n - \frac{1}{3}m$ 7
13. $5n - m$ 17
14. $2n - 11m$ $^-25$
15. $8n - 2m$ 26

A complete mathematical statement containing a variable may be called an *algebraic sentence.*

The sentence "A number increased by 5 is equal to 7" written in mathematical form would be $n + 5 = 7$. Thus if we wish to make a statement regarding a number whose value is not known, it can be written as an algebraic equation, letting a variable represent the unknown number. Study the following examples.

EXAMPLES

1. English Sentence: A certain number increased by 3 is equal to 11.
   Algebraic Equation: $n + 3 = 11$.
2. English Sentence: Twice a certain number decreased by 5 is 15.
   Algebraic Equation: $2n - 5 = 15$.

The algebraic equations in the above examples are true only when the proper value of the variable is chosen. Consequently these algebraic equations are conditional statements. The value or values of the variable which makes the algebraic equation a true statement is called the "solution set." If there is more than one element in the solution set, we call each element a "solution." We find the solution set of an equation when the directions ask us to "solve." You may wish to review Chapter Two to recall the terminology.

Write an equation to express the following statements. Solve.

1. Twice a certain number increased by 5 is equal to 13. $2n+5=13$; 4
2. 5 times a number decreased by 5 is equal to 20. $5n-5=20$; 5
3. One half a number increased by 7 will total 11. $\frac{1}{2}n+7=11$; 8
4. Twice Harry's age increased by 8 is equal to 40. $2n+8=40$; 16
5. The number 8 decreased by 3 times a certain number is equal to 2. $8-3n=2$; 2
6. One third of the class enrollment increased by 10 is equal to 20. $\frac{1}{3}n+10=20$; 30
7. 3 times a certain number decreased by nine is equal to zero. $3n-9=0$; 3
8. One fourth of a certain number increased by 7 times the same number is equal to 6. $\frac{1}{4}n+7n=6$; $\frac{24}{29}$
9. The product of 6 and a certain number, decreased by 12 is equal to 11. $6n-12=11$; $3\frac{5}{6}$
10. Half a number decreased by 11 is equal to 11. $\frac{1}{2}n-11=11$; 44
11. Twice a number is 7 more than 15. $2n=15+7$; 11
12. 18 decreased by twice a number leaves 10. $18-2n=10$; 4

As you have probably noticed, some solutions are more difficult to discover than others. In algebra we often find it necessary to solve equations considerably more complex than those in the previous problem set. Obviously it would be helpful if we could develop a systematic way to solve any first degree equation in one variable, no matter how complex it was. Basically we shall use our rules of operation in order to isolate the variable on one side of the equal sign. When this is done, we will have solved the equation. The next two pages will demonstrate some ways to do this and give you practice with more complex equations.

As you know, an algebraic equation is true if, and only if, the variable is assigned a certain value, or values which make the equation true. The value or values which result in true equations make up the solution set of the equation. We have also learned that if we add, subtract, multiply, or divide both sides of an equation by the same number (except zero), we will obtain a different but *equivalent equation*. This equivalent equation will have the same solution set as the original equation. Thus, if we can change the original equation to an equivalent equation in which the variable is isolated, our solution will be complete. Study the following examples.

EXAMPLES

1. *Solve:* $n - 7 = 13$ In order to isolate the variable n, we will add $^+7$ to both sides of the equation.

$$n - 7 + (^+7) = 13 + (^+7) \qquad \textit{Check: } 20 - 7 = 13$$
$$n = 20$$

The solution set is $\{20\}$.

2. *Solve:* $a + 5 = 24$ In order to isolate the variable a, we will subtract 5 from both sides of the equation. Is this operation equivalent to adding $^-5$ to both sides?

$$a + 5 - 5 = 24 - 5 \qquad \textit{Check: } 19 + 5 = 24$$
$$a = 19$$

The solution set is $\{19\}$.

3. *Solve:* $3a - 7 = 11$ In order to isolate the variable a completely, add 7 to both sides of the equation and then divide the resulting equivalent equation by 3.

$$3a - 7 + 7 = 11 + 7 \qquad \textit{Check: } 3(6) - 7 = 11$$
$$\frac{3a}{3} = \frac{18}{3}$$
$$a = 6$$

Check: $3(6) - 7 = 11$ Could we multiply both sides by the reciprocal of 3?

The solution set is $\{6\}$.

4. *Solve:* $\frac{1}{5}w - 3 = 5$ In order to isolate the variable w, we will add 3 to both sides of the equation and then multiply both sides of the resulting equation by 5.

$$\frac{1}{5}w - 3 + 3 = 5 + 3$$
$$5 \cdot \frac{1}{5}w = 5 \cdot 8$$
$$w = 40$$

The solution set is $\{40\}$.

Check: $\frac{1}{5}(40) - 3 = 5$

Is multiplying by 5 equivalent to dividing by $\frac{1}{5}$? yes

Using the previous examples, can you explain each step as you study these?

5. *Solve:* $4n + 7 = 2n - 11$
$$4n = 2n - 18$$
$$2n = {}^{-}18$$
$$n = {}^{-}9$$

Check: $4({}^{-}9) + 7 = 2({}^{-}9) - 11$
The solution set is $\{{}^{-}9\}$.

6. *Solve:* $5y - 16 = 3y + 4$
$$2y = 20$$
$$y = 10$$

Check: $5(10) - 16 = 3(10) + 4$
The solution set is $\{10\}$.

As you set up each equation and solve it, use the techniques developed in the Examples to isolate the variable.

1. $a + 7 = 38$ 31

2. $n - 5 = 1$ 6

3. $2n = 48$ 24

4. $2a - 5 = 17$ 11

5. $3x - 2 = 12$ $4\frac{2}{3}$

6. $3 + 2y = 9$ 3

7. $\frac{x}{32} = 35$ 1120

8. $3a - 8 = 11$ $6\frac{1}{3}$

9. $5a = a + 20$ 5

10. $3n - 5 = n + 13$ 9

11. $\frac{1}{2}r + 3 = 8$ 10

12. $3s - 5 = 2s - 3$ 2

13. $\frac{7r}{2} = 21$ 6

14. $2n + 8 = n - 30$ ${}^{-}38$

15. $2n - 7 = n - 6$ 1

16. $\frac{3n}{4} + 8 = 8 - \frac{n}{4}$ 0

17. $5a - 6 = 3a + 1$ $3\frac{1}{2}$

18. $\frac{2a}{3} + 5 = \frac{a}{3} + 7$ 6

19. $6s - 7 = 2s + 15$ $5\frac{1}{2}$

20. $4n + 7 = 2n + 13$ 3

21. $3\frac{1}{2}x = 14$ 4

22. $0.4k = 8$ 20

23. $2\frac{1}{3}x + 5 = \frac{2x}{3} + 20$ 9

24. $5x + 3 = 48$ 9

25. $4x - 6 = 10$ 4

Remember to check your answers in the remainder of the exercises.

26. $2a + 16 = 30$ *7*

27. $\frac{1}{2}x + 5 = 7$ *4*

28. $\frac{3x}{5} - 5 = 1$ *10*

29. $2b - 15 = 10$ *$12\frac{1}{2}$*

30. $\frac{2x}{3} + 1\frac{1}{3} = 5\frac{1}{3}$ *6*

31. $3y + 6 = 24$ *6*

32. $4x - 8 = x + 4$ *4*

33. $2a - 9 = 11 - 2a$ *5*

34. $2.5x - 25 = 25$ *20*

35. $5r = 24 - r$ *4*

36. $3.5b + 6 = b + 11$ *2*

37. $\frac{3x}{4} + 9 = \frac{x}{4} + 13$ *8*

38. $2\frac{1}{2}r = 3 - 2r$ *$\frac{2}{3}$*

39. $x - 14 = 27$ *41*

40. $\frac{a}{2} + 3 = 13$ *20*

41. $\frac{x}{12} = \frac{1}{3}$ *4*

42. $2s - 5 = 9$ *7*

43. $\frac{2x}{3} - 14 = 20$ *51*

44. $3t = 16 - t$ *4*

45. $3m + 4 = 7 - 2m$ *$\frac{3}{5}$*

46. $\frac{x}{6} = \frac{2}{3}$ *4*

47. $5x - 7 = 2x - 1$ *2*

48. $4n - 3 = 65$ *17*

49. $5r - 4 = 2r - 1$ *1*

50. $a + 7 = 10 - 2a$ *1*

51. $4r - 5 = 3 - 2r$ *$1\frac{1}{3}$*

52. $\frac{1}{2}s - 2 = \frac{1}{4}s + 7$ *36*

53. $7 - 2r = 9 - 5r$ *$\frac{2}{3}$*

54. $5a - 2 - a = 17$ *$4\frac{3}{4}$*

55. $2 + 7b = 17 - 3b$ *$1\frac{1}{2}$*

56. $\frac{3x}{4} - 7 = \frac{x}{4} + 11$ *36*

57. $\frac{3a}{5} = \frac{17}{11}$ *$2\frac{19}{33}$*

58. $2.4a - 3.7 = .4a + .3$ *2*

59. $5x - 3 = 17 - 5x$ *2*

60. $17 - 2x = 25 - 3x$ *8*

STEPS FOR SOLVING APPLIED PROBLEMS

1. Understand the problem.
2. Note what the problem asks for.
3. Look for hidden questions.
4. Estimate a reasonable answer.
5. Set up and solve the conditional sentence(s).
6. Check your answer.

To solve problems using equations, use the following five steps:

Step 1. Use a variable to stand for the unknown quantity. If there is more than one unknown quantity, usually the variable stands for the smallest.
Step 2. Express any other unknown quantity using the same variable.
Step 3. From the data given in the problem, set up an equation.
Step 4. Solve the equation.
Step 5. Check the answer.

EXAMPLES

1. Three times a certain number increased by 8 is equal to 17. What is the number?

Solution: Let n = the number

$$\begin{aligned} 3n + 8 &= \text{three times the number increased by } 8 \\ 3n + 8 &= 17 \text{ is the equation} \\ -8 \quad & \quad -8 \text{ (Subtract 8 from both sides)} \\ 3n &= 9 \\ \frac{3n}{3} &= \frac{9}{3} \text{ (Divide both sides by 3.)} \\ n &= 3 \quad \text{Does this check?} \end{aligned}$$

2. George weighs 10 lbs. more than Fred. The sum of their weights is 270 lbs. What does each boy weigh?

$$\begin{aligned} \text{Let } w &= \text{Fred's weight} \\ w + 10 &= \text{George's weight} \\ 2w + 10 &= \text{the sum of their weights} \\ 2w + 10 &= 270 \quad \text{is the equation} \\ 2w &= 260 \quad \text{(Subtract 10 from both sides)} \\ \frac{2w}{2} &= \frac{260}{2} \quad \text{(Divide both sides by 2)} \\ w &= 130 \quad \text{Fred's weight is 130 lbs.} \\ w + 10 &= 140 \quad \text{George's weight is 140 lbs.} \\ \text{Checking } 140 + 130 &= 270 \end{aligned}$$

Set down each step as in the examples above. Set up an equation for each of the following problems and solve.

1. The larger of two numbers is three times the smaller. If their sum is 84, find each number. *21 and 63*

2. Helen's age is six years less than twice Mary's age. The sum of their ages is 30 years. What is the age of each girl? *Helen, 12; Mary, 18*

3. Bill and Harry bought a motorcycle together. Harry paid \$75 more than Bill. If the total price was \$485, how much did each boy pay? *Bill, \$205 ; Harry, \$280*

4. The first side of a triangle is 2 inches longer than the second side. The third side is 10 inches longer than the first side. If the perimeter of the triangle is 53 inches, find the length of each side. *15", 13", 25"*

5. The number of girls in Mr. Allen's algebra class is 7 less than twice the number of boys. The class enrollment was 29 students. How many girls and how many boys were enrolled in the class? *12 boys and 17 girls*

6. If the second angle of a triangle has a measure 10 degrees more than the first, and the third angle has a measure 20° more than the first, what is the measure of each angle? (Remember: The sum of the measures of the three angles of any triangle is always 180°.) *50°, 60°, 70°*

7. The width of a rectangular lot is 30 feet less than the length. What are the dimensions of the lot if the perimeter is 460 feet? *100' wide ; 130' long*

8. If five times a number is decreased by 8 the result is the same as when the number is increased by 8. What is the number? *4*

9. The sides of a triangle are x, $x + 5$, and $x + 10$. The perimeter is 45 inches. What is the length of each side? *10", 15", 20"*

10. Mary and Jane together earned \$60. Their wages per hour were the same, but Mary worked twice as long as Jane. How much did each earn? *Mary, \$40; Jane, \$20*

11. Henry and Jim bought a car for \$360. Henry paid twice as much as Jim. How much did each pay? *Jim, \$120 ; Henry, \$240*

12. Mr. Jensen's orchard is four times as long as it is wide. It requires 600 rods of fencing to go around it. What are its dimensions? *60 rd. wide ; 240 rd. long*

13. During a period of three weeks, Bob Devlin worked part time. He earned \$7 more the first week than the second week. The third week he earned twice as much as the second week. He earned \$51 for the three week period. How much did he earn each week? *first week, \$18 ; second week, \$11 ; third week, \$22*

14. If a certain number is multiplied by 3, the result is the same as if 25 were added to half the number. What is the number? *10*

15. In a certain factory a machinist's weekly earnings are \$5 less than twice as much as those of his helper. Together they earn \$145. How much does each earn? *machinist, \$95 ; helper, \$50*

16. If twice a number is increased by 2, the result is the same as if one-half the number were increased by 20. What is the number? *12*

Consider the problem: Henry and George together have $27. Henry has twice as much as George. How much has each?

Using one variable, we could set up an equation representing George's money as n and Henry's money as $2n$, $n + 2n = 27$. Find n.

Some find it simpler to use two variables and two equations. That is:

Let x represent the number of dollars George has, and y the number of dollars Henry has.

We then know two things about the situation:

I $x + y = 27$ What English sentences are represented
II $y = 2x$ by these equations?

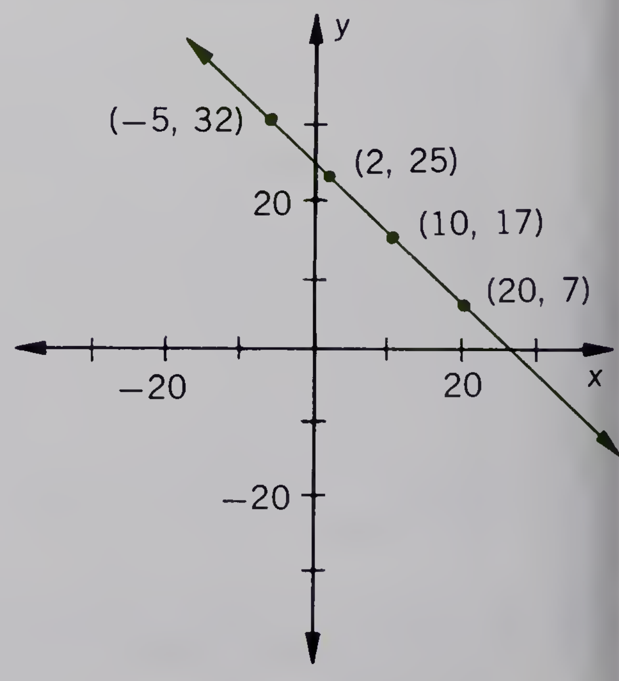
We need to use both equations to find the solution. Let us take them one at a time. First take equation I and set up pairs of values that make the equation true. If $x = 5$, then $y = 27 - 5$, or 22. Thus we can set up a series of pairs.

x	$^-7$	$^-6$	$^-5$	$^-4$	$^-3$	$^-2$	$^-1$	0	1	2	3	4	5
y	34	33	32	31	30	29	28	27	26	25	24	23	22

Check each pair and complete those for which only x is given.

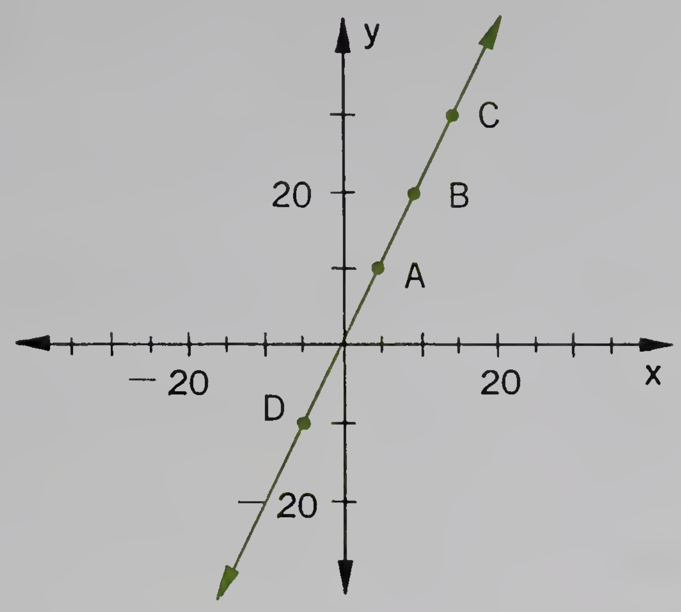
Rather than calculating each pair in the *solution* set, it is quicker to plot the coordinates on squared paper as in making a line graph. Use the vertical axis as the y-axis and the horizontal as the x-axis. Since we want to include negative as well as positive values, the origin (zero for both axes) is in the middle of the graph. Positive x values are to the right of the y-axis, and negative to the left. Positive y values are above the x-axis, and negative below. The horizontal x-axis and the vertical y-axis are called the *coordinate axes*.

Several pairs of values are shown for equation I. Since they lie along a straight line, it is necessary to plot only two to determine the line. Why? It is common practice, however, to plot a third as a check. The number of pairs of values for x and y that satisfy the equation is infinite. Therefore the line can be extended without limit in each direction. As you examine the line on the graph you can see the arrows indicate this fact.



1. Turning to equation II, $y = 2x$

What is the value for y if $x = 10?$ 20, $8?$ 16, $6?$ 12, $0?$ 0, $^{-}5?$ $^{-}10$

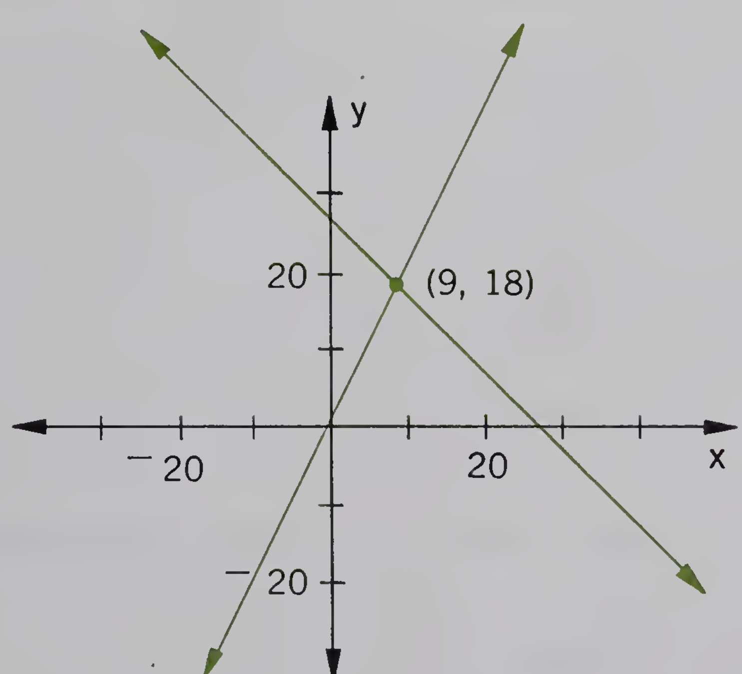


2. Several pairs of values have been plotted for equation II. Point A is how many spaces up from the x-axis? 10 How do you know that it represents $y = 10$? *The number of units up or down from the x-axis represents the value of y.*

3. Point A is how many units to the right of the y-axis? 5 How do you know that it represents $x = 5$? *The number of units to the right or left of the y-axis represents the value of x.*

4. Explain the location of points B, C, and D in the same manner. How many points are contained in this line? *infinite number*

5. In the graph the lines for equations I and II are both plotted on the same graph. What are the x and y values at the point of intersection? See the figure below. *$x = 9$; $y = 18$*



6. Do these values satisfy both equations? Do they satisfy the conditions of the original problem? Substitute the values and see. *yes*

While there are other methods for solving pairs of equations with two variables that are sometimes simpler to use, it is always possible to use the method of graphing. In general the procedures for graphing have three steps:

Step 1. For each equation, find at least three pairs of values that satisfy the equation.

Step 2. Plot the coordinates for the values and draw the line for each equation. (This line contains all pairs of real number values that satisfy the equation. We do not imply that a line always represents the conditions of the verbal problem.)

Step 3. Check the coordinates for the point where the lines intersect. Do they satisfy both equations?

Why use three pairs of values for each equation when two points are sufficient to determine a line? Why should the point of intersection be the solution to both equations? *The third pair of values is a check. The pair of values at the point of intersection is common to both equations.*

Use the steps in solving these equations by graphing.

1. $2x - y = 3$
$x + y = 6$ *(3,3)*

2. $3x + y = 4$
$x + 2y = 3$ *(1,1)*

3. $x - 2y = 5$
$3x + y = 1$ *(1,-2)*

4. $5x + y = 48$
$y = 3x$ *(6,18)*

5. $4x + y = 4$
$x - y = 1$ *(1,0)*

6. $2x + 3y = 7$
$x = 2y$ *(2,1)*

Use two variables and two equations to solve these problems.

7. Find two numbers whose sum is 12 and whose difference is 2. *7 and 5*

8. The perimeter of a rectangle is 20 feet. Twice the length is 8 feet more than the width. Find the length and width. (Use x = length, y = width.) *length, 6 ft.; width, 4 ft.*

9. Mary is four years older than Helen. The sum of their ages is 26 years. What is the age of each? *Mary, 15; Helen, 11*

10. How would you graph the equation $y = 2$ or $x = 7$? Remember that each point on the coordinate plane is located by an ordered pair. *See front.*

11. **a.** Determine the coordinates of the intersection of the graphs of $y = 3x + 2$ and $y = 2x$. *(-2,-4)*

b. $y = 2x$ and $y - 3x = {}^-2$ *(2,4)*

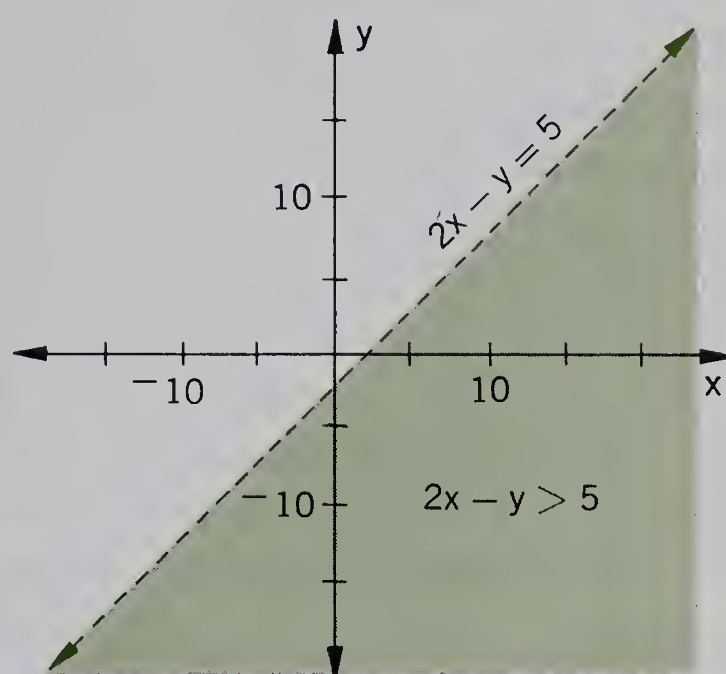
12. Determine the vertices of the triangle formed by the intersection of the graphs of $y + 2x = 3$, $y - 2x = 7$, and $x = {}^-4$. *(-1,5); (-4,-1); (-4,11)*

Often a problem is stated in terms of an inequality. Here also graphing provides a general method for solution. For example, suppose that twice Jim's money is over \$5 more than Eric has. The conditional statement is: $2x - y > 5$. What values of x and y satisfy this inequality?

If $x = 100$ and $y = 0$, then $2x - y > 5$. We can find as many pairs that satisfy this inequality as we wish. The solution set has an infinite number of pairs.

We can graph the solution set of $2x - y > 5$ using these steps:

1. Graph $2x - y = 5$



2. This line divides the entire plane of our graph into three parts. There are two half-planes, one on each side of the line, and the line itself. The line represents all pairs of values of x and y which satisfy $2x - y = 5$. One side of the line represents all pairs which satisfy $2x - y < 5$ and the other side represents all the pairs that satisfy $2x - y > 5$.
3. Which sides of the line of $2x - y = 5$ represents the solution set of $2x - y < 5$? ***left side***

To answer this question, take any point on either side of the line. Substitute the coordinates of this point into the expression $2x - y > 5$. Whether the coordinates satisfy the expression or not, tells us which side of the line represents the pairs of the solution set.

4. If the line itself is composed of points which are not on the solution set, we usually make it a broken line. If the line is part of the solution set, it is drawn solid. The side which does represent our set is then shaded.

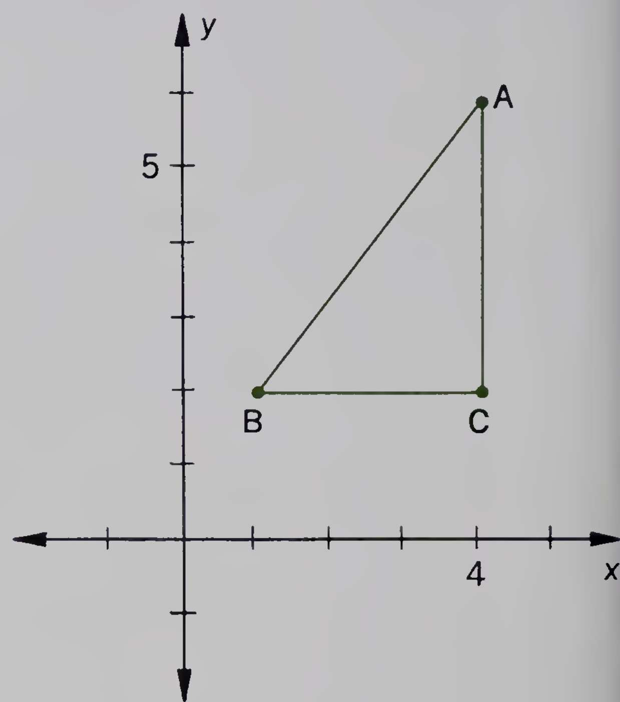
Any point in the shaded area has coordinates which satisfy $2x - y > 5$. Graph the solution set of: ***See front.***

1. $3x + y > 7$	**4.** $2x + 2y < -5$
2. $x - 2y < 3$	**5.** $3x - y \geq 9$
3. $5x + 2y > 9$	**6.** $2x + y \leq 7$

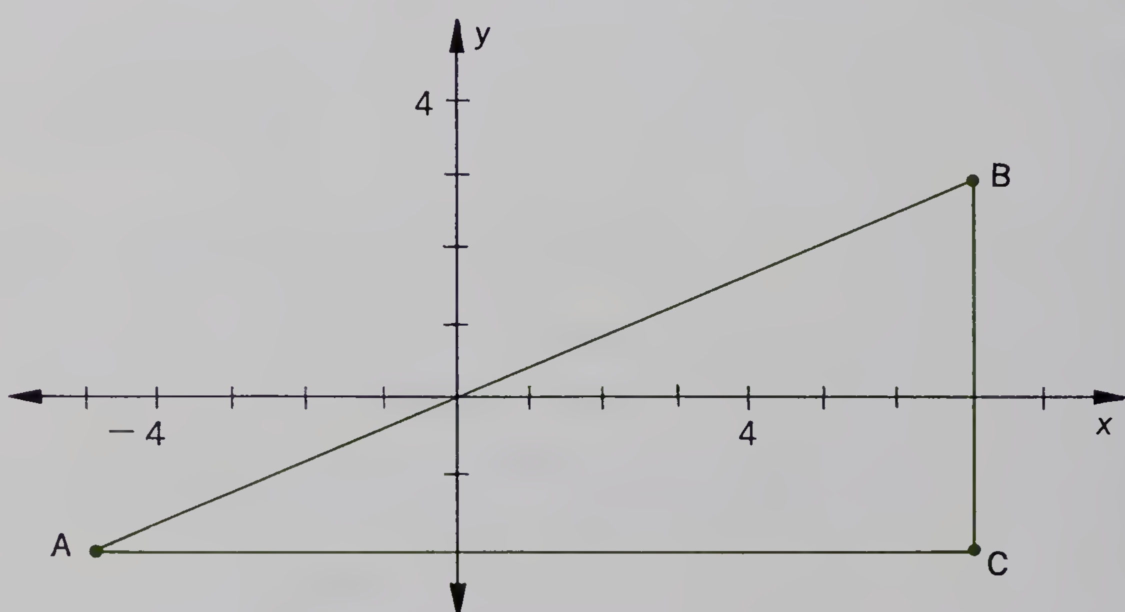
The Distance Formula

Let us again examine a graph on the coordinate plane. We have horizontal and vertical axes which are called the x-axis and y-axis, respectively. As we have learned previously, any ordered pair of real numbers can be plotted on this graph. Let us plot a point A with coordinates (4,6) and a point B with coordinates (1,2).

If we draw a horizontal line from B and a vertical line from A which meet at a point C, we form a right triangle. Can you verify that segment $\overline{BC}$ is $4 - 1$ or three units long, and $\overline{AC}$ is $6 - 2$ or four units long? If BC is three units and AC is four units, then the length of AB is $\sqrt{3^2 + 4^2} = \sqrt{25} = 5$ units by using the Pythagorean Theorem for right triangles. The origin $O(0,0)$ marks the intersection of the axes. Verify that $OB = \sqrt{5}$, $OC = 2\sqrt{5}$, and $OA = 2\sqrt{13}$. $OB = \sqrt{5}$; $OC = 2\sqrt{5}$; $OA = 2\sqrt{13}$

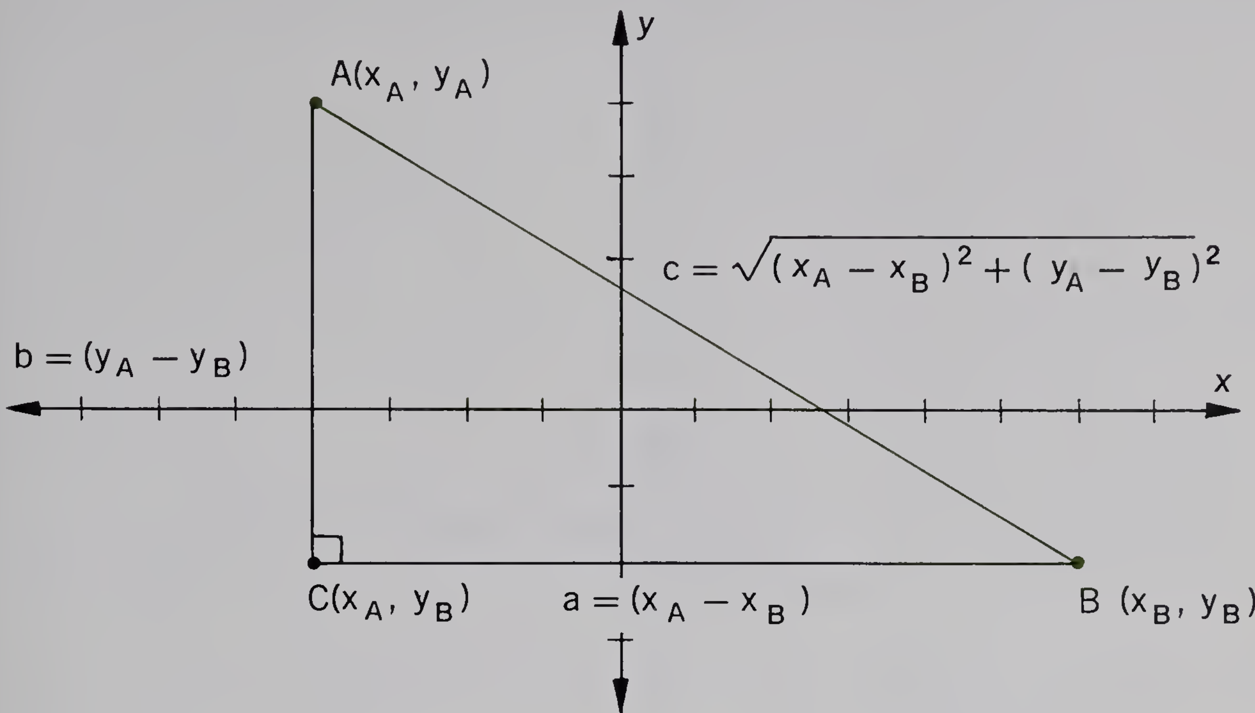


What is the length of BC in the figure below? Does $3 - (^{-}2)$ equal the length of BC? Does $7 - (^{-}5)$ equal the length of AC? If BC is five units long, and AC is 12 units long, find the length of AB by using the Pythagorean Theorem. 5 ; yes ; yes ; $AB = 13$



Now we shall generalize the data we have accumulated to give us a general formula for the distance between two points on a graph.

If the coordinates of point A are (x_A, y_A) and the coordinates of point B are (x_B, y_B), then the distance between points A and B is found by the formula $AB = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}$. In this formula, x_A refers to the x-coordinate of point A. It is read "x sub A." Read and explain the meaning of x_B, y_A, and y_B in the same way.



EXAMPLES

1. Find the distance between two points (3,2) and (7,5).

Solution: Let $x_A = 7$, $y_A = 5$, $x_B = 3$, $y_B = 2$.

Distance $= \sqrt{(7-3)^2 + (5-2)^2} = \sqrt{4^2 + 3^2} = \sqrt{25} = 5.$

Alternate Solution: Let $x_A = 3$, $y_A = 2$, $x_B = 7$, $y_B = 5$.

Distance $= \sqrt{(3-7)^2 + (2-5)^2} = \sqrt{(^-4)^2 + (^-3)^2} = \sqrt{25} = 5.$

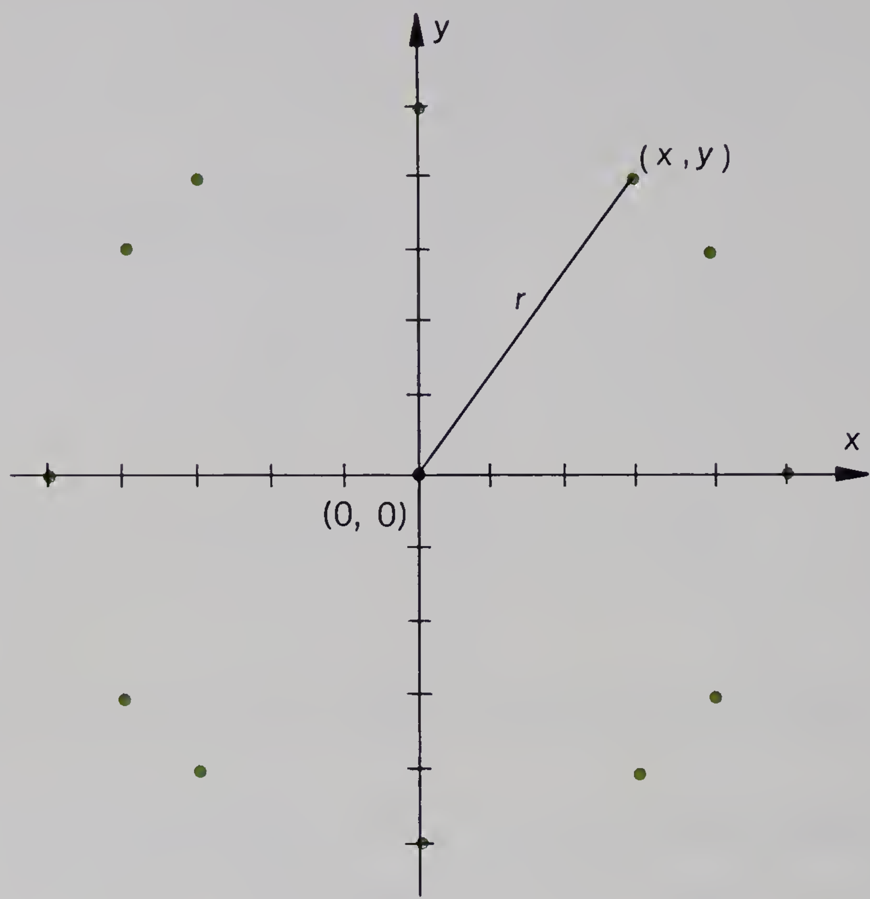
2. Find the distance between $R\ (^-2,7)$ and $S(3,^-5)$.

$RS = \sqrt{(x_R - x_S)^2 + (y_R - y_S)^2} = \sqrt{(^-2 - 3)^2 + (7 - (^-5))^2}$
$= \sqrt{(^-5)^2 + (12)^2} = \sqrt{25 + 144} = \sqrt{169} = 13$

Using the distance formula, find the distance between the following points on a coordinate graph. If the answers are not rational, express the answer with a radical.

1. $A(^-2,6)$ and $B(1,2)$ 5
2. $S(5,7)$ and $R(0,^-5)$ 13
3. (0,0) and (6,8) 10
4. (7,2) and $(3,^-6)$ $4\sqrt{5}$
5. (9,11) and $(^-1,1)$ $10\sqrt{2}$
6. (4,0) and $(7,^-3)$ $3\sqrt{2}$
7. $(^-3,^-7)$ and $(^-8,^-19)$ 13
8. (4,3) and $(^-3,^-4)$ $7\sqrt{2}$
9. $(6,^-6)$ and $(^-6,6)$ $12\sqrt{2}$
10. $(^-3,7)$ and $(^-3,15)$ 8

If we pick a point in the plane and plot several points in that same plane at a given distance from this point, it becomes obvious that the set of points will be a circle with the given point as center and the given distance its radius.



Now let us describe this set of points on a graph. We shall pick the origin (0,0) as the point, and let the given distance be r units. The coordinates of any point of the above set shall be (x,y). The condition required in the description of this set of points is that the distance from (x,y) and $(0,0)$ is always r units. Using the distance formula, we have:

$$\sqrt{(x - 0)^2 + (y - 0)^2} = r$$

Squaring both sides we have:

$$x^2 + y^2 = r^2$$

This is the equation of a circle whose center is the origin and whose radius is r units.

Now let us pick the point whose coordinates are (a,b) as the center, with the given distance from the center as r units. Then by the distance formula:

$$\sqrt{(x - a)^2 + (y - b)^2} = r$$

Squaring both sides:

$$(x - a)^2 + (y - b)^2 = r^2$$

This is the equation of a circle whose center is the point (a,b) and whose radius is r units.

EXAMPLE

Describe the set of points of the equation $(x - 7)^2 + (y - 3)^2 = 25$.

Solution: It is a circle whose center is the point (7,3) and whose radius is five units.

Describe the set of points represented by the equations in problems 1 to 10. See front.

1. $(x - 3)^2 + (y + 2)^2 = 4$

2. $(x + 7)^2 + y^2 = 16$

3. $(x + 2)^2 + (y - 3)^2 = 36$

4. $(x - 5)^2 + (y + 7)^2 = 25$

5. $x^2 + y^2 = 17$

6. $x^2 + (y - 2)^2 = 1$

7. $x^2 + (y + 7)^2 = 64$

8. $(x - 11)^2 + (y + 11)^2 = 144$

9. $(x - 7)^2 + y^2 = 5$

10. $(x + 2)^2 + (y - 7)^2 = \frac{1}{4}$

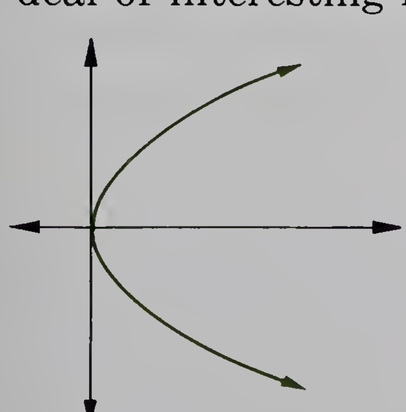
11. Can x or y be greater than 2 or less than $^{-}2$ in the equation $x^2 + y^2 = 4$? Why? No, x^2 or y^2 cannot be negative.

12. Graph this equation: $(x - 3)^2 + (y + 2)^2 = 4$ circle; center $(3,-2)$ $r=2$

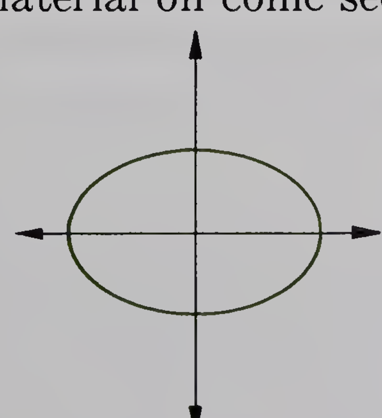
13. Graph this equation: $x^2 + (y + 4)^2 = 9$. circle; center $(0,-4)$ $r = 3$

14. Place the points (3,5) and (5,4) and $x^2 + y^2 = 36$ on the coordinate plane. Notice that one point is inside the circle. Substitute the values of each point in the circle equation. Do these results help you see how you can determine a point's location with respect to a circle? The point is inside the circle if $x^2+y^2<36$, and outside the circle if $x^2+y^2>36$.

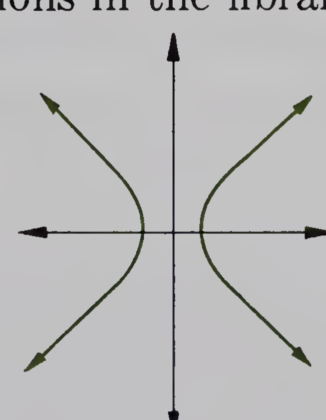
As you have noticed, the variables in the above equations are of the second degree. Since they are not first degree equations, their graphs are not straight lines, but are curved such as the circle. Various forms of equations in two variables of the second degree form a family of curves called *conic sections*. This family consists of the circle, the parabola, the hyperbola, and the ellipse. These curves have very many interesting properties and applications to everyday life. You will find a great deal of interesting material on conic sections in the library.



PARABOLA



ELLIPSE



HYPERBOLA

In Exercises 1, 3, 7, and 10 use two variables and solve by graphing. Solve the rest by whatever method you prefer.

1. The length of a rectangle is twice its width. Its perimeter is 24 rods. What are its dimensions? *l = 8 rd.; w = 4 rd.*

2. The base of an isosceles triangle is 8″. Its perimeter is 32″. What is the length of each of its equal sides? *12″*

3. Find two numbers whose sum is 8 and whose difference is -6. *7; 1*

4. The total proceeds from the sale of 215 tickets to a school play was \$330. Student tickets sold for \$1 and adult tickets were \$2. How many of each were sold? (Let x = the number of adult tickets, and y the number of student tickets.) What were the proceeds from the sale of adult tickets? from student tickets? *115 adults, \$230; 100 students, \$100*

5. Bill and Carl bought a two-way radio system together for a total of \$48. Bill paid as his share \$8 more than Carl. How much did each pay? *Bill, \$28; Carl, \$20*

6. If twice Harry's age is decreased by 5, the result is 31. How old is he? *18 yr.*

7. The larger of two numbers is 1 less than five times the smaller. Four times the smaller is 1 less than the larger. What are the numbers? *2; 9*

8. George and Harry work eight hours each, and their combined earnings are \$40. George is paid 50¢ per hour more than Harry. What does each earn per hour? *George, \$2.75; Harry, \$2.25*

9. The perimeter of a rectangle is 42 inches. If the length is twice the width, what are the dimensions of the rectangle? *l = 14 in.; w = 7 in.*

10. Divide 14 into two parts so that one part exceeds the other by 4. *9; 5*

11. The length of a rectangle is three times its width. Its perimeter is 72 inches. What are its dimensions? *l = 27 in.; w = 9 in.*

12. If twice a certain number is increased by 7, the result is 17. What is the number? *5*

13. A rope 360 feet long is to be cut into two parts so that the first part lacks 20 feet of being 3 times as long as the second. What is the length of each part? *265 ft.; 95 ft.*

14. Harold and Bill bought a used car together for \$250. Harold's share was \$75 more than Bill's. What was the share of each? *Harold, \$162.50; Bill, \$87.50*

15. If a certain number is increased by twice another number, the sum is 18. The first number is eight times the second. What are the numbers? $1\frac{4}{5}$; $14\frac{2}{5}$

16. Find two numbers whose sum is $^{-}100$ and whose difference is $+50$. $^{-}25$; $^{-}75$

17. A lot is three times as long as it is wide. The distance around the lot is 1500 feet. What are its dimensions? 187.5 ft. by 562.5 ft.

18. Four times a number decreased by 25 gives the same result as if the number were increased by 3. What is the number? $9\frac{1}{3}$

19. Jim and Henry bought a motor scooter for $96. Jim paid twice as much as Henry. How much did each pay? Jim, $64 ; Henry, $32

20. Each of the equal sides of an isosceles triangle measures 2″ more than the base. The perimeter is 64 inches. How long is each side? 20″; 22″; 22″

21. A 15′ board is to be sawed into two pieces so that one piece is 3′ longer than the other. What is the length of each piece? 6′; 9′

22. The sum of two numbers is 75. One number is four times the other. What are the two numbers? 15 ; 60

23. There are 40 more girls than boys in Mountain View High School. The total enrollment is 480. How many girls and how many boys are enrolled in the high school? 260 girls ; 220 boys

24. Eric has a 36″ steel bar he wishes to cut in two so that one piece is five times as long as the other. How long will each piece be? 6″; 30″

25. If five times a certain number decreased by 20 is equal to three times the number, what is the number? 10

26. If a certain number is multiplied by 5, the result is the same as if the number were increased by 16. What is the number? 4

27. A field is $1\frac{1}{2}$ times as long as it is wide. Its perimeter is 240 rods. What are its dimensions? l = 72 rd. ; w = 48 rd.

28. A machinist and his helper in a factory together earn $280 a week. The helper earns $\frac{3}{5}$ as much as the machinist. How much does each earn per week? machinist, $175; helper $105

29. John and Mike together sold 120 subscriptions to a magazine last week. If John had sold 15 more, he would have sold twice as many as Mike. How many subscriptions did each sell? Mike, 45 ; John, 75

30. Each of the equal sides of an isosceles triangle is three times as long as the base. The perimeter of the triangle is 21 inches. How long is each side? 3 in. ; 9 in. ; 9 in.

Part One

A. State whether the following statements are true or false.

1. The product of two negative numbers is a negative number. *F*
2. The *commutative law of addition* says that $a + b = b + a$. *T*
3. The difference between $^{-}8$ and $^{-}40$ on the number line is less than the difference between 0 and 8. *F*
4. $^{-}48$ divided by $^{+}6$ gives a quotient of $^{-}8$. *T*
5. $3ab$ means the product of 3, a, and b. *T*

B. Perform the indicated operations:

1. $\frac{(^{-}3)(^{-}7)}{(^{-}6)}$ $^{-}3\frac{1}{2}$
2. $(^{+}3)\ (^{-}6)\ (^{-}7)$ *126*
3. $(^{-}14) - (^{-}8)$ $^{-}6$
4. $3(7 - 9) - 2(8 - 5)$ $^{-}12$
5. $\frac{^{-}7}{^{-}14} + \frac{^{+}8}{16}$ *1*
6. $6a - 3a + b - 5b$ $3a - 4b$
7. $(^{-}4)\ (^{-}2)\ (^{-}6)\ (^{-}5)$ *240*
8. $(^{-}5 + 8) - (^{-}1 - 3)$ *7*
9. Subtract $^{-}6$ from 0 *6*
10. $(^{-}3) + (6) - (^{-}5) + (^{-}3)$ *5*
11. $\frac{7 \cdot 8 \cdot 6 \cdot 20}{^{-}5}$ $^{-}1344$
12. $2(6 - 3) - 3(8 - 2)$ $^{-}12$
13. $\frac{^{-}3}{8} + \frac{10}{16}$ $\frac{1}{4}$
14. Add $^{+}10$ to $^{-}34$ $^{-}24$
15. 5 decreased by 8 $^{-}3$
16. $(^{-}5)\ (^{-}3)\ (^{-}2)$ $^{-}30$

C. Translate to a mathematical phrase using x to represent the number.

1. A number increased by 7 $x + 7$
2. Half a number decreased by twice a second number $\frac{1}{2}x - 2y$
3. 8 decreased by twice a number $8 - 2x$
4. Four times a number divided by 3 $\frac{4x}{3}$
5. The difference between some number and 7 $x - 7$
6. The square of a number multiplied by 6 $6x^2$
7. One third of the difference between a number and $\frac{1}{2}$ of itself. $\frac{1}{3}(x-$

D. Translate these algebraic phrases into English: *See front.*

1. $x + 9$
2. $n^2 + 7$
3. $3x^2 - 7$

Part Two

A. List the numerals 1 through 20 on a sheet of paper. Examine each of the following statements carefully. If the sentence is true as stated, write T after the corresponding numeral on your paper. If it is not true, write 0.

1. The set of counting numbers is closed under addition. *T*
2. The commutative property holds under addition. *T*
3. Multiplication is distributive with respect to addition. *T*
4. An example of a negative irrational number is: $-\sqrt{13}$. *T*
5. The set of counting numbers is closed under subtraction. *O*
6. The commutative property holds under subtraction. *O*
7. An example of an irrational number is: $\sqrt{16}$. *O*
8. The number of irrational numbers is infinite. *T*
9. The set of positive and negative integers is closed under division. *O*
10. The commutative property holds under division. *O*
11. An example of a negative rational number is: $-\frac{5}{6}$. *T*
12. The set of rational numbers is closed under division. *O*
13. The set of negative integers is closed under subtraction. *O*
14. π is an irrational number. *T*
15. The set of positive and negative integers is closed under division. *O*
16. The commutative property holds under multiplication. *T*
17. Multiplication is distributive with respect to subtraction. *T*
18. Tangent 45° is a rational number. *T*
19. A fraction is a rational number. *T*
20. The set of positive and negative integers is closed under multiplication. *T*

B. Construct a number line 6 inches long. Label points corresponding to numbers 0 to 6, one inch apart. Mark off the quarter inches.

On the number line locate the points corresponding to: *See front.*

a. $\sqrt{2}$ **c.** $\sqrt{5}$ **e.** π **g.** sine 60°
b. $\sqrt{3}$ **d.** $\sqrt{10}$ **f.** tangent 60° **h.** cosine 60°

STEPS FOR SOLVING APPLIED PROBLEMS

1. Understand the problem.	2. Note what the problem asks for.	3. Look for hidden questions.
6. Check your answer.	5. Set up and solve the conditional sentence(s).	4. Estimate a reasonable answer.

Part Three

1. The sum of two numbers is 188. One number is three times as large as the other. What are the numbers? *47; 141*

2. There are 44 more girls than boys in the Wilson High School. The total enrollment is 376. How many girls and how many boys are there? *166 boys; 210 girls*

3. Henry has a 12-ft. board which he wishes to saw so that one piece will be 2 ft. longer than the other. How long should each piece be? *5'; 7'*

4. If a certain number is multiplied by 3, the result is the same as if the number were increased by 12. What is the number? *6*

5. The perimeter of a rectangular field is 480 rd. It is twice as long as it is wide. What are its dimensions? *80 rd. by 160 rd.*

6. Five times a certain number decreased by 35 is equal to twice the number increased by 10. What is the number? *15*

7. Harry has $25 more than Bill. If together they have $375, how much does each boy have? *Bill, $175; Harry,; $200*

8. Mary is three times as old as her younger sister. In five years, Mary will only be twice as old as her sister. What is the present age of each girl? *15 yr.; 5 yr.*

9. The perimeter of a triangle is 47″. The first side is 3″ longer than the second side. The third side is twice as long as the second side. How long is each side? *11"; 14"; 22"*

10. A board 28 ft. long is to be divided into 3 pieces such that the second piece is twice as long as the first, and the third is twice as long as the second. What is the length of each piece? *4'; 8'; 16'*

11. Mr. Jones has a pasture that is 5 times as long as it is wide. If the electric fence around the pasture is 4260 yards long, find the dimensions of the field. *355 yd. by 1775 yd.*

Part One

A. *Add:*

1. $\frac{3}{8}+\frac{1}{2}+\frac{5}{16}$ $1\frac{3}{16}$
2. $\frac{1}{6}+\frac{2}{3}+\frac{5}{12}$ $1\frac{1}{4}$
3. $2\frac{3}{5}+5\frac{5}{6}+3\frac{7}{8}$ $12\frac{37}{120}$
4. $10\frac{1}{2}+14\frac{11}{16}+12\frac{1}{4}+16\frac{2}{3}$ $54\frac{5}{48}$
5. $3\frac{3}{4}+\frac{5}{8}+32\frac{5}{16}+101\frac{1}{16}$ $137\frac{3}{4}$
6. 27.16 + 3.51 + .96 31.63
7. 15.8 + 75.39 + 1.087 + 227.96 320.237
8. 0.325 + 4.08 + 44.239 + 5.278 53.922
9. 7.81 + 324.63 + 6.056 + 17.863 356.359
10. 0.246 + 9.092 + 0.008 + 36.148 45.494

B. *Subtract:*

1. $\frac{7}{8}$ − $\frac{1}{2}$ $\frac{3}{8}$
2. 75.5 − 8.9   66.6
3. $\frac{15}{16}$ − $\frac{1}{8}$ $\frac{13}{16}$
4. 5.04 − 3.67   1.37
5. $\frac{5}{6}$ − $\frac{2}{3}$ $\frac{1}{6}$
6. 12.6 − 3.125   9.475
7. $23\frac{1}{3}$ − $16\frac{5}{6}$ $6\frac{1}{2}$
8. 16.5 − 3.766   12.734

C. *Multiply:*

1. $(\frac{3}{4})\cdot(\frac{8}{15})$ $\frac{2}{5}$
2. $(1\frac{3}{4})\cdot(48)$ 84
3. $(18)\cdot(\frac{5}{6})$ 15
4. $(\frac{5}{10})\cdot(\frac{8}{25})$ $\frac{4}{25}$
5. $(6\frac{1}{4})\cdot(\frac{12}{25})$ 3
6. $(16\frac{2}{3})\cdot(7\frac{1}{2})$ 125
7. $(\frac{3}{4})\cdot(12.40)$ 9.3
8. $(5.67)\cdot(24)$ 136.08
9. $(.25)\cdot(4.20)$ 1.05
10. $(.255)\cdot(3.16)$ .8058
11. $(.025)\cdot(.63)$ .01575
12. $(1.035)\cdot(6.6)$ 6.831

D. *Divide:* (Find quotients to the nearest hundredth)

1. $4\frac{1}{3}\div 6$ .72
2. $52\div 1\frac{3}{8}$ 37.82
3. $27\div\frac{3}{4}$ 36.00
4. $\frac{9}{14}\div\frac{3}{7}$ 1.50
5. $5\frac{1}{3}\div 9$ .59
6. \$357 ÷ 24 \$14.88
7. \$76.84 ÷ .63 \$121.97
8. \$213.00 ÷ 240 \$.89
9. 18.225 ÷ 45 .41

Part Two

A. *Solve:*

1. $x+19=37$ 18
2. $48-y=11$ 37
3. $76+y=72$ $^{-}4$
4. $85+m=97$ 12
5. $\frac{1}{2}x=18$ 36
6. $x+29=67$ 38
7. $52-x=27$ 25
8. $144\div c=9$ 16
9. $13x=169$ 13
10. $\frac{1}{3}x=24$ 72

B. Solve these proportions

1. $\frac{x}{9} = \frac{28}{63}$ 4
2. $\frac{19}{114} = \frac{5}{x}$ 30
3. $\frac{18}{x} = \frac{12}{4}$ 6
4. $\frac{x}{24} = \frac{17}{102}$ 4
5. $\frac{7}{12} = \frac{x}{48}$ 28
6. $\frac{27}{36} = \frac{x}{100}$ 75

C. Find n

1. 28 is $n\%$ of 35 80
2. 42 is 60% of n 70
3. 8 is 200% of n 4
4. 14 is $n\%$ of 40 35
5. 175 is 35% of n 500
6. 175% of 16 is n 28
7. 45 is 25% of n 180
8. 12 is $n\%$ of 90 13.3
9. 56 is 40% of n 140
10. 16.5 is $n\%$ of 55 30

D. Solve and check each equation

1. $x - 7 = 12$ 19
2. $3x + 2 = 11$ 3
3. $\frac{x}{5} = 8$ 40
4. $3x + 2 = x + 6$ 2
5. $\frac{1}{2}x = \frac{7}{8}$ $1\frac{3}{4}$
6. $2y + 9 = y + 12$ 3
7. $3x + 91 = 91$ 0
8. $7x + 3 = 63 + 4x$ 20
9. $5c - 3 = 13 - 3c$ 2
10. $\frac{x - 3}{5} = 6$ 33

Part Three

List the numerals 1 through 10 on your paper and indicate by letter, the correct answer to these multiple choice questions. *(circled)*

1. Multiplication can be thought of as:
   a. repeated subtraction
   b. repeated addition
   c. the inverse of addition
   d. the inverse of subtraction

2. In multiplication, the product is not affected by
   a. the size of the factors
   b. the order of factors
   c. the number of factors
   d. the decimal point location

3. The associative property holds for
   a. addition and subtraction
   b. addition and division
   c. addition and multiplication
   d. subtraction and division

4. The identity element for addition is
   a. 1
   b. 0
   c. $^{-}1$
   d. undefined

5. The best answer to $0 \div 5$ is
a. undefined b. 5 c. 0 d. 0.5

6. What property of numbers is illustrated by $5 + (3 + 7) = (5 + 3) + 7$
a. associative property c. distributive property
b. commutative property d. Pythagorean property

7. Which set of numbers is not closed under subtraction?
a. fractional numbers of arithmetic c. integers
b. rational numbers d. real numbers

8. A triangle that has no sides equal is called
a. isosceles b. equilateral c. scalene d. right

9. The polygon that has 5 sides is called the
a. hexagon b. octagon c. heptagon d. pentagon

10. The perpendicular distance from a vertex of a triangle to the opposite side is called
a. median b. angle bisector c. altitude d. centroid

Part Four

1. What is the result of each of the following operations?
24 a. Fifteen subtracted from the sum of 12 and 27
68 b. Twelve added to the product of 8 and 7
3 c. Sixty divided by twice the sum of 8 and 2
81 d. The difference of 12 and 9 multiplied by 27
16 e. The product of 18 and 8 divided by 9
20 f. The quotient of 108 divided by 9, increased by 8
84 g. Twelve less than the product of 16 and 6

2. If Camille is $(x - 3)$ years old today, represent her age 4 years from now in terms of x. $(x + 1)$

3. Three consecutive integers, x, $x + 1$, $x + 2$ have a sum equal to 63. Find x. 20

4. The rational number x when increased by $2\frac{1}{2}$ equals $6\frac{1}{4}$. Find x. $3\frac{3}{4}$

5. Three times a number increased by 7 equals 58. Find the number. 17

6. One half a number increased by 27 equals 34. Find the number. 14

7. One sixth of a certain number increased by 8 times the same number equals 98. Find the number. 12

8. The product of 8 and a certain number, decreased by 2 is equal to the product of 9 and 6. Find the number. 7

SPACE FIGURES

WORDS TO WATCH FOR

capacity	*yard*	*pentagonal prism*	*specific gravity*
cone	*cylinder*	*prism*	*sphere*
cube	*edge*	*pyramid*	*surface area*
cubic foot	*face*	*rectangular prism*	*triangular prism*
cubic inch	*lateral area*	*solid*	*volume*

Space figures are familiar features of our environment. Spheres, cylinders, and prisms are utilized in almost every area of human activity. Space figures possess certain geometric properties which distinguish them from other geometric figures, such as the line and plane and polygon. A space figure is determined by a set of points that forms a surface which completely encloses a portion of space. A sphere, cylinder, or prism is, therefore, only a shell. A model of a sphere is a tennis ball rather than a solid golf ball. A model of a cylinder is an open-ended can. A model of a cube is an empty carton whose edges are equal in length; the six faces are square surfaces.

Space figures offer a wide variety of problem situations. One problem will be to calculate the amount of space enclosed by a space figure. The space enclosed is called *volume.* Calculation of volume is a problem familiar to a contractor who is making a bid requiring the computation of the cost of excavation of a basement or the amount of concrete needed for a foundation. A similar problem arises when it is necessary to calculate how much a container, such as a can, storage tank, swimming pool, or freight car will hold. In this connection the volume is referred to as *capacity* and may be measured in cubic units or by some other standard measure of volume.

Another kind of problem occurs when it is necessary to measure the surface of a space figure. Such calculations are common for the painter, the sheet metal worker constructing a tank, or the contractor erecting a building. All are concerned with measurement of the *surface area* of the figure. In this chapter we shall be concerned with both types of problems. *See front for problems 1 through 6.*

1. List some additional examples of situations where measurement of volume or capacity, or of surface area, is required. What units of measurement are used in each instance?

2. List some examples of containers that occur in a classroom. Estimate the capacity of each.

3. Name some containers that are found in the grocery store or supermarket. What is the most common shape? What units of measurement are most commonly used?

4. List some situations where measurement of capacity is called for in the home. What units of measurement are commonly used?

5. List some materials that you might measure using these units of capacity:

a. teaspoon	**d.** gallon	**g.** cubic yard
b. quart	**e.** cubic foot	**h.** barrel
c. bushel	**f.** peck	**i.** liter

6. What unit of measure is used with each of these commodities?
a. water **b.** wheat **c.** crude petroleum **d.** gasoline

7. Water for use in the home is sold by the cubic foot. In Plainville the rate is 31¢ per 100 cubic feet, plus a monthly service charge of \$1.50. Mr. Adams's family used 130 cubic feet of water last month. What was the bill for the month? *\$ 1.90*

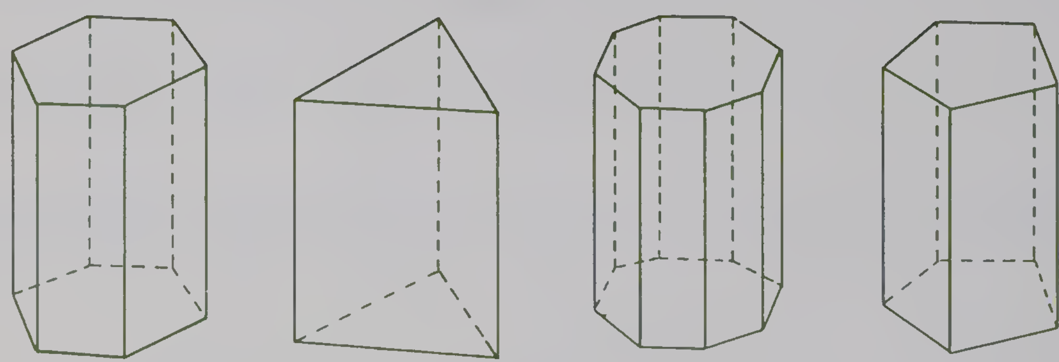
8. Gas for cooking and heating is also sold by the cubic foot. The Plainville gas rates are:
\$1.50 for the first 1000 cubic feet.
12¢ per 100 cubic feet for the second 1000 cubic feet.
10¢ per 100 cubic feet for the third 1000 cubic feet.
8¢ per 100 cubic feet for all over 3000 cubic feet. *\$4.18; 48 ¢*
Last month the Adams family used 3600 cubic feet of gas. What was the gas bill? How much more would 4200 cubic feet cost?

9. A gallon is 231 cubic inches. (A cubic foot is 12″ × 12″ × 12″.) How many gallons are there in a cubic foot, to the nearest tenth? *7.5 gal.*

10. A bushel is 2150 cubic inches. How many cubic feet are in one bushel, to the nearest tenth? *1.2 cu. ft.*

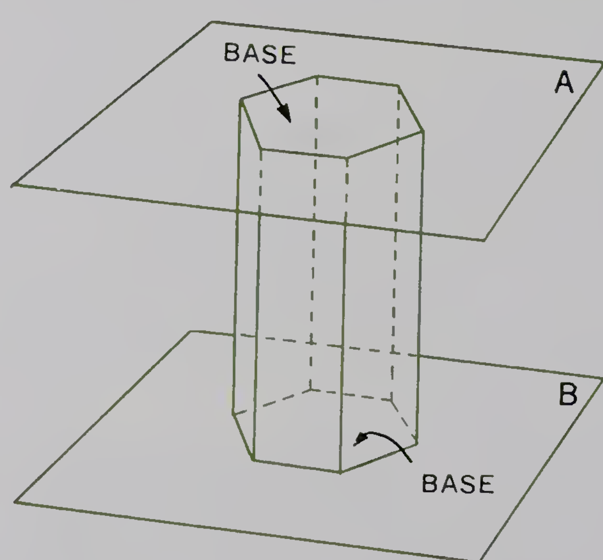
Some Properties of Prisms

A prism is one of the most familiar space figures. A cube is one example of a prism; a brick is another example. Prisms come in a variety of forms but all have basic identifiable properties. The following exercises are designed to reveal these properties.



1. Each prism has bases that are polygons. A prism is named according to the polygon that forms its bases. There is no trapezoidal prism illustrated above. Sketch a trapezoidal prism. *See front.*

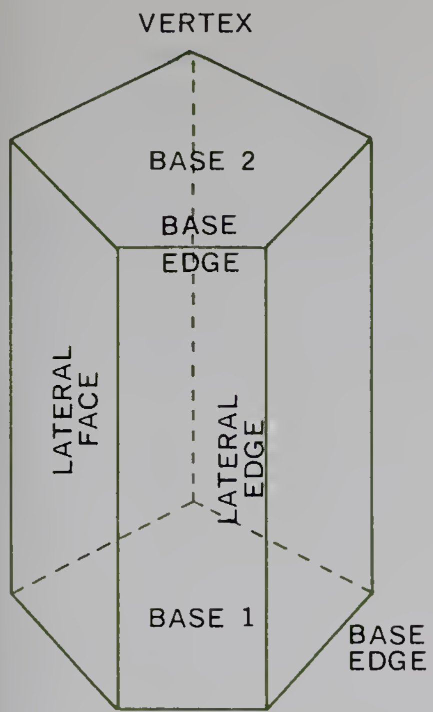
2. The two bases of a prism are in parallel planes. Using the expression ϕ or $\{\ \}$, complete this statement: Plane $A \cap$ plane $B = \square$ Write in words what the statement says. *Plane A and plane B do not intersect.*



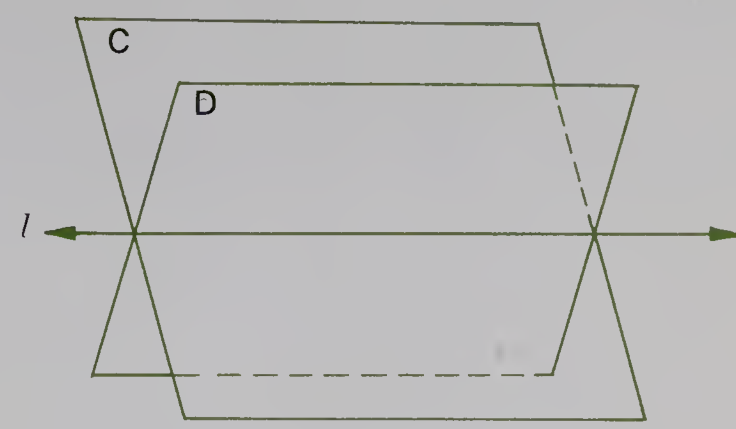
3. The two bases of a prism are congruent. What do you know about the size and shape of the bases? *They have the same shape and identical size.*

4. The parts of a prism are named at the top of the next page. You can see that a prism is bounded by plane figures. A plane is a set of points making up a flat surface. We can think of a desk top or a floor as representing part of a plane.

In the drawing below on the right you see two intersecting planes. What geometric figure is determined at their intersection? straight line



PARTS OF A PRISM

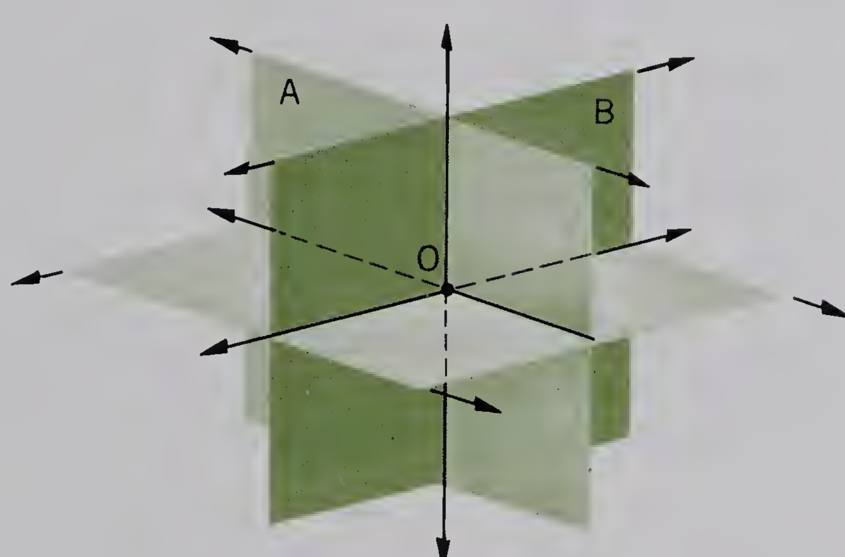


INTERSECTING PLANES

5. Complete this statement: Plane $C \cap$ plane $D = \square$ straight line

6. What is the name of the line on the prism determined by the intersection of two lateral surfaces? edge

7. What is the name of the line determined by the intersection of a lateral face and a base of the prism? base edge



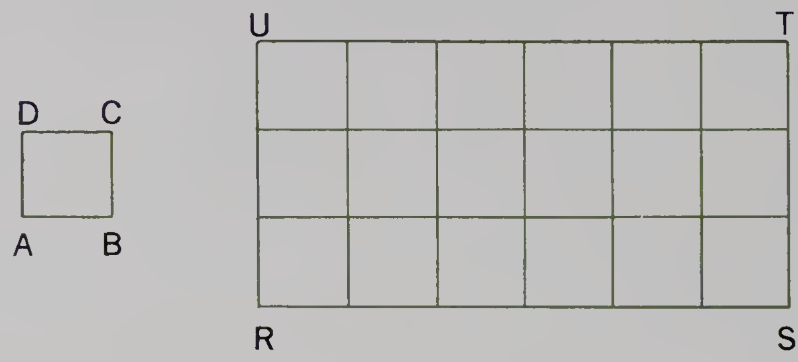
THREE INTERSECTING PLANES

8. The intersection of three planes can determine a point, as you can see in the drawing. How many points can you identify in your classroom that are determined by the intersection of three planes?

9. What is the name of a point on a prism formed by the intersection of the base and two lateral faces? How many such points (vertices) are there on a triangular prism? on an octagonal prism?
vertex ; 6 ; 16

The bases of a prism are polygons. To calculate the volume of a prism, we need to know the area of the region enclosed by the polygon forming the base. The area of any polygon is measured by the number of square units in the region enclosed by the polygon.

ABCD represents a unit of square measure. The measure of each side of *ABCD* is one unit of linear measure.



1. In *RSTU* the measure of the length of the rectangle is six units of linear measure. How many units of square measure are in a row along the base? 6 sq. units

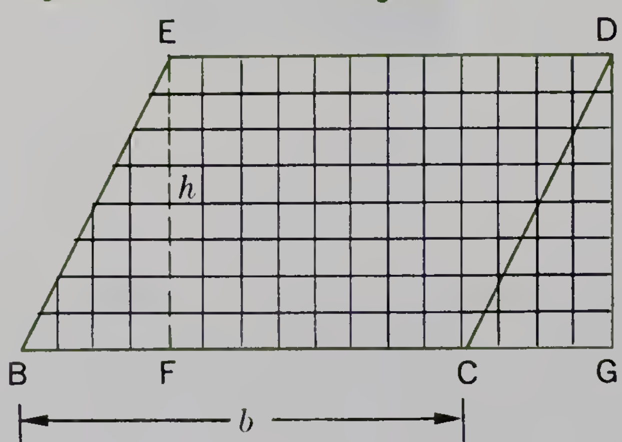
2. The measure of the width of the rectangle is three units of linear measure. How many rows of square measure are there? 3

3. How many units of square measure are in the region enclosed by the rectangle? 18 sq. units

4. Suppose the length is 9 and the width is 7. What is the measure of the area? Explain how you could find the area without counting. 63 ; length X width

5. The formula for the area of a rectangle is: $A = lw$. Explain the meaning of each of the variables. A = area ; l = length ; w = width

6. To find the area of a parallelogram, we can first convert it to a rectangle. Explain how we can do this. Move triangle BEF to occupy the position of triangle CGD as in the figure below.



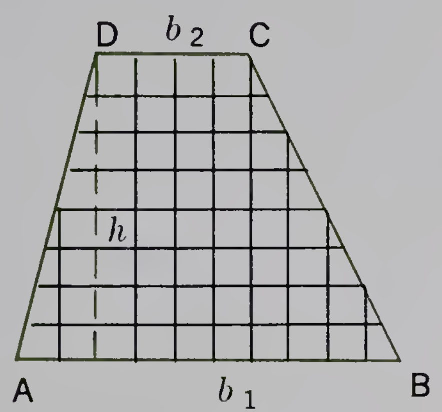
7. Can any parallelogram be converted to a rectangle with equal area? Sketch a few parallelograms and see if this is the case. yes

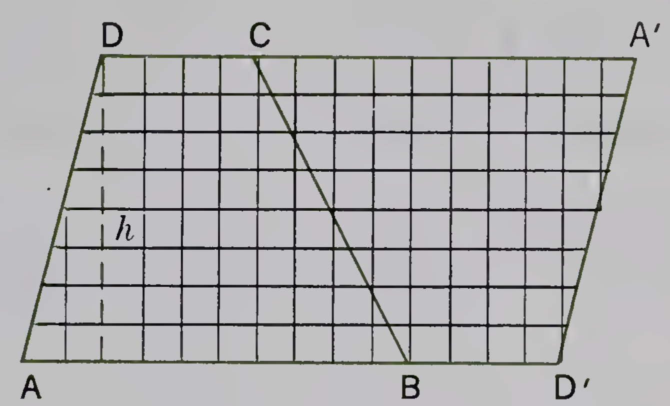
8. The formula for the area of a parallelogram is:

$$A = bh.$$ A = measure of the area

Explain the meaning of each of the variables. b = measure of the base
h = measure of the height

9. A trapezoid has two parallel bases, and two sides that are not parallel. As indicated below, if a congruent trapezoid is turned to rest on the other base and fitted against the original trapezoid, the two figures combine to form a parallelogram. What two measures are added to give the measure of the base of the parallelogram?
the sum of the two bases





10. What is the altitude of the parallelogram? the altitude of the trapezoid

11. Write an expression for the area of the parallelogram. $A = (b_1 + b_2) \times h$

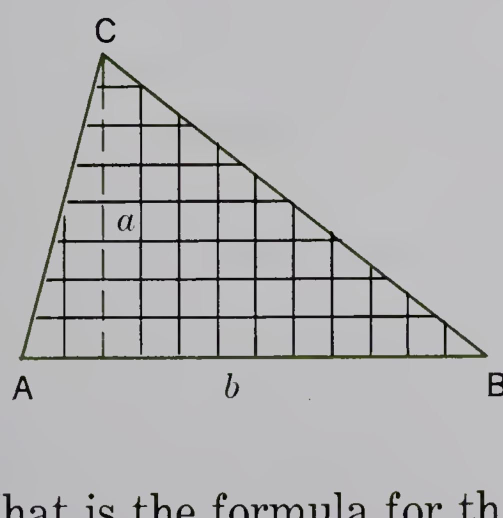
12. The original trapezoid is what fraction of the area of the parallelogram? $\frac{1}{2}$ Express each area in units of square measure. 56 ; 112

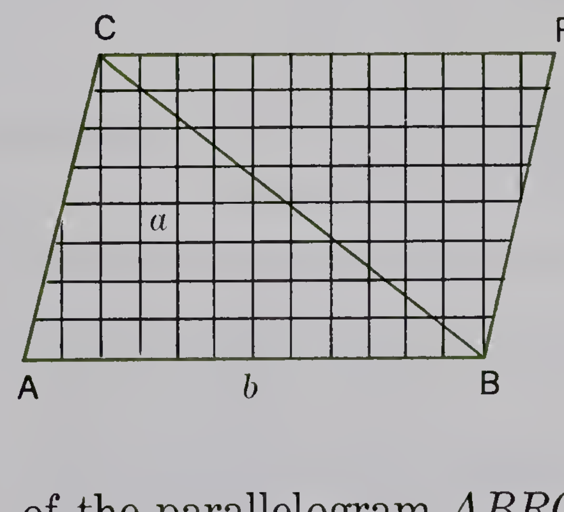
13. The formula for the area of a trapezoid is:

$$A = \frac{1}{2}h(b_1 + b_2).$$

Explain the meaning of each of the variables. See front.

14. The method used to obtain the formula for the area of a trapezoid is useful to obtain the formula for the area of a triangle. Explain what is done to obtain the formula. Refer to Exercises 9–12.
See front.





15. What is the formula for the area of the parallelogram $ABRC$?
$A = ba$

16. Write the formula for and find the area of the original triangle.
$A = \frac{1}{2}ba$

17. Calculate the area of each of these rectangles: See front.

Rectangle	*A*	*B*	*C*	*D*	*E*	*F*
Length:	9 in.	$18\frac{1}{2}$ in.	9.7 ft.	18 rd.	19 yd.	$18\frac{3}{8}$ in.
Width:	7 in.	6 in.	3.5 ft.	7 rd.	8 yd.	16 ft.

18. Calculate the area of each of these parallelograms. See front.

Parallelogram	*A*	*B*	*C*	*D*	*E*	*F*
Base:	90 ft.	17 yd.	13.4 ft.	17.2 in.	60 rd.	82 rd.
Altitude:	16 ft.	16 yd.	7.2 ft.	16 in.	20 rd.	60 rd.

19. Calculate the area of each of these trapezoids. See front.

Trapezoid	*A*	*B*	*C*	*D*	*E*	*F*
Base b_1:	27 ft.	19 yd.	36 in.	40 rd.	$20\frac{1}{2}$ ft.	$16\frac{2}{3}$ yd.
Base b_2:	53 ft.	31 yd.	54 in.	80 rd.	59 ft.	$33\frac{1}{3}$ yd.
Height:	15 ft.	16 yd.	20 in.	16 rd.	30 ft.	18 yd.

20. Calculate the area of each of these triangles. See front.

Triangle	*A*	*B*	*C*	*D*	*E*	*F*
Base:	27 ft.	50 ft.	13 in.	18 in.	9 yd.	20 rd.
Altitude:	16 ft.	20 ft.	8 in.	7 in.	8 yd.	16 rd.

Finding Areas Using Trigonometric Ratios

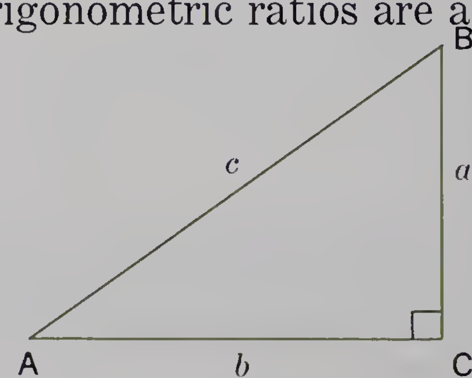
In finding the areas of polygons, you have been using measures of the sides, and in some cases, the altitudes of the polygons. You can also find the area of a polygon by using the trigonometric ratio of one of its angles. From this ratio you can determine the measure of an unknown altitude or side, or even both. Before we try this, let us review the trigonometric ratios. In stating trigonometric ratios we will now use A, B, and C to represent $m\angle A$, $m\angle B$, and $m\angle C$ respectively. The measure of the legs are represented by a and b and the measure of the hypotenuse is represented by c. In other words, tan ($m\angle A$) and tan A are treated as equivalent expressions.

In the right triangle ABC the trigonometric ratios are as follows:

$$\tan A = \frac{a}{b} = \frac{\text{opposite leg}}{\text{adjacent leg}}$$

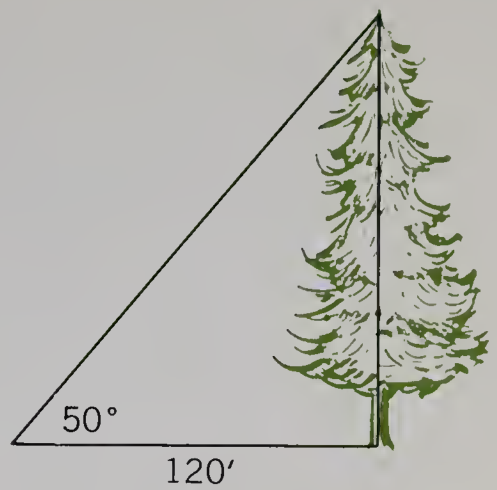
$$\sin A = \frac{a}{c} = \frac{\text{opposite leg}}{\text{hypotenuse}}$$

$$\cos A = \frac{b}{c} = \frac{\text{adjacent leg}}{\text{hypotenuse}}$$



EXAMPLES

1. At a distance of 120 feet, the angle of elevation of the top of a tree is 40°. How tall is the tree?



Explain how you know $\frac{x}{120} = \tan 40°$.

From the tables we find

$\tan 40° = 0.839$.

Then $\frac{x}{120} = \frac{839}{1000}$

and $x = 101$ to the nearest foot.

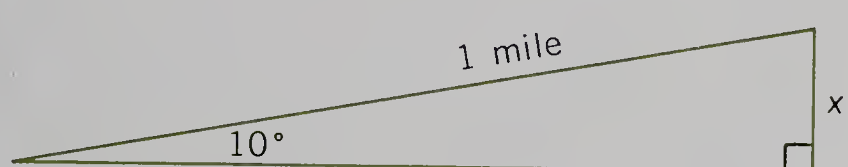
We round the answer to the nearest foot because the given measure of the base is to the nearest foot.

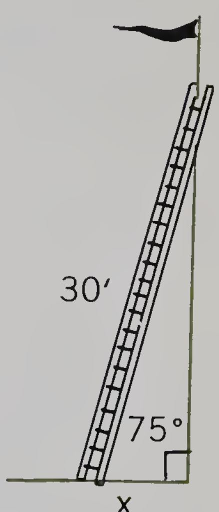
2. A mountain road has a 10° grade. How many feet does the altitude change in one mile? See the drawing below on the left.

Here we are concerned with the sine ratio. Explain how you can tell.

Then $\frac{x}{5280} = \sin 10°$. $\sin 10° = 0.174$.

$\frac{x}{5280} = \frac{174}{1000}$, and $x = 919$, to the nearest foot.





3. A ladder 30 feet long is placed against a wall so it makes an angle of 75° with the ground. How far from the wall is the foot of the ladder?

Here we use the cosine ratio. Why?

$\frac{x}{30} = \cos 75°$ $\cos 75° = 0.259$ Thus $\frac{x}{30} = \frac{259}{1000}$

Then $x = 8$ feet to the nearest foot.

---

1. At a distance of 280 feet, the angle of elevation of the top of a tree is 20°. How tall is the tree? *102 ft.*

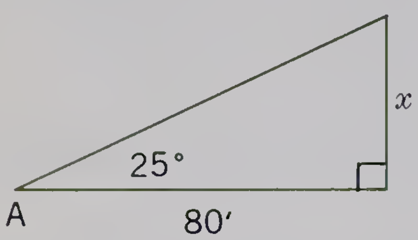
2. a. Steel cables support a large tower. If each cable is 85′ long and makes an angle of 70° with the ground, how tall is the tower?
   b. How far from the foot of the tower is the cable attached? *80 ft.*
   29 ft.

In each of the following exercises:

a. First decide which trigonometric ratio to use; then write an equation using this ratio.

b. Solve this equation for the indicated variable. *See front.*

1.



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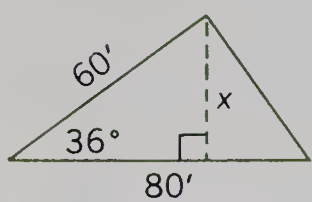

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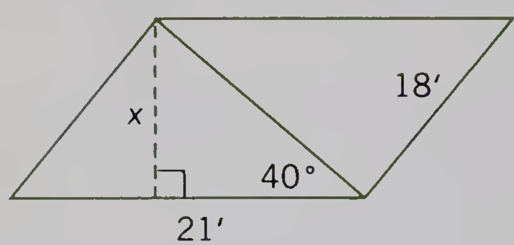

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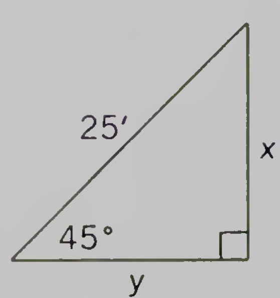

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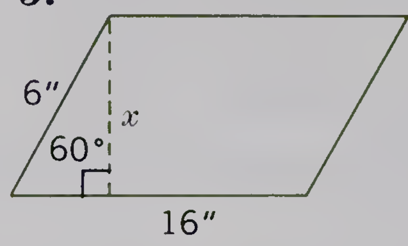

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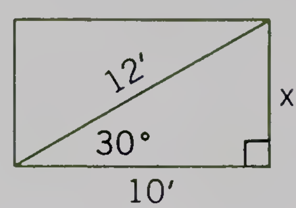

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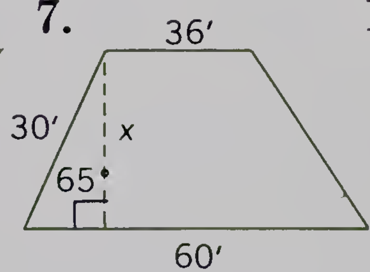

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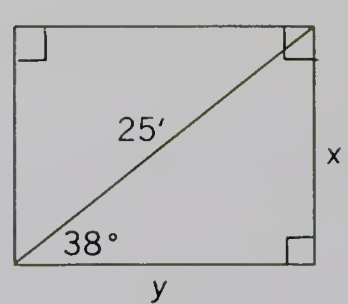

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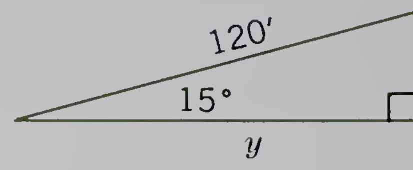

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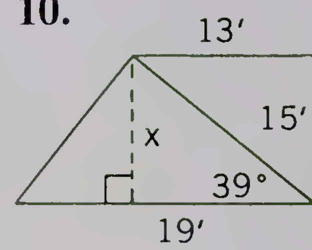

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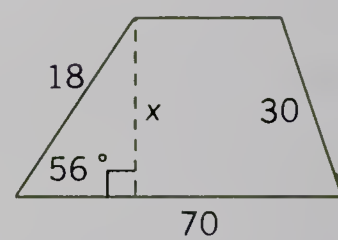

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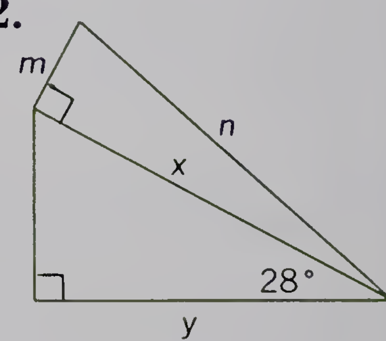
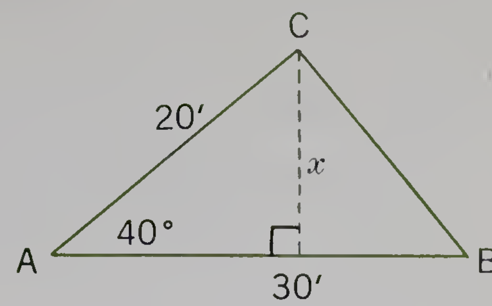


You know that you can find the area of a triangle if you are given the measures of the base and altitude. You can also find the area if you are given the measures of two sides and the included angle.

EXAMPLE

Find the area of triangle ABC.
We need to find the measure of the altitude of the triangle, marked x in the figure. This means that we are concerned with the sine ratio of 40°. Explain how you know this.



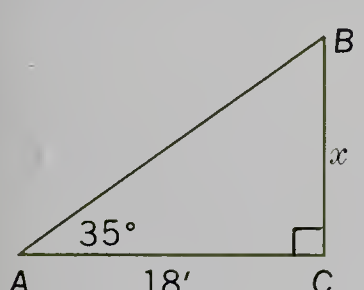
$$\frac{x}{20} = \sin 40°. \quad \sin 40° = 0.649$$

$$\frac{x}{20} = \frac{649}{1000}, \text{ and } x = 13.$$

What is the area of the triangle? *195 sq. ft.*

You are to find the area of each of the following figures. You will need trigonometric ratios to find missing dimensions. The figures are rectangles, parallelograms, or trapezoids. Identify a triangle which makes possible the use of trigonometry. To serve as a guide, the needed dimensions are indicated with variables. *See front.*

1.



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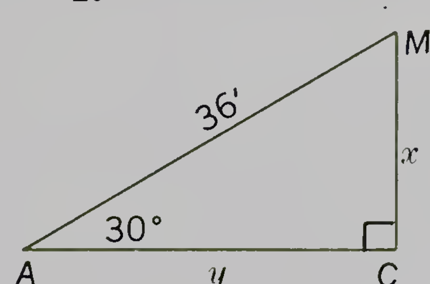

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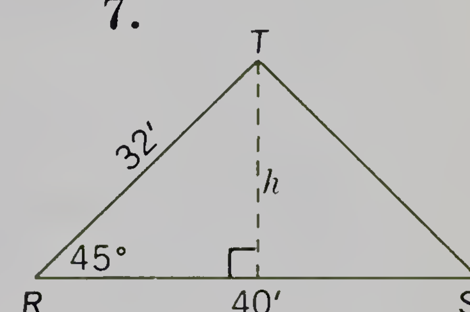

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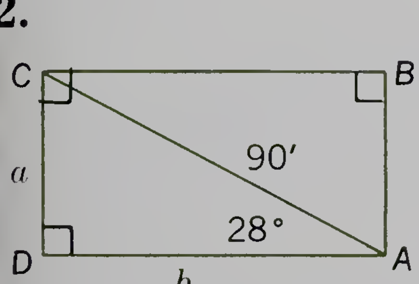

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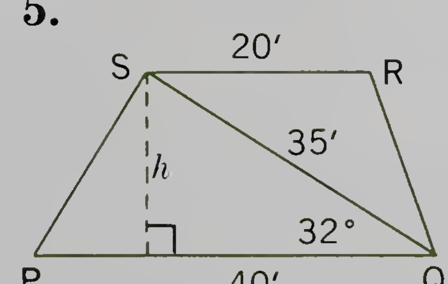

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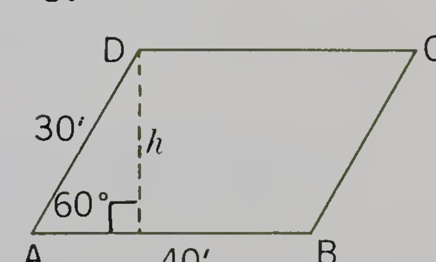

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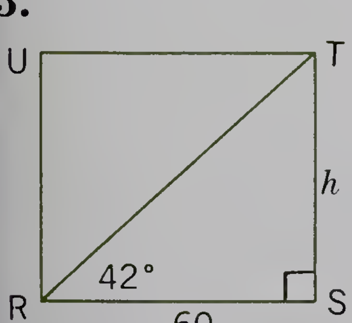

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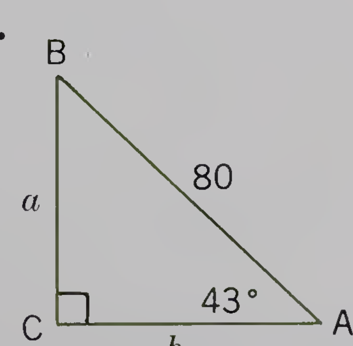

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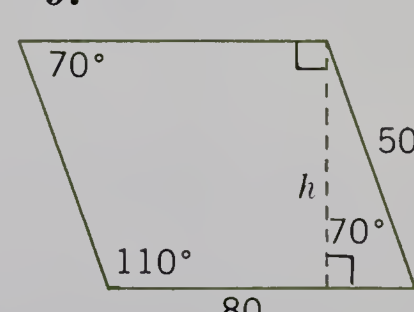
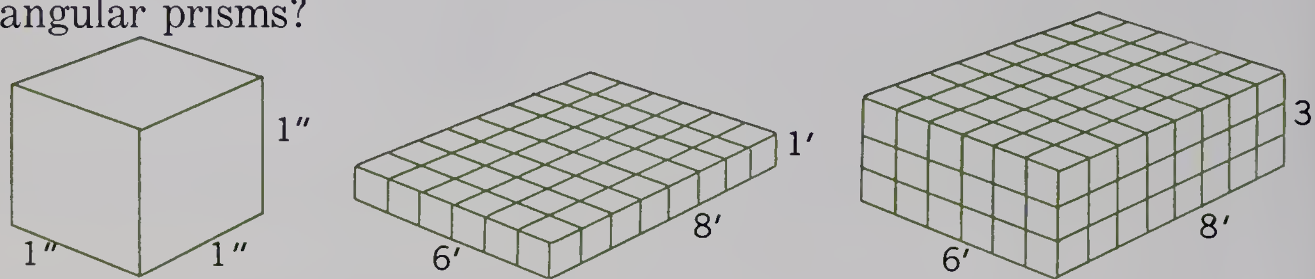


The space figures you have been exploring are examples of *right rectangular prisms.* A prism is a space figure whose lateral edges are parallel and whose bases are congruent polygons in parallel planes. A *rectangular* prism is so called because its bases are rectangles. In a right prism the lateral edges are perpendicular to the bases. A box is a common example of a right rectangular prism. Other examples are storage bins, television sets, pocket radios, and many types of refrigerators. Can you find examples of these latter items that are *not* right rectangular prisms? Can you think of other examples of objects that suggest right rectangular prisms?



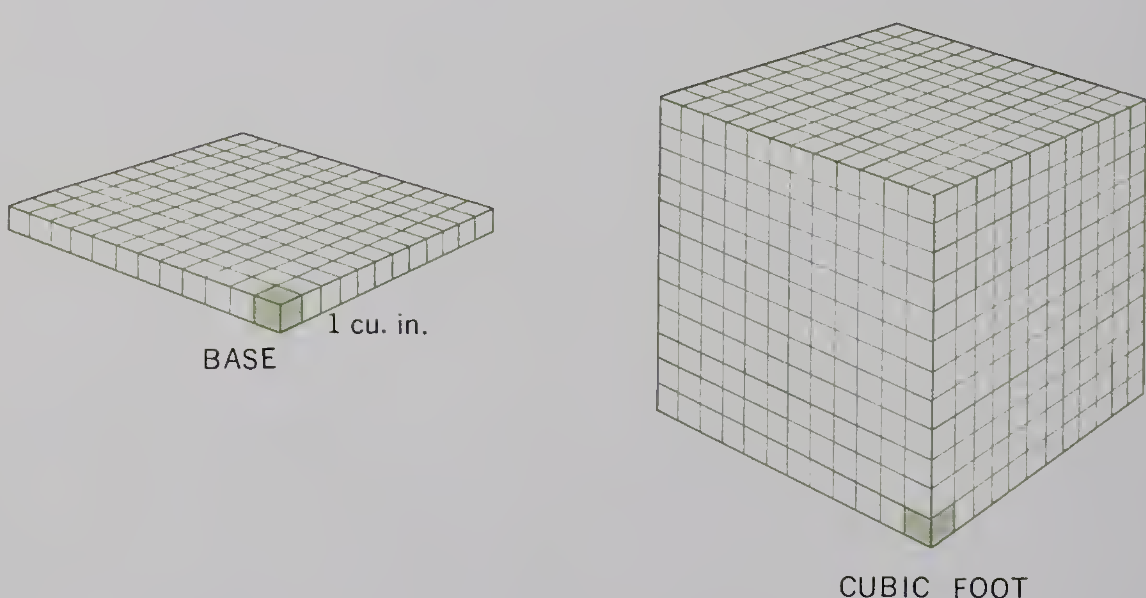
The volume of a space figure is defined as the amount of space it occupies. The measure of the volume is determined by finding how many cubic units can be fitted into the figure. A cubic unit is a cubic inch, cubic foot, cubic yard, and so on.

1. Suppose you wished to build a cubic foot block out of blocks each measuring one inch along each side. What would be the shape of the base? *square*

2. What is the area of the base? *144 sq. in.*

3. How many cubic inch blocks will it take to cover the base? *144*

4. How many layers of blocks will there be in a cubic foot? *12*



5. How many blocks, each a cubic inch, will it take to make up a cubic foot? *1728*

6. The number of cubic units in a right rectangular prism is equal to the product of the number of linear units in the length, width, and height of the figure. That is, $V = lwh$. From Exercises 2 and 4 explain why this is true. *Area of the base = l X w. Multiply this product by the height.*

7. It is also true that the number of cubic units in a right rectangular prism is equal to the product of the number of square units in the base times the number of linear units in the height of the figure. Compare Exercises 2 and 3 above and explain why this is true. *See front.*

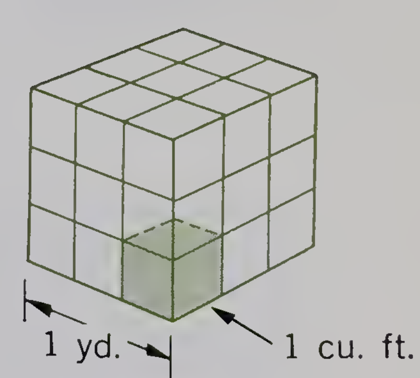
8. What is the measure in cubic feet of the volume of a room that is 15 feet long, 12 feet wide, and 8 feet high? *1440 cu. ft.*

9. What is the measure in cubic feet of the volume of a grain bin that is 8 feet long, 6 feet wide, and 4 feet deep? *192 cu. ft.*

10. A right rectangular prism is 18 inches long, 10 inches wide, and 4 inches high. What is the measure of its volume in cubic inches? cubic feet? *720 cu. in. ; $\frac{5}{12}$ cu. ft.*

11. How many cubic inches of water are in an aquarium that is 12 inches square at the base and is filled to a depth of 10 inches? *1440 cu. in.*

12. The amount of dirt excavated for a building is usually measured in cubic yards. How many cubic feet are in a cubic yard? How many cubic inches are in a cubic yard? *27 cu. ft. = 1 cu. yd. 46,656 cu. in. = 1 cu. yd.*



ONE CUBIC YARD

13. A contractor excavates a basement for a building. The excavation measures 45 feet in length, 27 feet in width, and 9 feet deep. How many cubic yards are excavated? *405 cu. yd.*

NOTE: You can express the measurements in yards before multiplying or you can divide the product by 27 after multiplying. Which is easier?

14. What will the excavation cost at \$11 per cubic yard? *\$4455*

15. Mr. Jensen's garage floor is 24 feet long and 18 feet wide. He plans to put in a concrete floor 6 inches thick. How many cubic yards of concrete will he need? NOTE: 6 inches is what fraction of a yard? *8 cu. yd.*

16. What will the concrete cost at \$16 a cubic yard? *\$128*

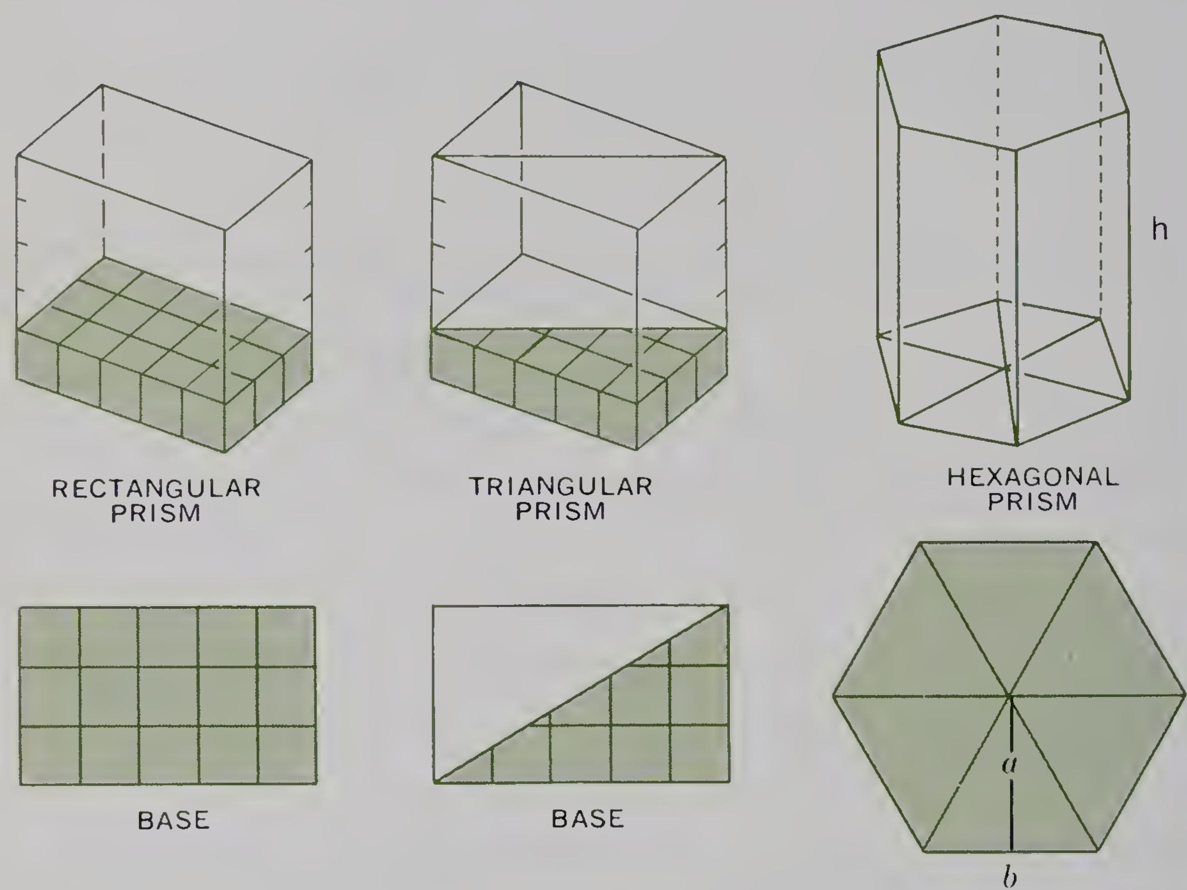
17. Mr. Jensen paid a man 15¢ a square foot to finish the garage floor after the concrete had been poured. What did it cost to finish? *\$64.80*

18. The foundation for the Henderson residence is 2 feet thick and 4 feet deep. It is 180 feet long. What is the cost of concrete at \$16 a cubic yard? *\$853.33*

You found that to obtain the measure of the volume of a right rectangular prism, you could use the formula

$$V = Bh$$

This same formula can also be used to find the measure of the volume of other right prisms. In the solids below, you can see how it applies to the triangular prism. The number of square units in the area of the base, either in the rectangular prism or the triangular prism is the same as the number of cubic units that can be fitted in the bottom layer. The number of linear units in the height tells how many layers there will be.



1. How many square units are in the area of the base in the rectangular prism? How many linear units are in the height? What is the measure of the volume? *15 ; 5 ; 75 cu. units*

2. How many square units are in the area of the base of the triangular prism? How many linear units are in the height? What is the measure of the volume of the triangular prism? *$7\frac{1}{2}$; 5 ; $37\frac{1}{2}$ cu. units*

3. The base of the hexagonal prism is made up of six congruent triangles. Express the area of each triangle in terms of a and b. What is the area of the base in terms of a and b? What is the volume?

189 cu. units

4. If $a = 7$ and $b = 9$, what is the measure of the area of the base?

2646 cu. units

5. If $h = 14$, what is the measure of the volume of the prism?

(ex. 3) $A = \frac{1}{2}ab$; $A = 6 \cdot \frac{1}{2}ab = 3ab$; $V = 3abh$

A. *Add:* Reduce all answers to simplest form.

1. $2\frac{1}{6}$, $5\frac{3}{4}$, $7\frac{1}{2}$ — $15\frac{5}{12}$
2. $3\frac{3}{8}$, $7\frac{1}{6}$, $8\frac{2}{3}$ — $19\frac{5}{24}$
3. $14\frac{2}{3}$, $11\frac{9}{16}$, $13\frac{1}{4}$ — $39\frac{23}{48}$
4. $5\frac{2}{7}$, $6\frac{1}{2}$, $3\frac{3}{4}$ — $15\frac{15}{28}$
5. $16\frac{2}{3}$, $9\frac{1}{4}$, $5\frac{2}{3}$ — $31\frac{7}{12}$

B. *Subtract:* Reduce all answers to simplest form.

1. $\frac{7}{8}$, $\frac{1}{2}$ — $\frac{3}{8}$
2. $15\frac{3}{4}$, $13\frac{7}{8}$ — $1\frac{7}{8}$
3. $4\frac{1}{6}$, $2\frac{1}{3}$ — $1\frac{5}{6}$
4. $9\frac{1}{2}$, $8\frac{11}{12}$ — $\frac{7}{12}$
5. $7\frac{2}{3}$, $6\frac{7}{8}$ — $\frac{19}{24}$

C. *Multiply:* Reduce all answers to simplest form.

1. $16 \cdot \frac{5}{6}$ $13\frac{1}{3}$
2. $5\frac{2}{3} \cdot 1\frac{4}{17}$ 7
3. $19\frac{1}{5} \cdot 1\frac{15}{16}$ $37\frac{1}{5}$
4. $15 \cdot 2\frac{4}{5}$ 42
5. $8\frac{1}{3} \cdot 3\frac{2}{5}$ $28\frac{1}{3}$
6. $2\frac{2}{3} \cdot 4\frac{7}{8}$ 13
7. $27 \cdot 3\frac{5}{9}$ 96
8. $13\frac{1}{3} \cdot 8\frac{5}{6}$ $117\frac{7}{9}$
9. $5\frac{8}{9} \cdot 6\frac{3}{4}$ $39\frac{3}{4}$

D. *Divide:* Reduce all answers to simplest form.

1. $\frac{9}{10} \div \frac{3}{5}$ $1\frac{1}{2}$
2. $\frac{3}{14} \div \frac{5}{7}$ $\frac{3}{10}$
3. $1\frac{4}{5} \div 2\frac{7}{10}$ $\frac{2}{3}$
4. $6 \div \frac{5}{12}$ $14\frac{2}{5}$
5. $1\frac{5}{9} \div 2\frac{4}{5}$ $\frac{5}{9}$
6. $3\frac{8}{9} \div 2\frac{4}{5}$ $1\frac{7}{18}$
7. $\frac{5}{6} \div 15$ $\frac{1}{18}$
8. $\frac{17}{35} \div \frac{5}{11}$ $1\frac{12}{175}$
9. $3\frac{1}{3} \div 16\frac{2}{3}$ $\frac{1}{5}$

E. *Add:*

1. $^-17 + (^-4)$ $^-21$
2. $^+35 + (^+6)$ 41
3. $^+2 + (^-19)$ $^-17$
4. $^-15 + (^+15)$ 0
5. $^-18 + (^+16)$ $^-2$
6. $^+13 + (^+19)$ 32

F. *Subtract:*

1. $^+17 - (^+15)$ 2
2. $^+21 - (^-21)$ 42
3. $^-14 - (^+3)$ $^-17$
4. $^-27 - (^-19)$ $^-8$
5. $^+35 - (^+35)$ 0
6. $^-18 - (^-5)$ $^-13$

If you need more practice, turn to the Practice Exercises on page 478.
If not, you may work in the Experts' Corner.

Specific Gravity

As you know, a cubic foot of one material does not necessarily weigh the same as a cubic foot of another. You could lift a cubic foot of wood, which weighs about 40 pounds, but you could not lift a cubic foot of gold, which weighs about 1200 pounds.

In order to express precisely the relative weights of various materials, the weight of an equivalent amount of water is taken as the standard of comparison. The ratio of the weight of any given substance to the same volume of water is called the *specific gravity* of the substance. At "standard conditions," that is, 39° temperature Fahrenheit and at sea level, a cubic foot of water weighs 62.4 pounds, to the nearest tenth of a pound. If substance A weighs 124.8 pounds per cubic foot, its specific gravity is 2.0. If a cubic foot of substance B weighs 31.20 pounds, its specific gravity is 0.5.

Specific gravity is usually expressed to the nearest hundredth.

SPECIFIC GRAVITY

Aluminum	2.68	Mercury	13.48	Wood: Ash	0.55*
Brick	2.00*	Lead	11.40	Oak	0.75*
Cast Iron	7.21	Silver	10.52	Hickory	0.85*
Copper	8.62	Steel	7.87	Pine	0.49*

* Varies. The given figure is average.

1. A cubic foot of marble weighs 170 pounds. What is its specific gravity, to two decimal places? 2.72

2. A cubic foot of brass weighs 530 pounds. What is its specific gravity? 8.49

3. Concrete has a specific gravity of 2.20. What does a cubic foot of concrete weigh? 137.3

4. The specific gravity of gold is 19.32. What is the weight of a cubic foot? 1205.6 lb.

The substance is lighter than water if the specific gravity is less than 1.

5. If the specific gravity of a substance is less than 1, it will float. Explain why this is true. Which of the substances listed will float? wood

6. Sea water weighs 3.4% more than an equal volume of pure water. What is the specific gravity of sea water? 1.034

7. When water freezes, its volume increases. The specific gravity of ice is 0.92. What is the weight of a cubic foot of ice? 57.4 lb.

A circular cylinder has as its bases two circles in planes parallel to one another. In this chapter, the term *cylinder* will be understood to mean a right circular cylinder, since these are the only types we shall consider.

A cylinder has two properties that make it especially useful for many purposes. Its form is more rigid, and for a given capacity it uses less material than a comparable prism. For this reason you will frequently find it used as a column to support a heavy load. It is also the most commonly used form for containers of liquids. You will find the supermarket shelves filled with cylindrically shaped cans, bottles, and even cartons. In industry the cylinder is used not only for storage tanks, but also for pipe lines and containers for compressed gases. In the home we find cylinders used as hot water tanks, glasses, coffee makers, fruit jars, and water pipes.

CYLINDER

1. List five examples of cylinders in your school or on the school grounds. *Answers will vary.*

2. For each example you listed, state whether it was used as: a container for liquids or gases, as a pipe for transporting liquids or gases, or as a support in construction. *Answers will vary.*

3. Make a similar list of cylinders you find around the home.

4. List some examples of cylinders that were not used for any of the purposes listed in Exercise 2. In each case see if you can explain the reason why the cylindrical shape was used. *Answers will vary.*

5. It is not uncommon to find shapes in nature that are approximately cylindrical. List some that you have observed. *Answers will vary.*

6. What is the shape of the top of a cylinder? *circle*

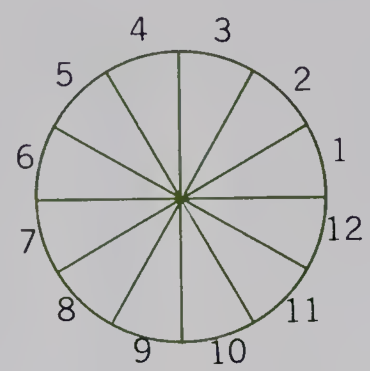
7. Compare the top and bottom of a cylinder in shape and size. *same shape and size*

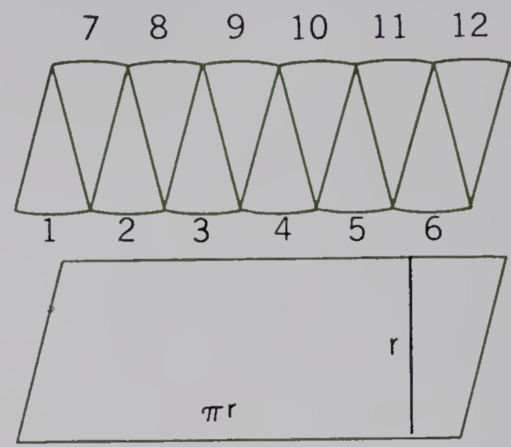
8. The top and bottom of a cylinder are called the bases of the cylinder. Are the bases parallel? *yes*

9. Refer to the definition of a prism on page 258 and explain why a cylinder is not a prism. *The base is not a polygon.*

As in rectangular solids, we will consider the volume and the total surface area of a cylinder.

The base of a circular cylinder (which is the only type we shall be concerned with) is a circle. To calculate the volume of a cylinder, you must calculate the area of the region enclosed by the base. The process of discovering the formula for the area of the circle is illustrated:





As you can see, the region enclosed by the circle is divided into 12 sectors by cutting along the diameters of the circle. When the six sectors are fitted into the other six, as indicated, a figure resembling a parallelogram is formed. As the number of sectors into which the circular region has been divided increases, the figure approximates a parallelogram.

The altitude of the parallelogram is equal to the measure of the radius of the circle. The measure of the base is half the measure of the circumference of the circle, or πr. Thus, the area of the parallelogram can be expressed as $\pi r \cdot r = \pi r^2$. Since the area of the parallelogram is equal to that of the circle from which it is derived, the formula for the area of the circle is also $A = \pi r^2$.

EXAMPLE

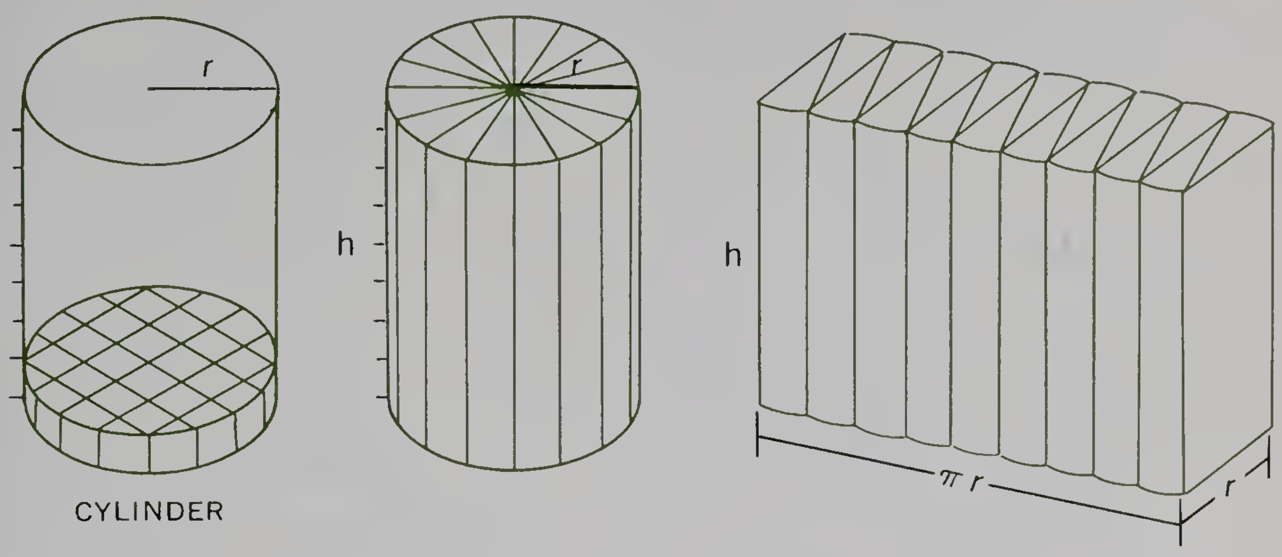
What is the area of a circle whose diameter is 16 feet?
Since a diameter is 16 feet, a radius is 8 feet. The formula is: $A = \pi r^2$
For the value of π we can use 3.14 or $3\frac{1}{7}$, whichever is most convenient.
Here we will use 3.14, as 8 is not divisible by 7. Then $r^2 = 64$ and $(64) \cdot (3.14) = 200.96$.
Since the diameter was given to the nearest foot, $A = 201$ square feet.

Find the area (r = radius, d = diameter, C = circumference).

1. $r = 14$ in. *616 sq. in.*
2. $d = 20$ in. *314 sq. in.*
3. $C = 6.42$ ft. *3 sq. ft.*
4. $r = 21$ ft. *1386 sq. ft.*
5. $C = 62.8$ in. *314 sq. in.*
6. $d = 40$ in. *1256 sq. in.*
7. $r = 12$ ft. *452 sq. ft.*
8. $d = 14$ ft. *154 sq. ft.*
9. $r = 9$ ft. *254 sq. ft.*

By examining the cylinder, you can see that it appears reasonable that the volume should be calculated in the same way as the volume of a prism; that is

$$V = Bh$$



CYLINDER

You can see that a cylinder can be approximated by a prism just as a circle can be approximated by a parallelogram.

1. What is the formula for the area of the base of a cylinder? *$A = \pi r^2$*

2. Suppose the radius of the base of the cylinder above is 5″. What is the area of the base? (Use $\pi \approx 3.14$.) *79 sq. in.*

3. If the height of the cylinder is 8″, what is its volume? *632 sq. in.*

4. Another way of writing the formula for the volume of a cylinder is:

$$V = \pi r^2 h$$

Study the diagrams and write the meaning of the formula. *See front.*

5. Find the volume of a cylinder with a radius of 2′ and height of 6′. (Use $\pi \approx 3.14$.) *75 cu. ft.*

6. Find the volume of a cylindrical tank that has a diameter of 20 inches and a height of 18 inches. (Use $\pi \approx 3.14$.) *5652 cu. in.*

7. Use $\pi \approx 3.14$ and find the volume of the following cylinders:

a. Height, 6 inches; radius = 4 inches *301 cu. in.*
b. Height, 2 feet; diameter = 3 feet *14 cu. ft.*
c. Height, 30 feet; base = 124 square feet *3720 cu. ft.*
d. Height, 24 feet; circumference = 12.56 feet *301 cu. ft.*
e. Height, $1\frac{1}{2}$ yards; radius = 2 yards *19 cu. yd.*

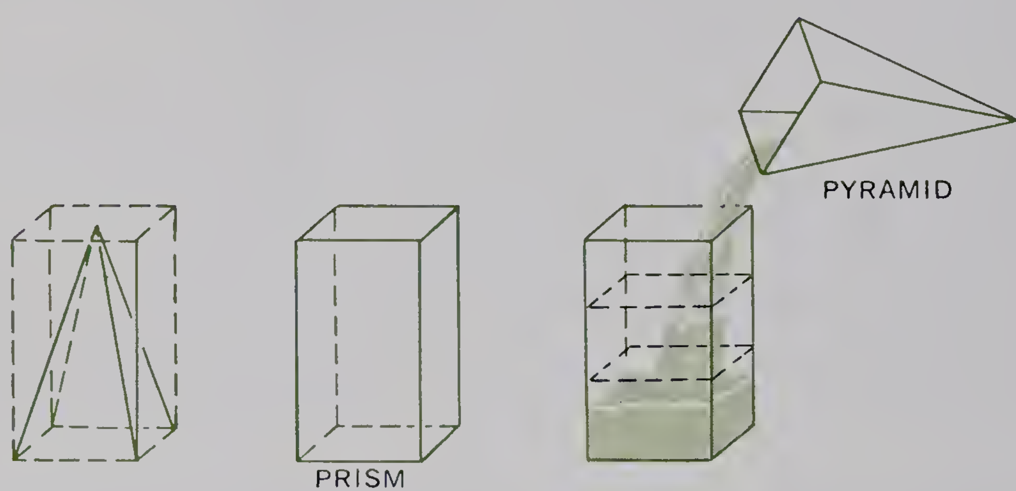
8. A cylindrical lawn roller has a 7″ radius and is 30″ wide. It weighs 50 pounds when empty. How many gallons of water will it hold? *20 gal.*

The pyramid and the cone are both familiar forms. The most widely known models of pyramids are the pyramids of ancient Egypt. Storage bins in elevators are upside down pyramids which come to a point at the bottom so they can be emptied readily.

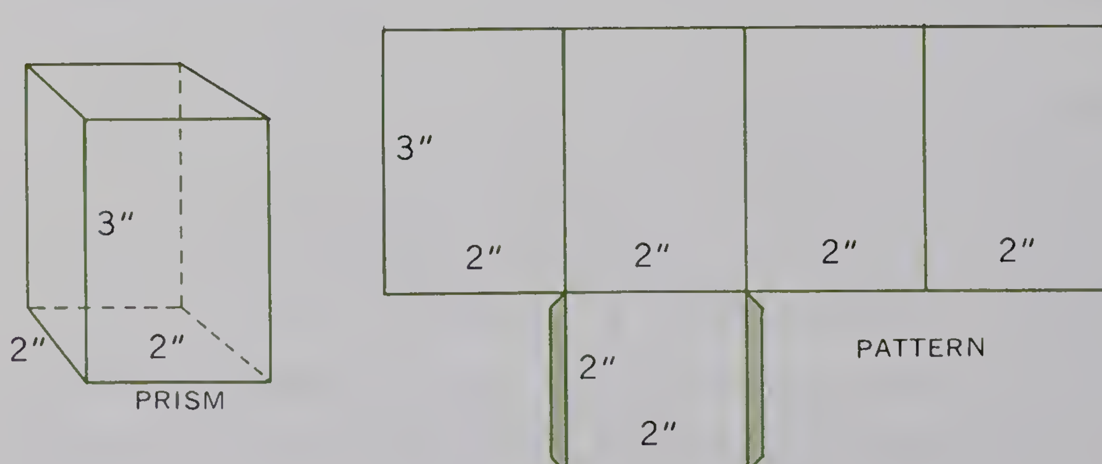
You are familiar with the cone in the common paper drinking cup, and more recently, in the nose cone of a space missile.

The lateral faces of a pyramid are all triangles. The base of the pyramid may be any kind of polygon. The pyramid is named according to the shape of its base. The pyramids of Egypt are called *square pyramids* because their bases are square.

It is easy to calculate the volume of a pyramid because it is $\frac{1}{3}$ the volume of a prism having the same altitude and a base of the same size. You can readily show this with an experiment, as illustrated, by making models of the prism and the pyramid. Then, by filling the pyramid with sand and pouring it into the prism, you will find that it takes three pyramids of sand to fill the prism.

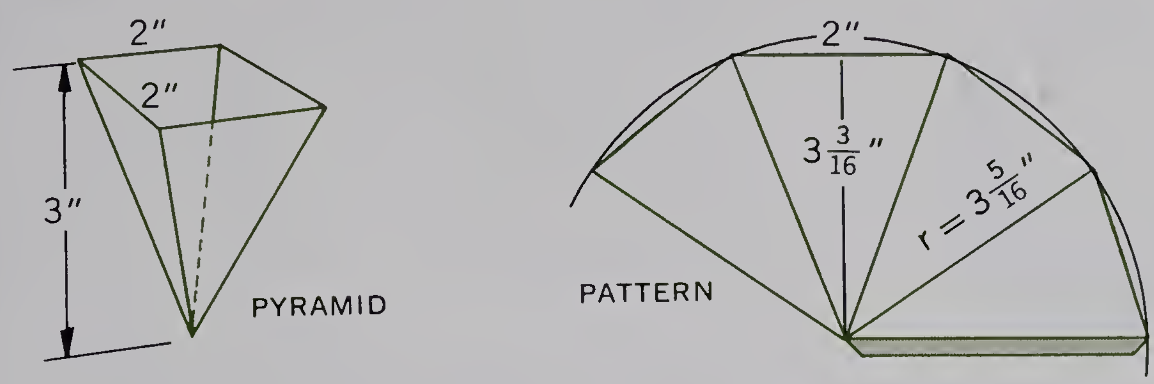


A prism is shown below, together with the pattern, in scale drawing. Draw the pattern full size on cardboard. The colored tabs $\frac{1''}{4}$ to $\frac{3''}{8}$ wide are for pasting. Fold the patterns along the lines and paste together with the tabs inside.



The pattern for the pyramid, as you can see, consists of four isosceles triangles lying side by side.

The diagrams below show how the Pythagorean Theorem was used to find first the altitude of the pyramid and then the length of a lateral edge. To draw the pattern, after the length of the edge has been calculated, set the compass for the length of the edge and draw a semicircle. Then set the compass to the length of one side of the base and lay off the bases of the triangles.

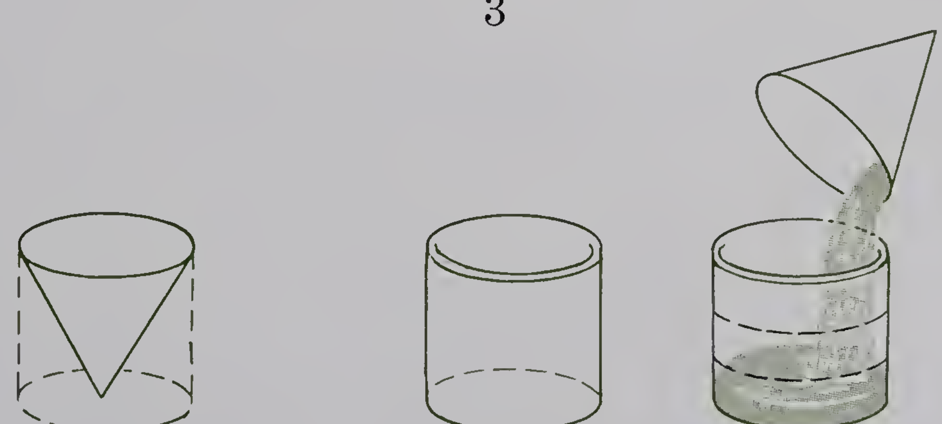


You now have a pyramid and prism with the same altitude and bases of the same size and shape with which you can show that the formula for the volume of a pyramid is:

$$V = \frac{1}{3}Bh$$

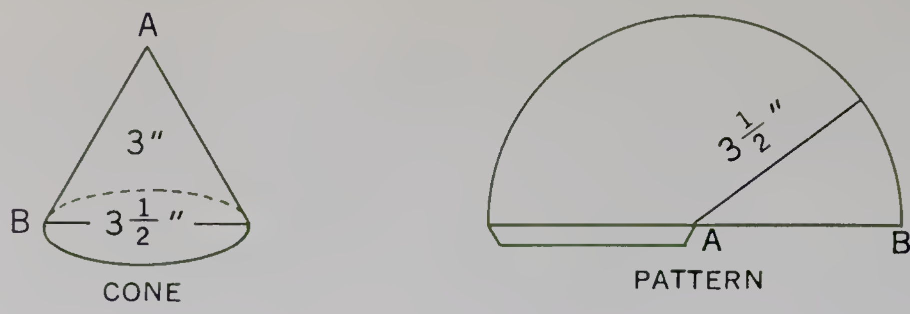
An experiment similar to the one with prisms and pyramids may be carried out to show that the volume of a cone is the volume of a cylinder having the same base and altitude. On the following page you will find scale drawings of patterns which you can use to make a cone and cylinder to carry out the experiment, demonstrating that the volume of a cone can be calculated with the formula:

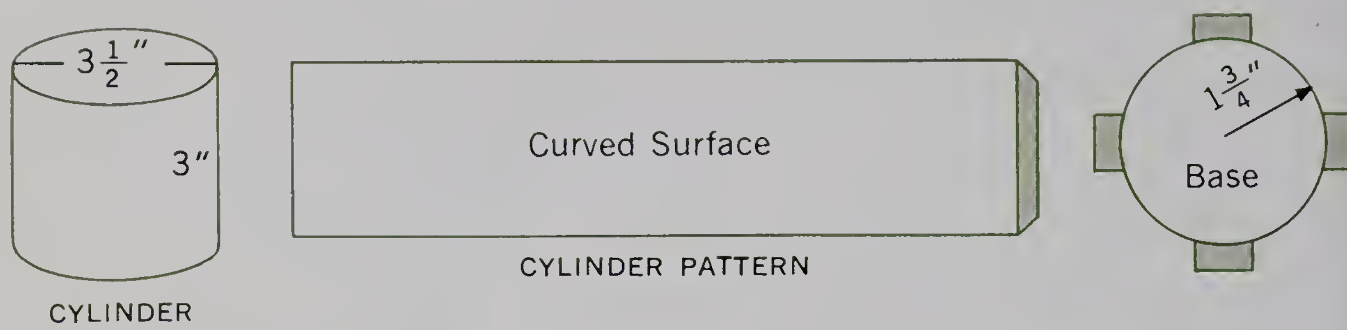
$$V = \frac{1}{3}Bh$$



1. What is the volume in cubic inches of the rectangular prism on the previous page? *12 cu. in.*
2. What is the volume of the pyramid at the top of the page? *4 cu. in.*

3. What is the volume of the cylinder? of the cone? 28.875 cu. in.; 9.625 cu. in.

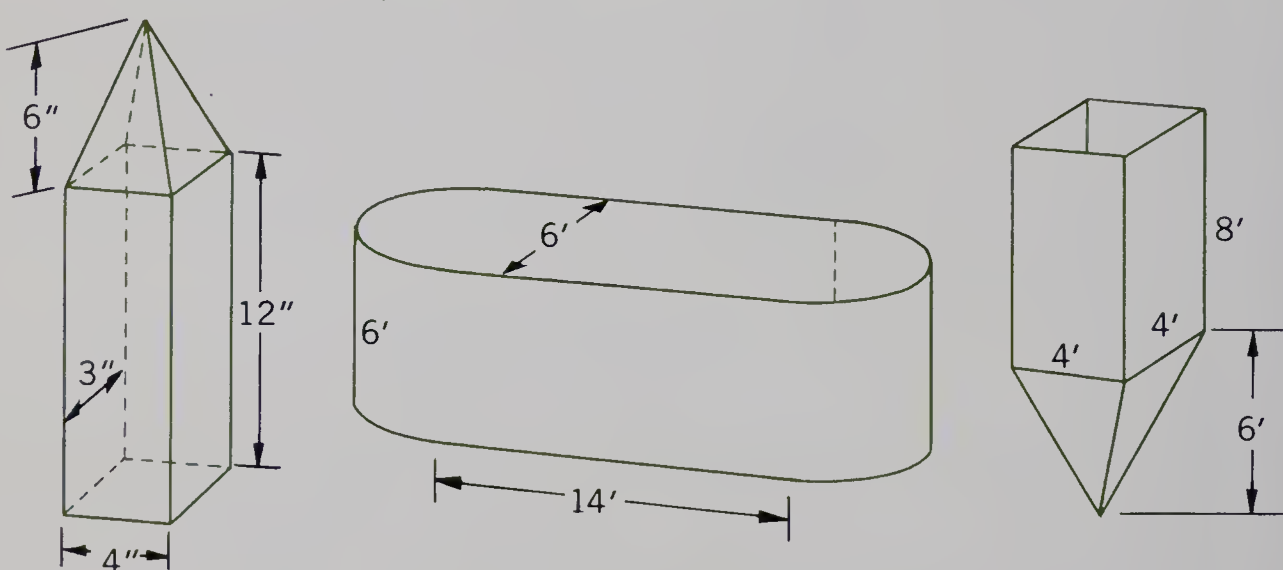




4. One of the famous pyramids of Egypt is the pyramid of Gizeh. The base is a square 755 feet on a side. The height is 481 feet. How many cubic feet of stone does it contain if it is solid? 91,394,008 cu. ft.

5. Many containers are made up of several different kinds of figures, whose volumes must be calculated separately and then added together. Calculate the volume of each of the containers.

168 cu. in.; 674 cu. ft.; 160 cu. ft.



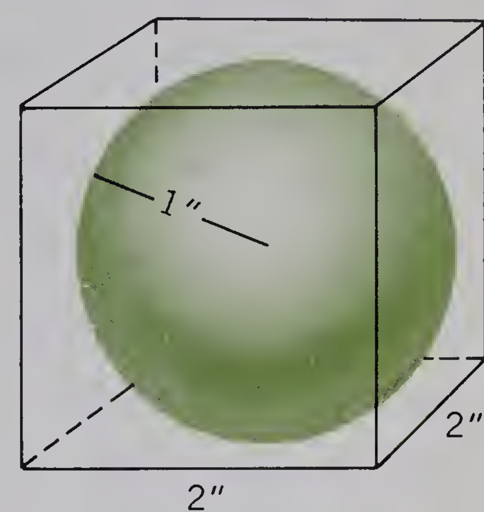
SPECIAL PROJECTS

a. Measures for a bushel or a gallon are not usually cubic in form. You can construct a cube whose capacity is approximately a gallon if you make a cardboard cube with one face open, whose edges are $6\frac{1}{8}$″ long.

b. A box whose shape is a cube with edges $12\frac{7}{8}$″ long contains approximately a bushel. Is a bushel greater or less than a cubic foot? Find out how a bushel of wheat is legally defined.

The sphere has a number of important characteristics. For purposes of construction, it has the minimum surface area for a given volume. It is also a rigid form of construction. For these reasons the spherical shape is generally used in tanks to hold gases under pressure. A sphere rolls with a minimum of friction. This accounts in part for its use in ball bearings, golf balls, baseballs, basketballs, soccer balls and so on.

Notice how the volume of a sphere can be calculated experimentally, comparing it to a cube whose edge is equal in length to that of the diameter of the sphere. If the cubical container were filled with water, it would contain 8 cubic inches. If the sphere were immersed, it would displace a little more than half the water–actually 4.2 cubic inches, to the nearest tenth of a cubic inch.



The volume of a sphere can be calculated from the formula:

$$V = \frac{4}{3}\pi r^3$$

EXAMPLE

The official baseball has a radius of about 1.5″. What is its volume?

$$V = \frac{4}{3} \cdot (3.14) \cdot (1.5) \cdot (1.5) \cdot (1.5)$$

$$V = 14.1 \text{ cu. in. (to the nearest tenth)}$$

Check and see if you get the same answer.

Solve these exercises using $\pi \approx 3.14$.

1. The official tennis ball has a diameter of 2.4″. What is its volume in cubic inches? *7.2 cu.in.*

2. The diameter of a golf ball is 1.7″. What is its volume? *2.6 cu.in.*

3. A spherical gas container is 16′ in diameter. How many cubic feet of gas will it hold? *2143.6 cu.ft.*

4. Find the volume of these spheres: *904 cu.in.* *523 cu.ft.*

a. $d = 4''$ *34 cu.in.* **c.** $r = 3'$ *113 cu.ft.* **e.** $d = 12''$ **g.** $r = 5'$

b. $r = 8''$ **d.** $d = 14'$ **f.** $r = 9'$ **h.** $r = 4.5'$

2143.6 cu. in. *1436 cu.ft.* *3052 cu.ft.* *382 cu.ft.*

5. If you double the radius of a sphere, its volume will be how many times as great? *8* Try some numerical examples before answering.

The measure of capacity of a container is the number of liquid or dry units of capacity it will contain. Frequently we need to know the capacity of a bin in bushels to determine how much grain it will hold. Or we may need to measure the capacity of a water tank or a gasoline tank in gallons. The most common measures of dry measure and liquid measure are these:

Dry measure		*Liquid measure*	
1 bushel (bu.)	4 pecks	1 gallon (gal.)	4 quarts
1 peck (pk.)	8 quarts	1 quart (qt.)	2 pints
1 quart (qt.)	2 pints	1 pint (pt.)	16 fluid ounces

The units of capacity are defined in terms of units of cubic measure. In most states a bushel is legally defined as 2150.42 cubic inches. 1 gallon is 231 cubic inches.

302.42 cu. in. too little

1. You can see from these definitions that a liquid quart is not equal to a dry quart. If a dealer measured 8 gallons of grain as a bushel, how many cubic inches too little or too much would there be?

2. How many cubic inches are in 1 peck? in one pint dry measure?
537.61 cu. in.; 33.6 cu. in.

3. An approximation that is sufficiently accurate for most purposes is 1 bushel $\approx$ 1.25 cubic feet. How many cubic inches is 1 bushel more or less than 1.25 cubic feet? 9.58 cu. in. less

4. Another approximation that is useful is 7.5 gal. $\approx$ 1 cu. ft. How many cubic inches more or less than 1 cubic foot are 7.5 gallons?
4.5 cu. in. more

5. A cubic foot of water (at sea level and at 39° Fahrenheit) weighs about 62.4 lb. What does 1 gallon weigh, to the nearest tenth of 1 lb.? 8.3 lb.

6. An old saying is that "a pint's a pound the world around." What does 1 pint of water weigh, to the nearest tenth of 1 lb.? 1.0 lb.

7. A fluid ounce is a measure of capacity, not of weight. To the nearest tenth of 1 ounce, what does a fluid ounce of water weigh?
1.0

8. A rectangular bin measures 12 feet in length, 6 feet in width, and 5 feet in height. How many cubic feet does it measure in volume?
360 cu. ft.

9. How many bushels of wheat will it contain? 289 bu.

10. Mr. Edwards has a rectangular water tank in his barn whose base measures 3 feet by 6 feet, and whose height measures 5 feet. How many cubic feet does it measure in volume? 90 cu. ft.

11. How many gallons of water will it contain when filled? *673 gal.*

12. A freight car measures 40 feet in length (inside measurement) and 8.5 feet in width. If it is filled to a depth of 6 feet with wheat, how many bushels will it hold? *1632 bu.*

13. A bushel of wheat weighs 60 pounds. How many tons of wheat are in the car in Exercise 12? *49 tons*

14. How many tons of wheat are in the car if it is filled only to a depth of 4 feet? (There are two ways of working this problem.) *32.7 tons*

15. Mr. Jones built a water trough for his cattle in the form of a rectangular prism that measured 12 feet in length, 3 feet in width, and 3 feet in height. What was its gallon capacity? *810 gal.*

16. A quart milk carton has a square base measuring 3 inches on each side. How high should it be to hold a quart? *6.4 in.*

17. A paper drinking cup is shaped like an inverted cone. Its diameter measures 4 inches, and its height 4 inches. How many of the cups can be filled from a quart measure? *3.5*

18. The grain hopper in an elevator is shaped like an inverted pyramid. The base is square, and the measure of each side of the base is 5 feet. The measure of the depth is 8 feet. How many bushels of grain will it hold? *53.3 bu.*

19. Centerville has a spherical water tank. The measure of a diameter is 18 feet. How many gallons will it hold if filled to capacity? *22,891 gal.*

20. Mr. Smith purchased a portable steel grain storage bin, in the form of a cylinder with a circular base. The measure of a diameter of the base is 8 feet, and the measure of the height is 10 feet. How many bushels of grain will it hold if it is filled? *402 bu.*

21. When Mr. Adams put running water in his house and barn on the farm, he piped water from the windmill to a rectangular tank upstairs in the house. The tank measured 6 feet on each side of the square base, and 5 feet in height. How many gallons of water will it hold? *1350 gal.*

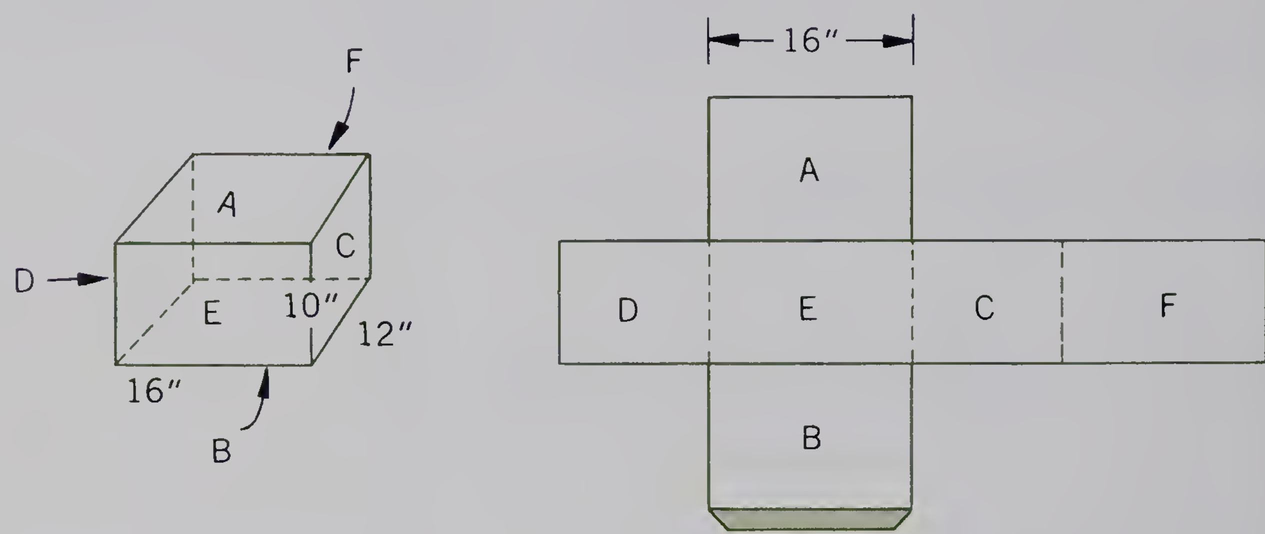
22. In the barn Mr. Adams has a tank which measures 6 feet on each side of a square base, and 8 feet in height. What is its capacity in gallons? *2160 gal.*

23. What will the water in each of Mr. Adams's tanks weigh when full? *11,205 lb. and 17,928 lb.*

24. Mr. Adams's grain bin measures 9 feet in length and 6 feet in width. It is filled with oats to a depth of 5 feet. How many bushels of oats are in the bin? *216 bu.*

Roy and Mike were assigned the responsibility for making ballot boxes for the student body elections. In order to have them reasonably neat in appearance, they found three cardboard boxes of identical size, and proceeded to cover them with colored paper.

1. The boxes were right rectangular prisms, 16 inches in length, 12 inches in width, and 10 inches in height. What were the dimensions of the pieces of paper the boys cut to cover the top and bottom of the boxes? 16" x 12"

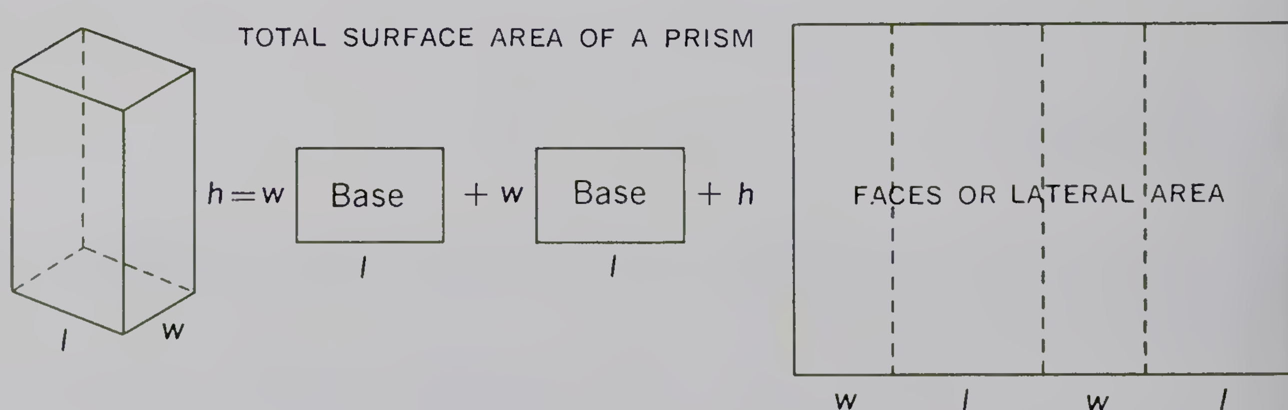


2. Roy planned to cover the four *lateral faces* with one sheet of paper What are the dimensions of the sheet of paper? 56" x 10"

3. What is the area of the two sheets of paper used to cover the top and bottom of the box? 192 sq. in.

4. What is the area of the sheet of paper used to cover the lateral faces of the box (D, E, C, and F)? 560 sq. in.

5. What is the total area of the paper used to cover the three boxes? 2832 sq. in. At $2\frac{1}{2}$¢ a square foot, how much did the paper cost? 50¢



The area of the faces of a prism (not including the bases) is its lateral area L. The area of the lateral surfaces L is equal to the perimeter of the base times the height, $L = Ph$. The total surface area T

of the prism includes the area of the bases. Thus a formula for the total surface area of a rectangular prism is:

$$T = 2lw + 2lh + 2wh = 2lw + Ph$$

Can you show that these formulas are equivalent using the distributive and associative properties? $T = 2lw + h(2l + 2w)$
$P = 2l + 2w$; thus $T = 2lw + Ph$

6. What is the area of paper needed to cover a box 20 inches long, 15 inches wide, and 10 inches high? 1300 sq. in.

7. Find the total surface area T of the following rectangular prisms:

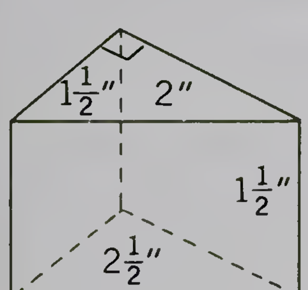
Prism	*Length of Base*	*Width of Base*	*Height*
A	20 in.	14 in.	10 in. 1240 sq. in.
B	3 ft.	2 ft.	2 ft. 32 sq. in.
C	24 in.	18 in.	6 in. 1368 sq. in.
D	6 ft.	$4\frac{1}{2}$ ft.	$2\frac{1}{2}$ ft. 106.5 sq. ft.

8. What is the lateral area L of a bin that measures 20 feet long, 10 feet wide, and 8 feet high? 480 sq. ft.

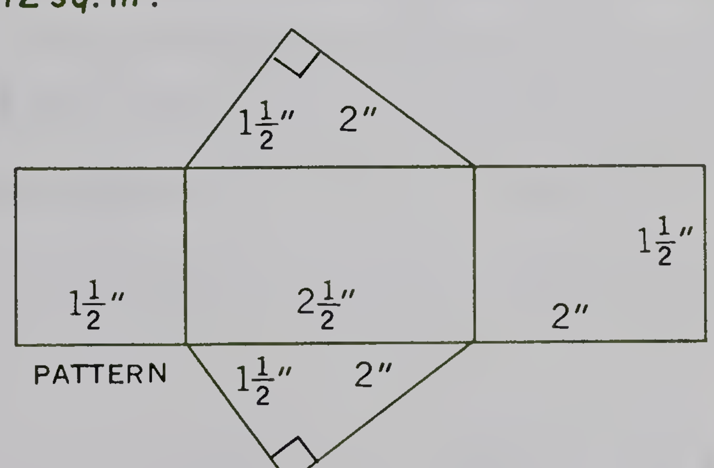
9. What will it cost to paint the four walls and ceiling of a room that measures 13 feet long, 9 feet wide, and 8 feet high? The cost of painting is 32¢ per square foot. \$ 150.08

10. Illustrated below is a triangular prism that has been cut along the edges indicated. You can see it flattened out. Calculate the area of each of the lateral faces and the bases. 3 sq. in.; $3\frac{3}{4}$ sq. in.; $2\frac{1}{4}$ sq. in.
3 sq. in.

11. Calculate the total area T. 12 sq. in.



TRIANGLE PRISM



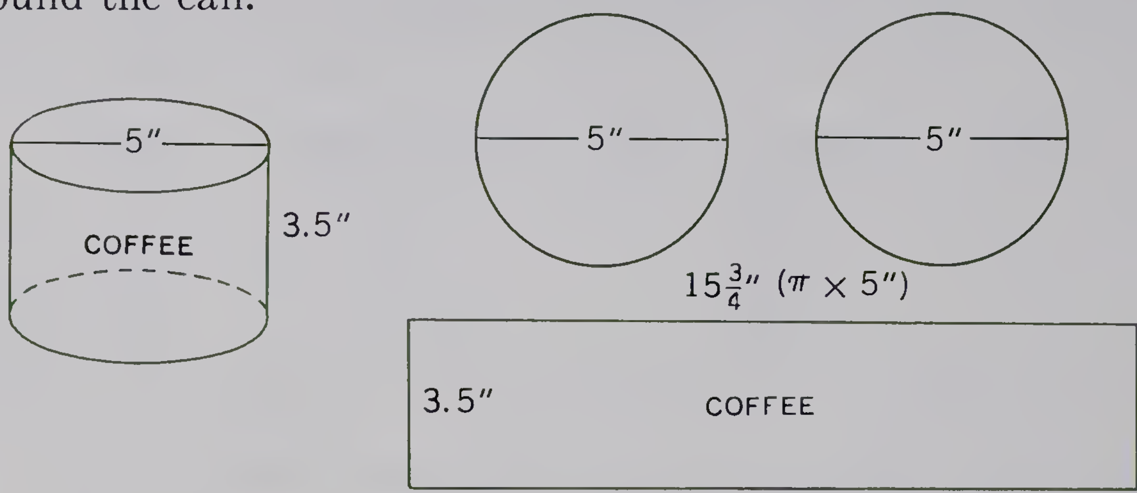
PATTERN

12. Find the total surface area T of each of the following triangular prisms. The shape of the base is that of a right triangle.

Prism	*Leg*	*Leg*	*Hypotenuse*	*Height*
A	4 in.	3 in.	5 in.	8 in. 108 sq. in.
B	5 in.	12 in.	13 in.	4 in. 180 sq. in.
C	6 in.	8 in.	10 in.	12 in. 336 sq. in.

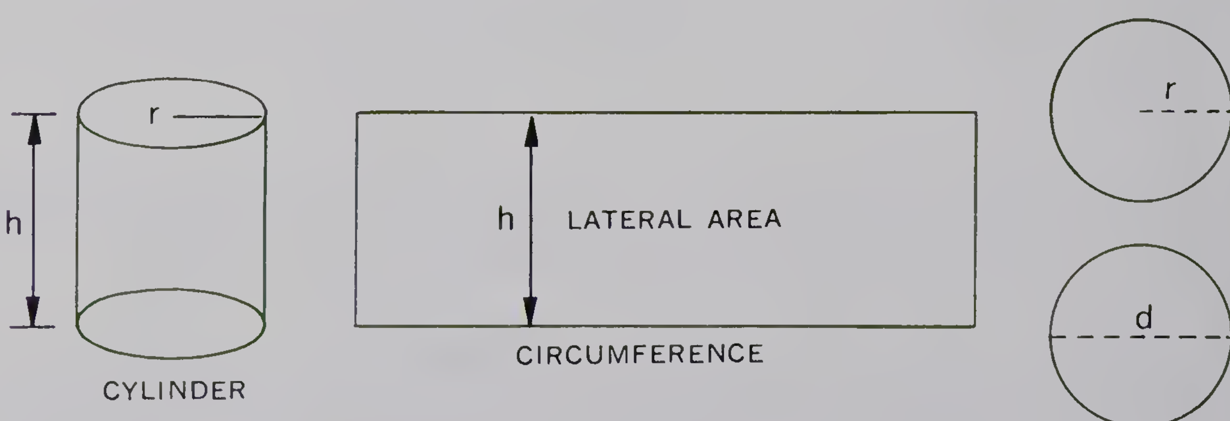
When Sally and Joan decorated for a party, they planned to make imitation drums by covering coffee cans with paper.

1. The top and bottom of the can were circles with the measure of a diameter of 5 inches. What was the circumference of the can? *15.70 in.*
2. Sally found that a rectangular sheet of paper was needed to go around the can.



   The height of the can corresponds to what dimension of the rectangle? *width*

3. The circumference of the can corresponds to what dimension of the rectangle? *length*
4. The height of the coffee can was 3.5 inches. What is the area of the rectangular piece of paper required to go around the can? *55.0 sq. in.*
5. To cover the total surface of the can with paper, Sally and Joan used the circles for the top and bottom. What is the area of each circle? *19.6 sq. in.*
6. What is the total area of the surface of the cylinder? *94.2 sq. in.*
7. The formula for the total surface of a cylinder is $2\pi r^2 + \pi dh$. State and explain the rule expressed by the formula. Does the total surface of a cylinder equal $2\pi r(r+h)$? Explain. *See front.*



8. What is the total surface area of a can that has a radius of 1.4 inches and a height of 4 inches? ($\pi \approx \frac{22}{7}$) *47.5 sq. in.*

9. The *lateral surface* of a cylinder is the curved surface around the cylinder and does not include the surface area of the top and bottom bases. Find the lateral surface of a cylindrical silo 32 feet high and 16 feet in diameter. ($\pi \approx 3.14$) 1608 sq. ft.

10. A water tank in the shape of a cylinder has an open top and rests on a concrete base. The height of the tank is 28 feet and the diameter is 12 feet. How much will it cost to paint the tank at 35¢ per square foot? (Use lateral area.) ($\pi \approx \frac{22}{7}$) \$ 369.60

11. What will it cost to paint the lateral surface of a silo that is 30 feet high and 14 feet in diameter at 25¢ per square foot? ($\pi \approx \frac{22}{7}$) \$ 330

12. State a formula for finding the lateral surface of a cylinder. $L.S. = \pi dh$

13. How many square feet of metal are required to make a ventilating pipe 30 feet long and 18 inches in diameter? 141.3 sq. ft.

14. A one-pound can of coffee is 4 inches high and 5 inches in diameter. If the height is doubled, will it hold twice as much coffee? If its diameter is doubled? yes ; 4 times as much coffee

15. If the height remains the same, and the diameter is $7\frac{1}{4}$ inches, how many pounds of coffee will it hold to the nearest tenth? ($\pi \approx 3.14$) 2.1 lb.

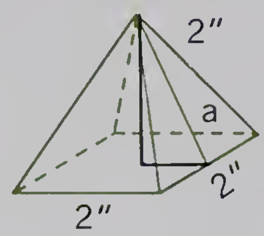
16. Calculate the total surface area of the cans in Exercise 14 (after the height was doubled) and in Exercise 15. Which uses less material, and how much? The can in exercise 15 uses 9 square inches less.

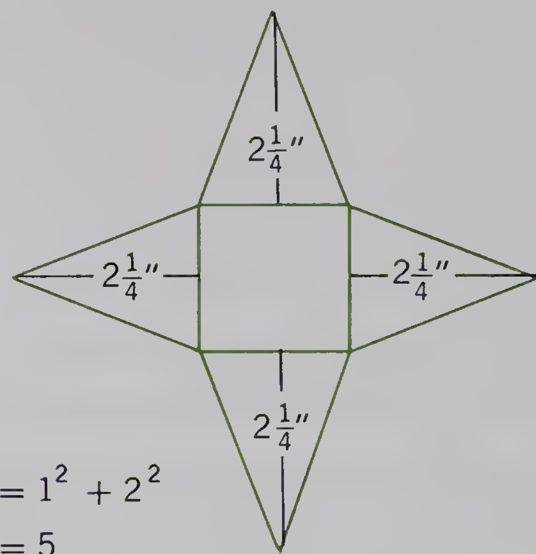
SPECIAL PROJECTS

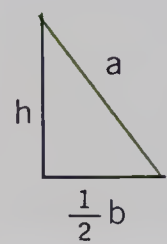
1. A manufacturer of sheet metal cans has three models. Model A is in the form of a rectangular prism and measures 15 in. × 7 in. × 11 in. Model B is also a rectangular prism which is 21 in. × 11 in. × 5 in. The third model is a cylindrically shaped can which has an 8-inch diameter and a height of 23 inches.
   a. Find the volume of each can.
   b. Find the square inches of metal in each can (total surface area).
   c. Explain which would require the least materials to manufacture.

2. Two cans of fruit juice have exactly the same height of 10 in. The radius of one can is 2 inches and the radius of the second is 3 inches.
   a. The radius of the second can is how many times as great as the radius of the first can?
   b. The second can holds how many times as much juice as the first?
   c. The surface area of the second can is how many times as great as the surface area of the first?
   d. What do you conclude about increasing the volume of a circular cylinder?

By examining the patterns on Pages 274, 276, and 280, you can see how you could draw them on cardboard, fold along the lines, and paste them together, to form solids. You should leave tabs on some of the edges for pasting.







$a^2 = 1^2 + 2^2$

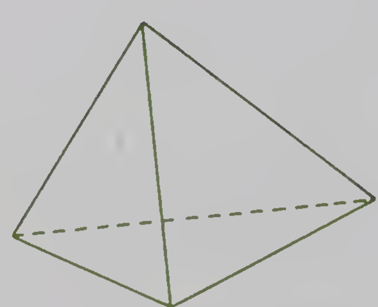
$a^2 = 5$

$a = \sqrt{5} = 2.26$

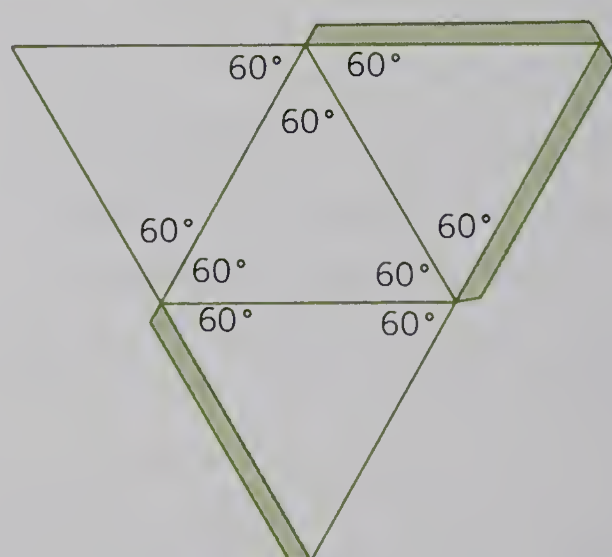
$a = 2\frac{1}{4}$

There are several interesting figures that you might like to make. Try to make a model of a pyramid. You will need to use the theorem of Pythagoras to find the altitude of the triangles that form the sides.

Another interesting figure is the regular tetrahedron, which has four faces, each of which is an equilateral triangle.



TETRAHEDON MODEL

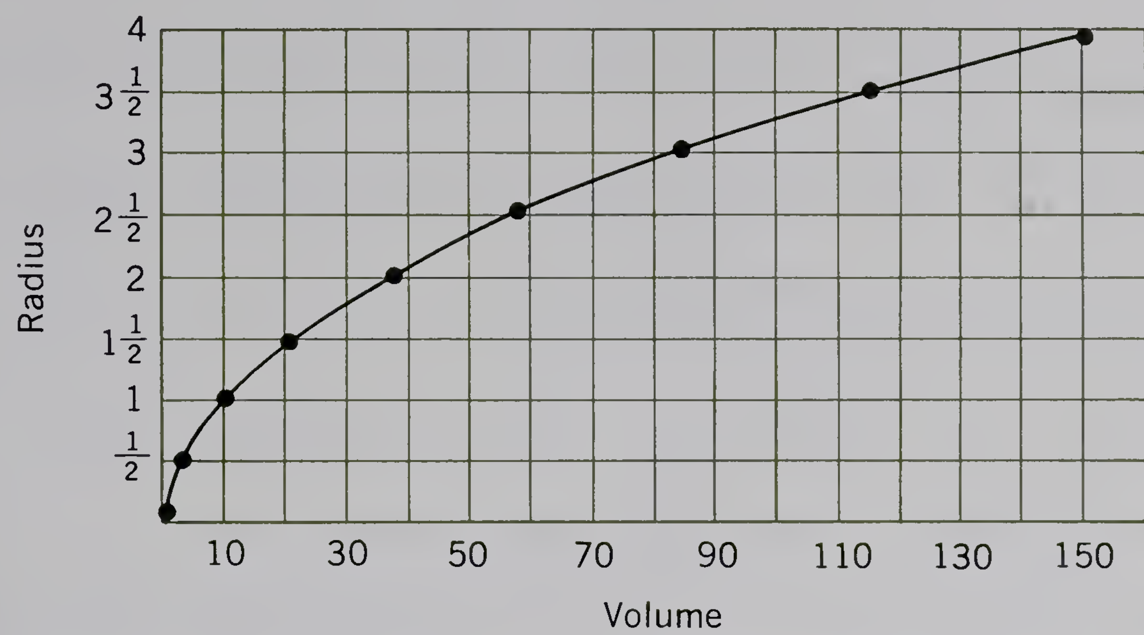


By examining the models of your figures, and drawing sketches as necessary, answer these questions:

1. How many vertices does a triangular prism have? *6* a rectangular prism? *8*
2. Can you tell how many vertices a hexagonal prism has without counting? *12* Explain how you can tell for any prism. *Double the number of the vertices of the base in the figure.*

The graph below shows how the volume of a cylinder changes as the radius changes. Notice that in all cases the height of the cylinder remains constant at 3 inches.

CHANGE IN VOLUME OF A CYLINDER
3 INCHES HIGH AS THE RADIUS CHANGES



1. Copy and complete the table below using the graph. Read the graph as accurately as you can:

Column I		Column II		Column II ÷ Column I
Radius	*Volume*	*Radius*	*Volume*	*(To the nearest tenth)*
1″	10 cu. in.	2″	38 cu. in.	3.8
2″	38 cu. in.	4″	150 cu. in.	3.9
$1\frac{1}{2}$″	20 cu. in.	3″	85 cu. in.	4.3
1″	10 cu. in.	3″	85 cu. in.	8.5
1″	10 cu. in.	4″	150 cu. in.	15.0

Study the figures in the third column of the table and answer the following questions. If you are a good detective, you can discover some interesting facts.

2. From your figures in the table used in Exercise 1, can you write a statement which tells how the volume changes when the radius is doubled? When the radius is doubled, the volume is quadrupled.

3. Can you write a statement telling how the volume changes when the radius is tripled? When the radius is tripled, the volume increases 8 times.

4. Tell how the volume changes when the radius is multiplied by 4.

5. How does the volume of a cylinder change as the radius of the cylinder increases, height remaining constant?

The volume increases 16 times.

Using the Associative Property

By use of the associative property you can frequently find ways of multiplying and dividing by fractions of multiples of 10 without use of a pencil.

EXAMPLES

1. Find $18 \cdot 50$. Since $50 = \frac{1}{2} \cdot 100$, we know $18 \cdot 50 = 18\left(\frac{1}{2} \cdot 100\right)$
   Applying the associative property, $\left(18 \cdot \frac{1}{2}\right) \cdot 100 = 9 \cdot 100$, or 900

2. Find $360 \div 75$. Since $75 = \frac{3}{4} \cdot 100$, we know $360 \div 75 =$
   $360 \div \left(\frac{3}{4} \cdot 100\right) = \left(360 \div \frac{3}{4}\right) \div 100$, or $480 \div 100$

Use the associative property to regroup factors. Then find the answers mentally. Show the grouping on your paper.

1. $240 \cdot 25$ *6000*
2. $160 \cdot 75$ *12,000*
3. $120 \cdot 500$ *60,000*
4. $160 \cdot 125$ *20,000*
5. $16 \cdot (7.5)$ *120*
6. $125 \cdot 400$ *50,000*
7. $320 \cdot 125$ *40,000*
8. $1800 \div 50$ *36*
9. $1200 \div 75$ *16*
10. $315 \div 25$ *12.6*
11. $24 \div 75$ *$\frac{8}{25}$ or .32*
12. $360 \div 50$ *$7\frac{1}{5}$ or 7.2*
13. $90 \div 75$ *$1\frac{1}{5}$ or 1.2*
14. $400 \div 50$ *8*
15. $348 \cdot 25$ *8700*

16. A field is 40 rd. wide and 75 rd. long. What is its area? *3000 sq. rd*

17. A lot is 125′ long and 96′ wide. Find the area. *12,000 sq. ft.*

18. A bin is 25′ long, 12′ wide, and 8′ deep.
   a. How much is $8 \cdot 12$? *96*
   b. How would you multiply your answer by 25? *Multiply by $\frac{1}{4}$ and add two zeros.*
   c. How many cubic feet does the bin contain? *2400 cu. ft.*

19. The base of a circular fish pond has an area of 100 sq. ft.
   a. If it is filled to a depth of 1 ft., how many cubic feet will it contain? *100 cu. ft.*
   b. How many gallons is this? (1 cu. ft. = 7.5 gal.) *750 gal.*
   c. How many gallons will the pond hold if it is filled to a depth of 2 ft.? *1500 gal.*
   d. How deep must the pond be to hold 3000 gallons? *4 ft.*

20. A can with a base 10″ square is 16″ high. How many more cu. in. does it contain than two cans of the same height, each with a base 5″ square?

a. What is the area of the base of the larger can? 100 sq. in.
b. What is its volume in cu. in.? 1600 cu. in.
c. What is the area of the base of one smaller can? 25 sq. in.
d. What is the volume of one smaller one? 400 cu. in.
e. What is the volume of two of the smaller cans? 800 cu. in.
f. How much less is this than the volume of the larger can? 800 cu. in.

21. How many gallons will a tank contain that is 5′ wide, 15′ long, and is filled with water to a depth of 16′?

a. What is the area of the base of the tank? 75 sq. ft.
b. What is its volume in cubic feet? 1200 cu. ft.
c. Each cubic foot is 7.5 gallons. How much is 7.5 times the answer you found in part b? 9000 gal.

22. The base of a cylindrical water tank has an area of 50 sq. ft. How deep must it be filled to hold 800 cubic feet of water? (Divide 800 by 50. Why?) 16 ft.

23. A field that contains 1200 sq. rd. is 75 rd. long. How wide is it? 16 rd.

24. A room is 16′ wide, 25′ long, and 10′ high.

a. What is the perimeter of the room? 82 ft.
b. What is the area of the side walls of the room? (Neglect the openings, such as doors and windows.) 820 sq. ft.
c. What will it cost to paint the walls at 25¢ per sq. ft.? $205
d. What are the dimensions of the ceiling? 16′ X 25′
e. How many square feet are there in the ceiling? 400 sq. ft.
f. What will it cost to paint the ceiling at 25¢ per sq. ft.? $100
g. What is the total cost of painting the walls and ceiling? $305

25. The water tank in Clearville is cylindrical, 18′ in diameter, and 25′ high. How many gallons of paint will it take to paint the lateral surface, if one gallon of paint covers 250 sq. ft.? Count a fraction of a gallon as a whole gallon. 6 gal.

26. How many square feet of sheet metal will it take to make a gasoline tank 8′ high and 4′ in diameter, closed on both ends? 126 sq. ft.

27. Which has the greater capacity, a cylinder 7 feet high with a diameter of 20 feet or a rectangular solid 6 feet high with base dimensions of 20 feet by 20 feet? Let $\pi \approx 3\frac{1}{7}$. How many gallons will they hold? cylinder, 2200 cu. ft., 16,500 gal.; rectangular solid, 2400 cu. ft., 18,000 gal.

28. How many gallons will a cylindrical gas tank hold if it has a diameter of 14 inches and a height of 36 inches? 24 gal.

STEPS FOR SOLVING APPLIED PROBLEMS

1. Understand the problem.	**2.** Note what the problem asks for.	**3.** Look for hidden questions.
6. Check your answer.	**5.** Set up and solve the conditional sentence(s).	**4.** Estimate a reasonable answer.

1. A contractor agrees to excavate a basement 24′ long, 18′ wide, and 12′ deep at \$5 a cubic yard. What will it cost? *\$ 960*

2. A triangular prism has a base which is a right triangle, with sides 10″ and 24″ and hypotenuse 26″. The prism is 3′ high. What is its volume in cubic inches? *4320 cu. in.*

3. A tin can is 6″ high. It has a diameter of 4″. How much more or less will it hold than two cans each 6″ high with bases each 3″ in diameter? *9.42 cu. in. less*

4. A milk carton is 4″ square on the base. How high, to the nearest $\frac{1}{8}$″, must it be to hold half a gallon? *$7\frac{2}{8}$″*

5. Mr. Jones built a cylindrical tank in the field to store his grain until he could ship it to market. It is 8′ high and 12′ in diameter. How many bushels will it hold? *723.5 bu.*

6. The city water tank is 20′ in diameter. It is 25′ high. How many gallons will it hold? *58,875 gal.*

7. A coal bin 12′ long, 6′ wide is filled with hard coal to a depth of 5′. A ton of hard coal is about 35 cu. ft. How many tons of coal are in the bin? *10.3 tons*

8. A freight car is to be loaded with 1000 bushels of grain. The floor of the car is 36′ × 8.5′. How many feet high will the grain be loaded? *4.1 ft.*

9. Jim has a pail whose base has a diameter of 1 foot. He wishes to place marks on the sides to measure gallons. How far apart, to the nearest $\frac{1}{4}$″, should the marks be? *2.0″*

10. Mike is making a flower box in the form of a rectangular prism with an open top. The base is 5″ × 12″. It is to be 4″ high. How many square inches of sheet copper will Mike need, not allowing for waste? *196 sq. in.*

Problems on volume and capacity contain many hidden questions. You may need to change the units of volume or capacity, or you may need to calculate volume or capacity, then calculate the cost. You will find several types of hidden questions in these problems.

In each exercise, first state the hidden question, or questions, then answer them, in solving the problem.

1. Mr. Ellsworth is a contractor. He has been asked to estimate the cost of excavating a basement 40′ × 35′ × 12′. He figures that he should charge \$3.50 per cubic yard. What should his estimate be? *\$ 2177.70*
2. Mr. Henderson has a grain bin whose floor measures 12′ × 6′. It is filled with wheat to a depth of 4′. Wheat is selling at \$2.10 per bushel. How much is the wheat worth? *\$ 483.84*
3. A gasoline tank is 4′ in diameter and 12′ high. It is filled with gasoline. What is the gasoline worth at 30¢ a gallon? *\$ 339.12*
4. Water weighs 62.4 lb. per cu. ft. Steel weighs 0.3 lb. per cubic inch. Steel weighs how many times as much as water? *8.3*
5. One thousand gallons of water are discharged into a fish pond 6′ in diameter. How deep will the water be? *4.72′*
6. Mary plans to paper the walls and ceiling of her room, which is 10′ × 13′ × 8.5′ high. The paper costs 15¢ a square foot. How much will the paper for her room cost? *\$ 63.15*
7. Mr. Smith plans to put 9″ of gravel on his drive which is 90′ long and 12′ wide. What will the gravel cost at \$5 a cubic yard? *\$ 150*
8. At 50¢ a square yard, how much will it cost to paint the curved surface of a cylindrical tank that is 40′ high and 12′ in diameter? *\$ 83.73*
9. A lawn roller is 2′ in diameter and 30″ long. How many square feet does it cover in five revolutions? *78.5 sq. ft.*
10. The roof of a tunnel is shaped like a half cylinder. Its diameter is 40′, and it is $\frac{1}{2}$ mile long. How much would it cost to paint the inside of the roof of the tunnel at 40¢ a square yard? *\$ 7368.53*
11. Jim put a block of wood on the turning lathe. The block was 5″ square and 9″ long. He turned it into a cylinder 3″ in diameter and 9″ long. How many cubic inches of wood did he cut away? *161.4 cu. in.*
12. The weight of ice is .92 times as much as the weight of an equal volume of water. What is the weight of a cube of ice 10″ on each edge? *33.3 lb.*

STEPS FOR SOLVING MATHEMATICAL PROBLEMS

1. Understand the problem.
2. Analyze the data.
3. Discover new facts.
4. Follow up and verify promising leads.
5. Review your solution.

1. How can you measure out exactly 7 quarts of water, using only a 5-quart and an 8-quart pail?
HINT: If you start by filling the 8-quart pail and filling the 5-quart pail from it, the remainders in each step will look like this:

STEP	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
Remaining in 8-quart pail	3	0	6	1	0			
5-quart pail	5	3	5	5				

In Step 1 the 5-quart pail was filled from the 8-quart pail, leaving 3 quarts in the latter. The 5-quart pail was then emptied, and in Step 2 the 3 quarts from the 8-quart pail were poured into the 5-quart pail, leaving the 8-quart pail empty. What happened in Step 3? Continue the process. *See front.*

2. Suppose you reverse the operation, filling the 8-quart pail from the 5-quart pail, until 2 quarts are left in the 8-quart pail. How many steps does it take? List them. *See front.*

3. Note that in the table for Exercise 1 you are actually seeking a multiple of 8 that, when divided by 5, leaves a remainder of 4. Can you explain why this is true? *See front.*

4. Explain, in a similar manner, the mathematical relationship revealed in the table in Exercise 2. *See front.*

5. Using a 3-quart measure and a 5-quart measure, explain, with a table, how you could measure out 7 quarts of water. *See front.*

6. With a 7-quart pail and a 4-quart pail, how can you measure out 5 quarts of water? *See front.*

7. Explain why you could not measure 5 quarts of water using a 9-quart and a 6-quart measure. Make up 3 similar puzzles. Solve.
There is always a differential of 3 quarts.

In a secret message letters may be replaced by numbers to form a cipher. In the Example we replace some of the digits with letters. You can readily restore the missing digits by studying the number relations and finding clues.

EXAMPLE

Clue 1: Since the last digit in the answer is 0, the second digit in the multiplier is 5. (Why can't it be 0?) Then what digit is E?

Clue 2: Then the first digit in the multiplicand must be 3, and the partial product when multiplied is 1880.

Clue 3: Then the last two digits in the second partial product must be 04, to give the answer ending in 920. That means that the first digit in the multiplier is 4.

So the exercise was $\begin{array}{r} 376 \\ \underline{45} \end{array}$ Check the answer.

$$\begin{array}{r} A\,7\,6 \\ \underline{S\,D} \\ 1\,8\,Q\,E \\ \underline{H\,D\,E\,S} \\ H\,S\,9\,2\,0 \end{array} \qquad \begin{array}{r} 376 \\ \underline{45} \\ 1880 \\ \underline{1504} \\ 16{,}920 \end{array}$$

1. Here is another. It is easier than it looks. First find the clues.

Clue 1: When the last digit in the multiplicand was multiplied by 5 there was 4 to carry. Then it must be 8 or 9.

Clue 2: When it was multiplied by 2 the product was 18. Why? Then is Q, 4 or 9? Complete the work.

$$\begin{array}{r} 489 \\ \underline{25} \\ 2445 \\ \underline{978} \\ 12{,}225 \end{array} \qquad \begin{array}{r} Q\,8\,A \\ \underline{2\,5} \\ 2\,4\,4\,W \\ \underline{A\,7\,8} \\ 1\,F\,2\,2\,W \end{array}$$

2.
$$\begin{array}{r} 2\,E\,9 \\ \underline{G\,E} \\ Q\,A\,Q \\ \underline{A\,A\,H\,G} \\ A\,S\,N\,N\,Q \end{array} \qquad \begin{array}{r} 239 \\ \underline{53} \\ 717 \\ \underline{1195} \\ 12{,}667 \end{array}$$

4.
$$\begin{array}{r} 2\,9\,4 \\ \underline{8\,E} \\ 2\,3\,G\,S \\ \underline{S\,D\,G\,S} \\ S\,G\,E\,7\,2 \end{array} \qquad \begin{array}{r} 294 \\ \underline{88} \\ 2352 \\ \underline{2352} \\ 25{,}872 \end{array}$$

6.
$$\begin{array}{r} 9\,3\,3 \\ \underline{E\,Q} \\ G\,W\,6\,5 \\ \underline{H\,G\,W\,G} \\ H\,D\,F\,0\,5 \end{array} \qquad \begin{array}{r} 933 \\ \underline{85} \\ 4665 \\ \underline{7464} \\ 79{,}305 \end{array}$$

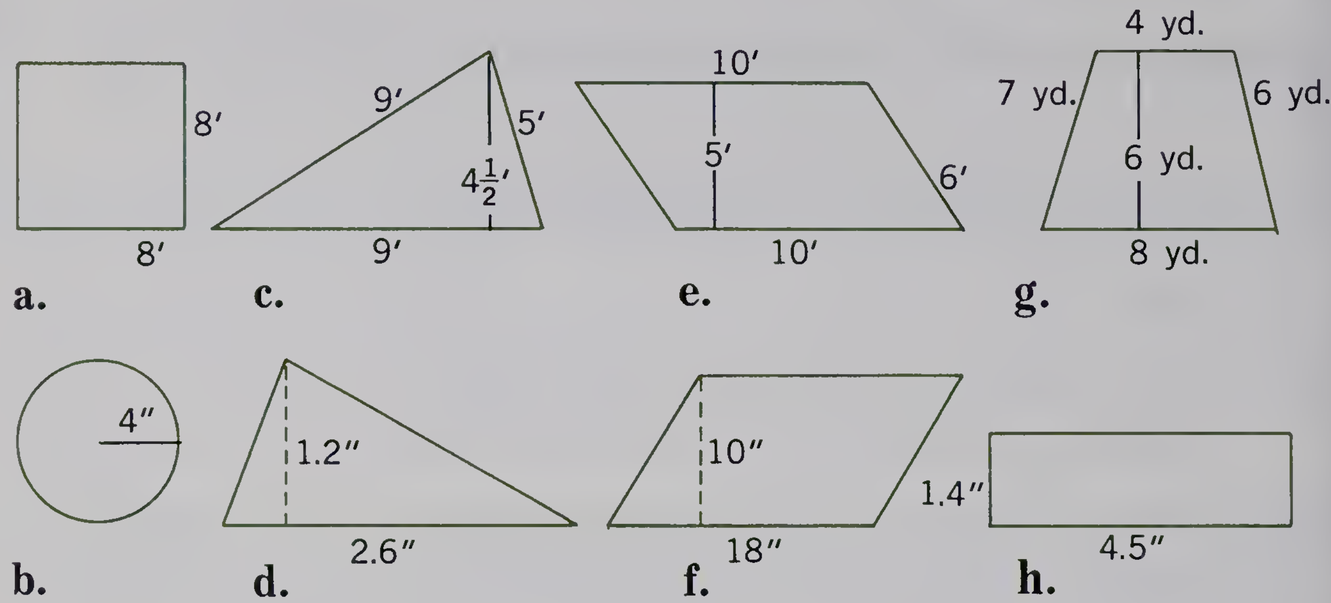
3.
$$\begin{array}{r} 3\,1\,H \\ \underline{Q\,A\,S} \\ 2\,5\,Q\,F \\ Q\,Q\,F\,H \\ \underline{6\,W\,F} \\ S\,A\,H\,A\,0 \end{array} \qquad \begin{array}{r} 315 \\ \underline{278} \\ 2520 \\ 2205 \\ \underline{630} \\ 87{,}570 \end{array}$$

5.
$$\begin{array}{r} E\,0\,7 \\ \underline{S\,G} \\ I\,F\,S\,3 \\ \underline{W\,2\,4\,E} \\ W\,J\,E\,F\,3 \end{array} \qquad \begin{array}{r} 207 \\ \underline{69} \\ 1863 \\ \underline{1242} \\ 14{,}283 \end{array}$$

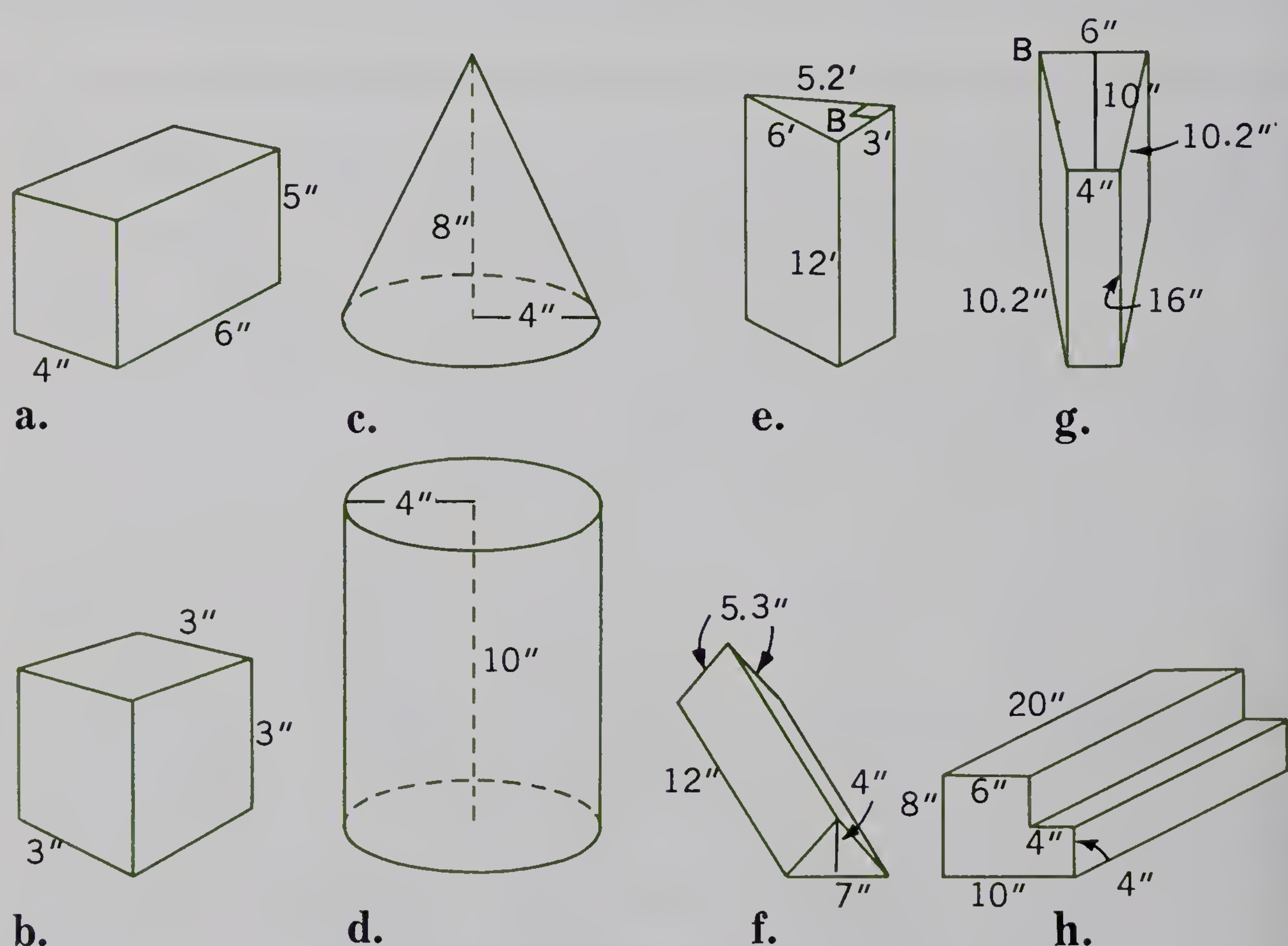
7.
$$\begin{array}{r} R\,S\,T \\ \underline{S\,2} \\ 8\,M\,8 \\ \underline{R\,S\,T} \\ 5\,P\,D\,8 \end{array} \qquad \begin{array}{r} 419 \\ \underline{12} \\ 838 \\ \underline{419} \\ 5028 \end{array}$$

Part One

1. For each of the following figures: *See front.*
A. Write the formula for finding its area. B. Calculate its area.



2. For each of the figures below: A. Write a formula for finding its volume. B. Calculate the volume. *See front.*



3. For each of the figures above except the cone: A. Write the formula for finding the total surface area. B. Calculate the total surface area. *See front.*

Part Two

List the numerals 1 through 16 on a sheet of paper. Then read each of the following statements carefully. If it is always true exactly as stated, write + before the number of the statement on your paper. If it is not true, write 0.

1. A cube has six faces. T
2. The surface area of a pyramid is the same as its lateral area. 0
3. A triangular pyramid has four triangular surfaces. T
4. To find the surface area of a cylinder, you can use the formula: $A = 2\pi r(h + r)$ T
5. The volume of a sphere is $\frac{1}{3}$ the volume of a cylinder whose base has the same radius. 0
6. If you double the height of a cylindrical container, you double its capacity. T
7. If you double the diameter of a cylindrical container, you double its volume. 0
8. If one sphere has twice the diameter of another, its volume is four times the volume of the other. 0
9. The number of bushels a bin will hold is about 0.8 times the number of cubic feet in its capacity. T
10. A tank with a capacity of 132 cu. ft. will hold exactly a ton of water. 0
11. The surface area of a cylinder with a given volume is the same as the surface area of a rectangular prism with the same volume. 0
12. A cube is a prism. T
13. The surface area of a sphere is less than the surface of any other figure with the same volume. T
14. A pentagonal prism has one more face than a hexagonal prism. 0
15. A pyramid is a prism. 0
16. A tank with a capacity of 1000 cubic feet will hold about 7500 gallons of water. T
17. Doubling the dimensions of a cube will increase the volume to twice what it was originally. 0
18. The total surface area of a cube is increased by 4 times its original area by doubling the dimensions. 0
19. A 2 cubic foot container will hold 15 gallons when filled. T

STEPS FOR SOLVING APPLIED PROBLEMS

1. Understand the problem.	2. Note what the problem asks for.	3. Look for hidden questions.
6. Check your answer.	5. Set up and solve the conditional sentence(s).	4. Estimate a reasonable answer.

Part Three

1. What is the volume of a rectangular prism that measures 18 inches long, 6 inches wide, and 5 inches high? *540 cu.in.*
2. What is the capacity, in bushels, of a bin that is 14 feet long, 8 feet wide, and 10 feet high? *896 bu.*
3. How many cubic feet in $12\frac{2}{3}$ cubic yards? *342 cu. ft.*
4. A hexagonal prism has a base area of 162 square inches and a height of 21 inches. What is the volume of the prism? *3402 cu. in.*
5. Sunshine cookies are packaged in boxes measuring $6'' \times 3'' \times 3''$. How many packages of cookies can be packed in a carton $2\frac{1}{2}$ feet long, $1\frac{1}{2}$ feet wide, and $1\frac{1}{2}$ feet high? *180*
6. A cylindrical water tank has a radius of 2 feet and stands 3 feet high. How many gallons of water will the tank hold? *282.6 gal.*
7. How many bushels are in a conical pile of grain that has a diameter of 15 feet and a height of 12 feet? *565.2 bu.*
8. What is the area of walls and ceiling in a closet that measures 9 feet long, 4 feet wide, and 8 feet high? *244 sq. ft.*
9. What is the total surface area of an oxygen tank shaped as a cylinder that has a radius of 6 inches and a height of 36 inches? *1582.6 sq.in.*
10. What will it cost to paint the lateral surface of a water tank in the shape of a cylinder that has a diameter of 16 feet and a height of 24 feet? Cost of painting is 28¢ per square foot. *$ 337.61*
11. A conical pile of hard coal is 9 ft. high with a 6 ft. radius. If 35 cubic feet weigh one ton, how many tons are in the pile? *9.7 tons*
12. How many bushels of corn are there in a rectangular bin that is 12 feet long, 10 feet wide, and 10 feet high? *960 bu.*
13. Double the dimensions in Exercise 12; find how many bushels. *7680 bu.*

CHAPTER EIGHT

CONSTRUCTION

WORDS TO WATCH FOR

bid	*estimate*	*scale drawing*
board feet	*scale*	*specifications*

Certain mathematical skills are needed in construction projects. If you plan to enter the construction industry or if you ever become involved in planning, building, or remodeling a home, you will find application for all the mathematics you have learned. Let us follow a typical project and see what problem-solving situations arise.

There were five members of the Clark family. The three children were growing up, and it was becoming increasingly clear that more room was needed in the house. One evening Mr. Clark said, "Why don't we remodel our garage into a combination family and rumpus room? We can build a carport to shelter the car. We can have fun planning together, and doing most of the work ourselves, we can save a great deal of money." All agreed enthusiastically that it would be an exciting family project.

"The first job," continued Mr. Clark, "is to collect some information." We need to know how much flooring to buy and how much paneling is necessary for the walls. Measurements for these materials must be carefully taken and recorded if they are to be correct and useful." Mr. Clark drew a sketch of the garage, indicating dimensions needed, and showing where windows, doors, and a fireplace should be placed.

Jerry and Jim, the two boys, had to take the measurements and calculate the areas. Using a steel tape, they determined the floor measurements to be 18 feet in width and 20 feet in length.

1. They planned to cover the cement floor with tile that comes in 8-inch squares. How many pieces are needed to cover the floor? 810
2. The east wall is 18 feet in width, and the room is to be 8 feet in height. The east wall has no windows. What is the wall's area in square feet? 144 sq. ft.
3. The fireplace planned for the east wall is to be built of brick, 4 feet in width, and extend up to the ceiling. How many square feet of wall space will it occupy? 32 sq. ft.
4. Except for the fireplace the east wall will be paneled. How many square feet of paneling will be needed? 112 sq. ft.
5. Two windows are planned for the south wall. Each window is to be 3 feet in width and 4 feet high. How many square feet will the windows take up? 24 sq. ft.
6. The remainder of the south wall will be covered with sheet rock. How many square feet of sheet rock will be needed, if the south wall is 20 feet in length? 136 sq. ft.
7. The north wall, adjoining the kitchen, is also 20 feet in length, and 8 feet high. They planned to have a folding door leading to the kitchen in this wall. This door will be 3 feet wide and $6\frac{1}{2}$ feet high. A window measuring 3 feet square is also planned. How many square feet would be occupied by the door and window? 28.5 sq. ft.
8. How many square feet of sheet rock is needed for the north wall? 131.5 sq. ft.
9. The west wall was to have no doors or windows, but was to be covered with paneling. How many square feet of paneling would be needed? 144 sq. ft.
10. Prepare a drawing to represent this room and save the information to help you work out the next section.

COMMITTEE PROJECT

Suppose the Clarks planned to use the west wall for book shelves and built-in furniture.

1. Prepare a scale drawing of the west wall, including space for books, record player, television, and other equipment needed in a rumpus room.
2. Construct a large model or plans of the Clarks' project. Show windows, doors, and fireplace.
3. Organize committees to:
   (a) plan the furniture in the Clarks' new room, (b) plan the interior decorating, and (c) plan the lighting arrangement in the room.

Mrs. Clark and the Clarks' daughter, Susie, decided to check prices on the various materials for the family room. Susie listed the parts of the room and the kinds of materials that might be used. She and her mother then took the list with them to check prices. *360 sq. ft.*

1. Their first stop was at a linoleum store to price flooring materials. How many square feet of flooring was needed for the $18' \times 20'$ floor?

2. They priced various kinds of tile for the family to consider for the floor. The prices per square foot were:
   asphalt tile, 28¢; cork tile, 55¢; and asbestos tile, 31¢.
   Compare the cost of covering the floor with each type of material.
   $100.80; $198; $111.60

3. Mrs. Clark and Susie next went to a building materials company to find the cost of materials for the fireplace. The clerk gave them the following prices: *new, $30; used, $42; tile, $93.60*
   new brick, 5¢ each; used brick, 7¢ each; and tile brick, 13¢ each.
   Mr. Clark estimated that they would need 600 bricks or 720 tile bricks. How much more would it cost to build the fireplace with the tile brick than with the new brick? *$63.60*

4. They priced two types of paneling for the two walls. Mahogany sold for 29¢ per square foot and knotty pine sold for 31¢ per square foot. Compare costs of paneling for the east and west wall. Allow four extra square feet for cutting, etc. *mahogany, $75.40; knotty pine, $80.60*

5. Mrs. Clark found the total cost of sheet rock would be $10.92. The necessary tape, nails, and paint would cost $11.75 more. A contractor offered to plaster and paint the walls for $40.00. How much would they save by using the sheet rock and doing the job themselves? *$17.33*

6. For other fixtures and materials, Mrs. Clark and Susie visited a lumber company. Here they found the sash for their three windows would cost $20.36. The folding door would cost $28.95. Venetian blinds for the windows would cost $8.40 each for the two 3×4 windows and $7.00 for the 3×3 window. What would be the total bill for these items? *$73.11*

7. The ceiling has the same measurements as the floor. How many square feet are there in its area? (See Exercise 1 above.) *360 sq. ft.*

8. To cover the ceiling of their room, the clerk at the lumber yard suggested accoustical tile at 18¢ per square foot or plain ceiling tile for $13\frac{1}{2}$¢ per square foot. Compare the total cost of each of these types of tile. *accoustical tile, $64.80; plain tile, $48.60; plain tile, $16.20 less*

9. At a family meeting that night, the Clarks decided to use

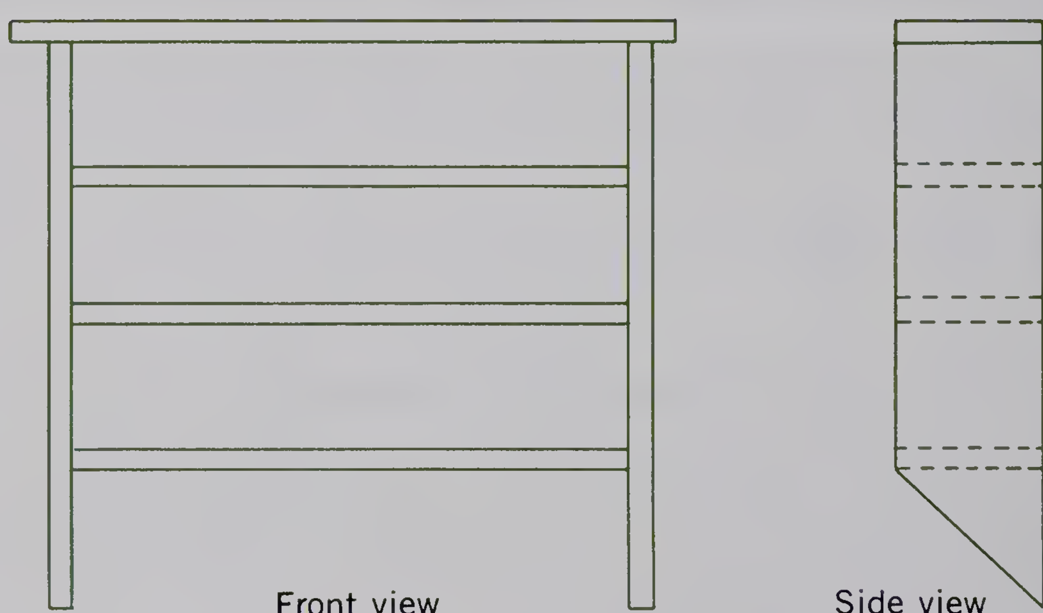
For the floor	Cork tile	$ 198.00
For the fireplace	New brick	30.00
For the paneled walls	Mahogany paneling	75.40
For the other walls	Sheet rock, painted	22.67
For the ceiling	Plain ceiling tile	48.60
For other fixtures and materials (Exercise 6)		73.11
		$ 447.78

What is the total cost of these materials? (You will need to go back to Exercises 1 through 8 for these figures.)

Anyone who is planning to build a house, an automobile, or any other object of many parts has to have some sort of a plan to follow. Without such a plan, it would be very difficult to determine quantities needed and to predict the cost. Without a plan, the final product might end up quite different from what was intended.

For proper planning, architects and designers make *scale drawings* of a project. The scale of a drawing or blueprint tells what fraction the length of a line on a drawing is of that part on the object. If an architect, in planning a home, made a drawing whose dimensions were exactly $\frac{1}{64}$ of actual size to be built, he would say that the scale of his drawing was $\frac{1}{64}$ or 1 to 64.

Examine Jerry's plan for making a bookshelf to be fastened on the wall of his study. The scale is $\frac{1}{16}$ of an inch = 1 inch.



Bookshelf

10. Use your ruler to find the answers to these questions:
a. How long is the top? **b.** How long is each shelf? **c.** How wide is the board used for each shelf? **d.** Are the distances between shelves all the same? **e.** What would be the dimensions of a board large enough to make one of the sides? a. 36"; b. 30"; c. 8"; d. yes; e. 8" X 31"

A. Find n:

1. $\frac{n}{6} = \frac{8}{24}$ 2

2. $\frac{15}{5} = \frac{n}{25}$ 75

3. $\frac{n}{6} = \frac{12}{3}$ 24

4. $\frac{15}{n} = \frac{21}{7}$ 5

5. $\frac{18}{32} = \frac{27}{n}$ 48

6. $\frac{2}{14} = \frac{n}{21}$ 3

7. $\frac{n}{16} = \frac{6}{4}$ 24

8. $\frac{75}{n} = \frac{36}{12}$ 25

9. $\frac{n}{100} = \frac{5}{8}$ 62.5

10. $\frac{16}{n} = \frac{4}{5}$ 20

11. $\frac{7}{9} = \frac{11}{n}$ $14\frac{1}{7}$

12. $\frac{n}{100} = \frac{9}{16}$ $56\frac{1}{4}$

13. $\frac{14}{26} = \frac{7}{n}$ 13

14. $\frac{13}{n} = \frac{1}{2}$ 26

15. $\frac{n}{100} = \frac{4}{25}$ 16

16. $\frac{5}{18} = \frac{n}{36}$ 10

B. *Multiply:*

1. $18 \times .45$ 8.1

2. 28×3.2 89.6

3. 1.56×5.1 7.956

4. 3.15×2.4 7.56

5. 15.5×1.16 17.98

6. 4.29×4.8 20.592

C. *Divide:*

1. $5.4 \div .18$ 30

2. $22.5 \div .075$ 300

3. $.144 \div 2.4$ .06

4. $3.9 \div .013$ 300

5. $1.029 \div 4.9$ .21

6. $33.6 \div 21$ 1.6

D. Copy and complete the following:

1. 15% of \$70 is n \$10.50

2. 3 is n% of 50 6

3. 1.8% of \$60 is n \$1.08

4. 15 is 75% of n 20

5. 125% of 50 is n 62.5

6. 12 is n% of 96 12.5

7. 400 is n% of 225 177.8

8. $\frac{4}{5}$% of 560 is n 4.48

9. 30% of n is 21 70

10. 28% of 150 is n 42

11. 12 is 150% of n 8

12. n is 120% of 60 72

13. 90% of n is 135 150

14. n is 0.3% of 250 .75

15. 56 is 175% of n 32

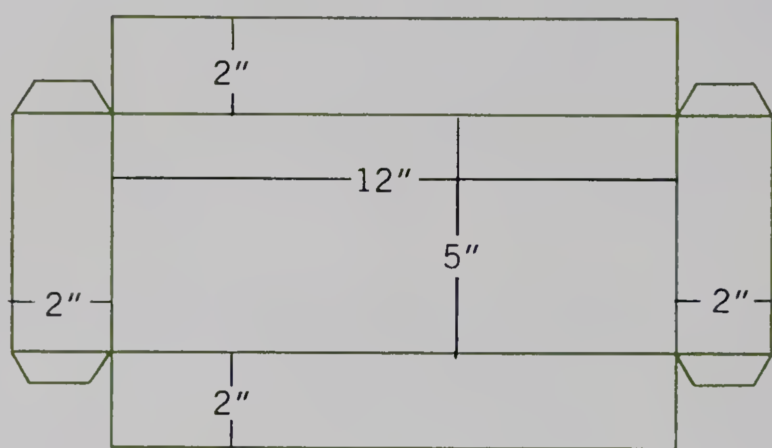
16. 1.7% of 80 is n 1.36

If you need more practice, turn to the Practice Exercises on page 495 and following. If not, you may work in the Experts' Corner on the following page.

Mathematics in the Sheet Metal Shop

People who work with sheet metal have to be able to make drawings and calculate areas. Before making an article, it is usually necessary to make a drawing or a sketch to show how it will look when it is cut out of sheet metal. Then the area can be calculated to determine how much material is needed. (Use $\pi \approx 3.14$)

1. Joe made a rectangular flower box, open at the top. It was 12″ long, 5″ wide, and 2″ deep. The drawing shows how Joe plans to make it of sheet copper. How large will the rectangular sheet be before Joe cuts out the pattern? 9″ X 16″



2. Martin decides to make a flower box similar to Joe's, but smaller. It is to be 5″ long, 3″ wide, and $2\frac{1}{2}$″ deep. He plans to make it of galvanized iron. Prepare a sketch like the one Joe made to show the pattern. Notice the $\frac{3}{8}$″ pieces on each edge of the ends which are for soldering the corners.

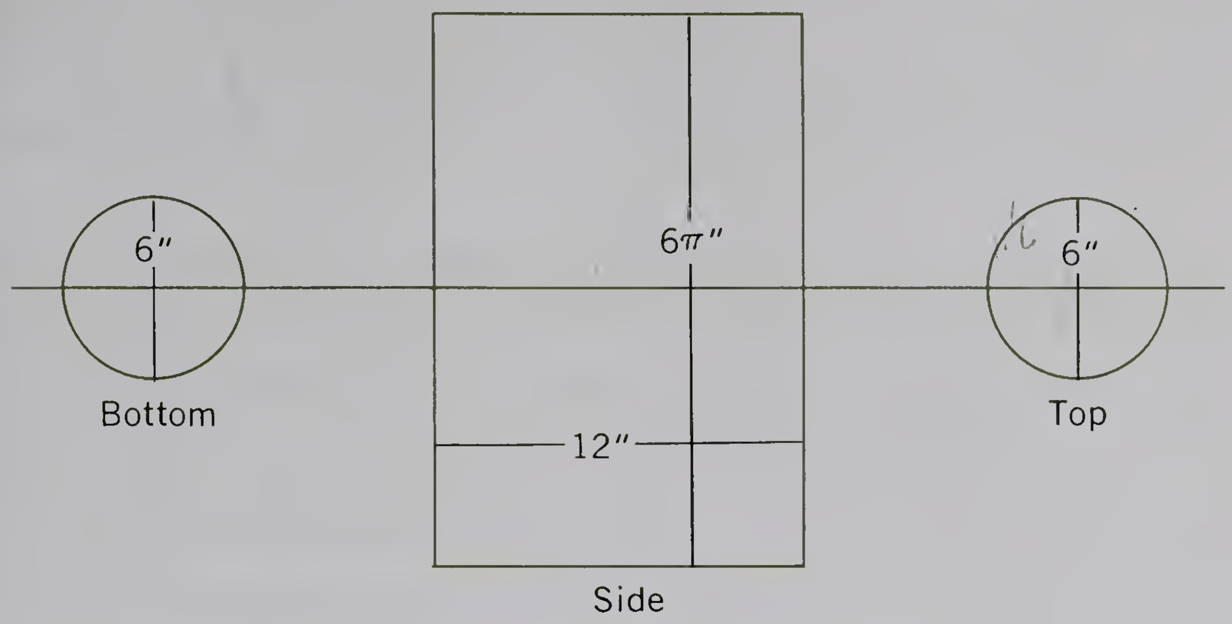
3. What are the dimensions of the rectangular piece of galvanized iron that Martin will need? How many square inches of sheet iron will this piece contain? 8″ X 10″; 80 sq. in.

4. Mr. Jensen needs a circular cover for the end of a pipe 12″ in diameter. How many square inches of galvanized iron will he use if the cover is cut from a 12″ square? How much will he waste? 113.04 sq. in.; 30.96 sq. in.

5. Mr. Jones is installing a ventilation pipe 9″ in diameter. He wishes to put steel bands around it for reinforcement. How long will each band be if he adds 2″ to the length of each band for fastening it? 30.26 in.

6. If he plans to use 12 bands, how many feet of steel band will Mr. Jones require? 30.26 ft.

7. Jim is making a tin cookie can, 12″ high and 6″ in diameter. He made a drawing to show how it would look when the top and bottom were cut out and before the side was curved into a cylinder. What are the dimensions of the side? 12″ x 18.84″



28.26 sq. in.; 28.26 sq. in.; 226.08 sq. in.

8. Calculate the area in square inches of the top, bottom, and side.

9. The formula for the total area of the surface of a cylinder is:

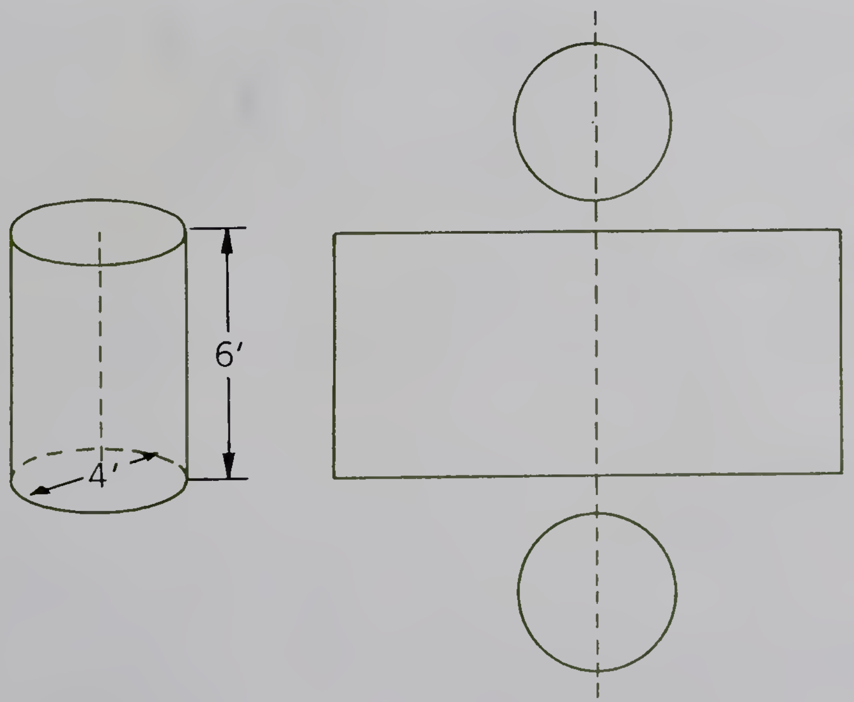
$2\pi r^2$, area of top and bottom

$$A = 2\pi r^2 + 2\pi rh, \text{ or } 2\pi r(r + h)$$ $2\pi rh$, area of side

Look at the three parts of the cylinder in Exercise 8 and explain where each part of the formula comes from.

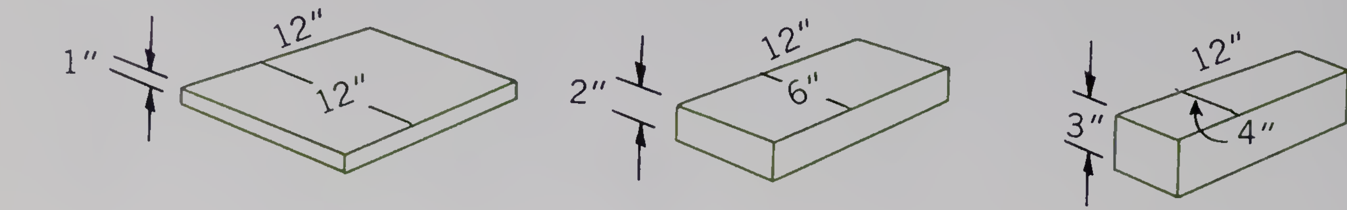
10. Use this formula to calculate the number of square feet of sheet iron used to make a barrel 3′ high and 2.5′ in diameter. 33.36 sq. ft.

11. Mr. Anderson is making an oil tank in the shape of a cylinder as indicated. How many square feet of sheet steel are in the two ends? 25.12 sq. ft.

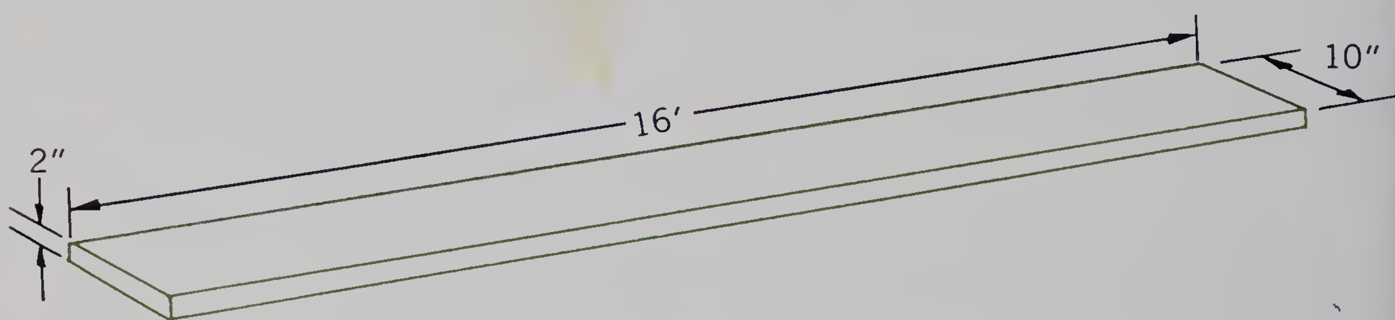


12. What is the total number of square feet of sheet steel in the tank? 100.48 sq. ft.

Ordinary lumber is usually sold at so much per board foot. A board foot of lumber is the amount of lumber contained in a board 1 foot long, 1 foot wide, and 1 inch thick. Here are some examples.



In ordering lumber the thickness, width, and length are given in that order. The thickness and width (called the cross-sectional dimensions) are given in inches, and the length of the board is given in feet. Thus, the board below is described as $2'' \times 10'' \times 16'$ and means a board $2''$ (2 inches) thick, $10''$ wide, and $16'$ (16 feet) long. To find the number of board feet in any board, divide the product of the thickness and width in inches by 12, and multiply the answer by the length of the board expressed in feet.

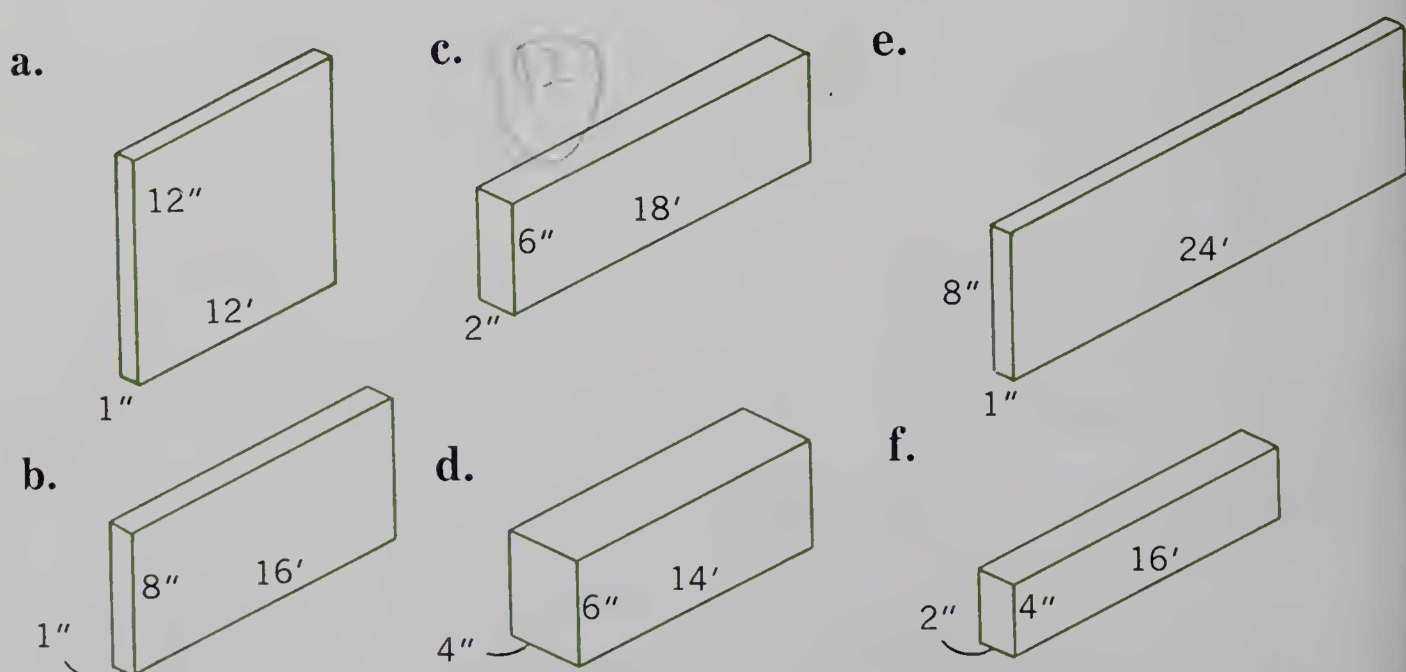


1. Verify that each piece of lumber above contains 1 bd. ft.

2. What is the cost of 120 bd. ft. of lumber at \$.12 per bd. ft.? *\$14.40*

3. Find the cost of 3500 bd. ft. of redwood siding at \$.24 per bd. ft. *\$840*

4. How many board feet are there in each of the boards shown below?

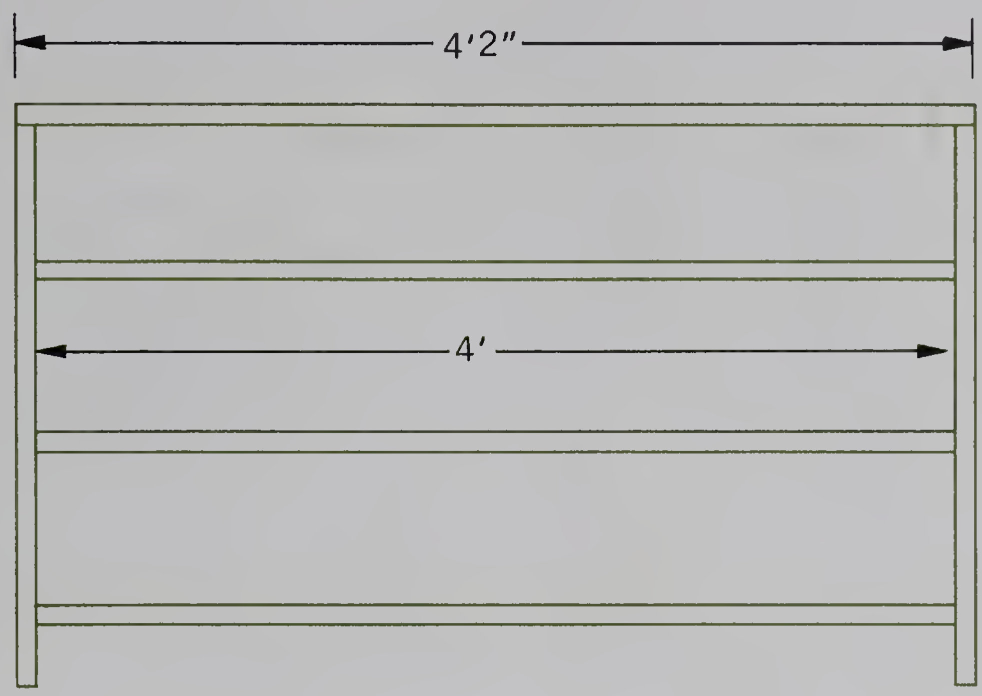


a. 12 bd. ft. b. $10\frac{2}{3}$ bd. ft. c. 18 bd. ft. d. 28 bd. ft. e. 16 bd. ft. f. $10\frac{2}{3}$ bd. ft.

GEORGE ADAMS BUYS LUMBER

George Adams decided to build some shelves for his room. He wanted to make the shelves from 1″ × 10″ lumber and use the same material for the sides. First he made a front view drawing of his project.

1. One inch represents how many feet on George's drawing? $1\frac{1}{3}$ ft.



2. How long are the two side pieces? 2′6″
3. What is the length of the three lower shelves? 4′
4. The lumber salesman told George that he could only sell lumber to the even foot. How long a board did George have to buy for the top shelf? 6′
5. How many board feet of 1″ × 10″ did George buy for the entire project? (Round off to the nearest board foot.) 20 bd. ft.
6. Large quantities of lumber are usually priced at so much per 1000 board feet. This is abbreviated as so much per M. If the price of lumber is $110 per M, what is the cost of 1 bd. ft.? $.11
7. What is the cost of 4500 bd. ft. of lumber priced at $90 per M? $405
8. Mr. Abrams bought 6000 bd. ft. of lumber for $720. What was the price per M? $120
9. What was George's cost for his lumber priced at $140 per M? $2.80
10. Mr. Donaldson purchased 9500 bd. ft. of lumber priced at $130 per M. What is the cost? $1235
11. If the price of lumber is $125 per M, what is the cost of 1 bd. ft.? $12\frac{1}{2}$¢
12. If 7000 bd. ft. of lumber cost $1050, what was the price per M? $150

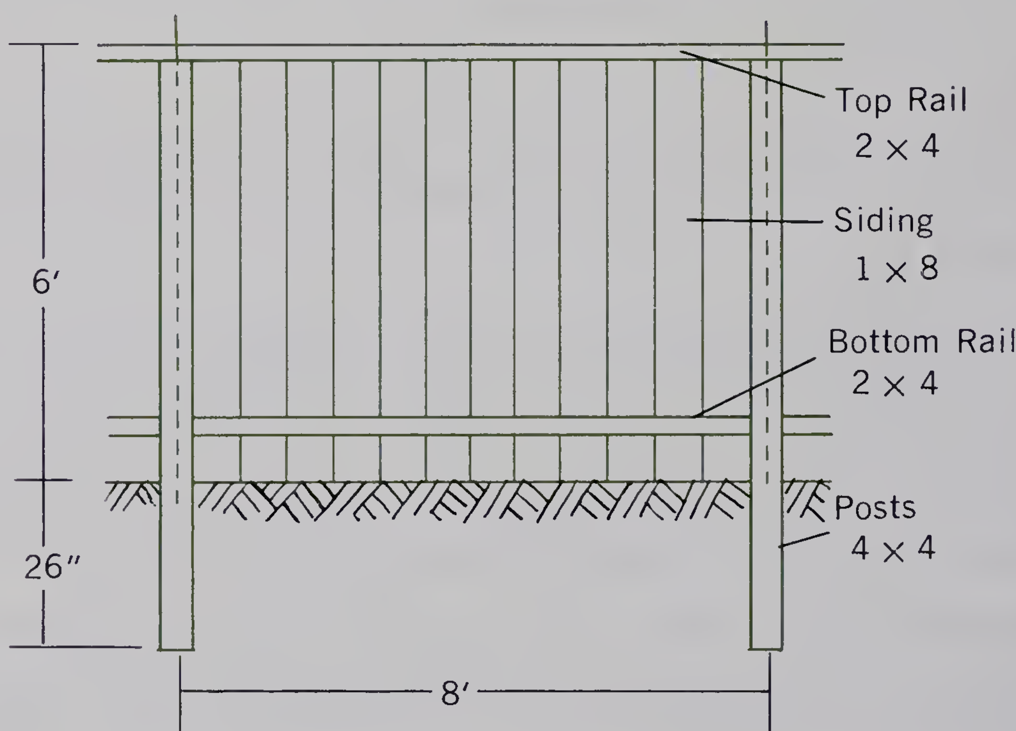
1. Mr. Adams needed to have a new fence built along the back of his yard. George wanted to earn some money, so he gave his father a price for which he would do the job. In other words he decided to *bid* on the job. His father gave him the following *specifications* for the fence:

- The fence shall be 96′ long.
- The fence shall be 6′ high.
- The fence shall have a bottom and top rail of 2″ × 4″.
- The posts shall be 8′ apart from center to center and be 4″ × 4″.
- The posts shall go into the ground 26″.
- The fence boards shall be 1″ × 8″ running vertically.
- All the lumber shall be rough redwood.

From these specifications George prepared the sketch shown below for his father's approval. Examine the diagram carefully.

How long must each post be to give George a fence 6 feet above the ground? 8′

For 96 feet of fence, how many posts will George need? 13 (Remember there must be a post every 8′, including one at each end.)



2. How many pieces of 2″ × 4″ × 8′ will George need for the top and bottom rails? 24

3. George decided to buy his 1″ × 8″ siding in 12-foot lengths. How many pieces of 1″ × 8″ × 12′ boards will the fence require? 72

4. George found that he could have the post holes dug for \$.75 each. What would his cost be for digging the holes at this price? \$ 9.75

5. George allowed \$1.80 in his bid for nails. How many pounds of nails could he buy for this amount at \$.12 per lb.? 15 lb.

6. If the posts cost \$140 per M, the siding cost \$160 per M, and the rails cost \$115 per M, what was the total cost of the lumber? \$126.29

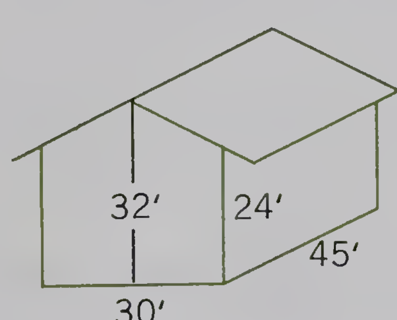
7. George decided that he would allow $\frac{3}{4}$ of the cost of the lumber for his labor charge in his bid. How much did he allow for his labor? \$94.72

8. To allow for unforeseen miscellaneous expenses, George decided to add 10% of all items, including labor, for "contingencies." How much did he allow for contingencies? \$23.26

9. Copy and complete this cost breakdown for George's bid

Item	*Cost*
Digging holes	9.75
Posts	19.41
Rails	14.72
Siding	92.16
Nails	1.80
Labor	94.72
TOTAL	232.56
Contingencies (10% of total)	23.26
AMOUNT OF BID	255.82

10. Which has the larger area, a triangle whose base is 14″ and whose altitude is 9″, or a square 8″ on a side? How much larger? square; 1 sq. in.



11. Mr. Mason is planning to paint a barn, shown above. How many square feet of surface is on each end of the barn? 840 sq. ft.

12. How many square feet are there on the two sides (total)? 2160 sq. ft.

13. A gallon of paint will cover 250 square feet. How much will the paint cost at \$4 a gallon? (Count a fraction of a gallon as a gallon.) \$64

14. Jim plans to paint the walls and ceiling of a room, 9′ × 13′ and 8′ high. A gallon of paint will cover 275 square feet of surface, and costs \$5 a gallon. How much will the paint cost? (Disregard doors and windows, and count a fraction of a gallon as a gallon.) \$10

Very often a home owner uses the basement in his home for storage, for the furnace, for a workshop, or for a recreation room. The floor and sides of a basement are usually concrete. The cost of a basement also includes the expense of excavating the dirt where the basement is to be located. The charge for this work is usually based on a certain price per cubic yard.

1. How many cubic yards of dirt must be excavated for a basement 24 feet long, 18 feet wide, and 6 feet deep? (You can either change the dimensions to yards, or figure volume in cubic feet; then divide by 27. Why 27?) 96 cu. yd. ; 1 cu. yd. = 27 cu. ft.

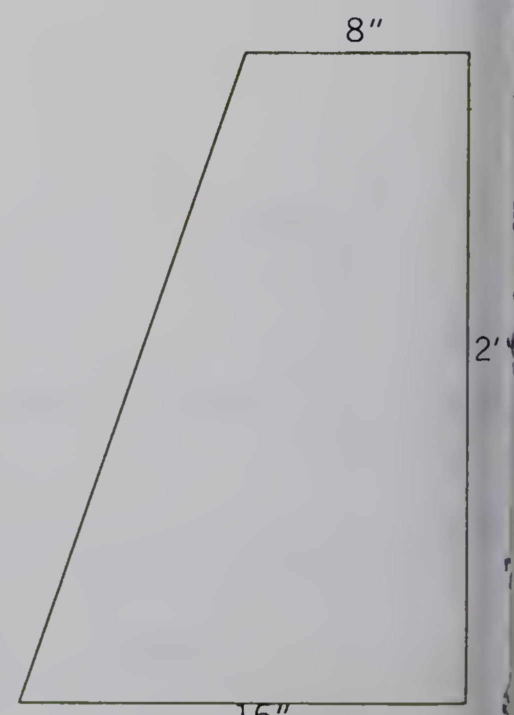
2. What would the excavation cost in Exercise 1 at $3.10 per cubic yard? $ 297.60

3. Even when no basement is needed, some excavation is required to level off the ground for building. What is the charge for leveling off an 1800 sq. foot floor area of a house to an average depth of 6 inches at $2.80 per cubic yard? $ 93.33

4. Mrs. Ralston wanted new topsoil for her front lawn. How many cubic yards did she need to cover the area 60′ × 36′ with 6″ of new topsoil? 40 cu. yd.

5. The foundation of the Jensen residence is 2′ thick and 2′ deep. The foundation is 180 feet long. What is the cost of the concrete for this foundation at $17.20 per cubic yard? (Here it will be easiest to find cubic feet first; then divide.) $ 458.67

6. The foundation of the Williams' residence is thicker at the bottom than at the top, as shown on the right. What is the area of the end shown in the figure, in square feet? 2 sq. ft.



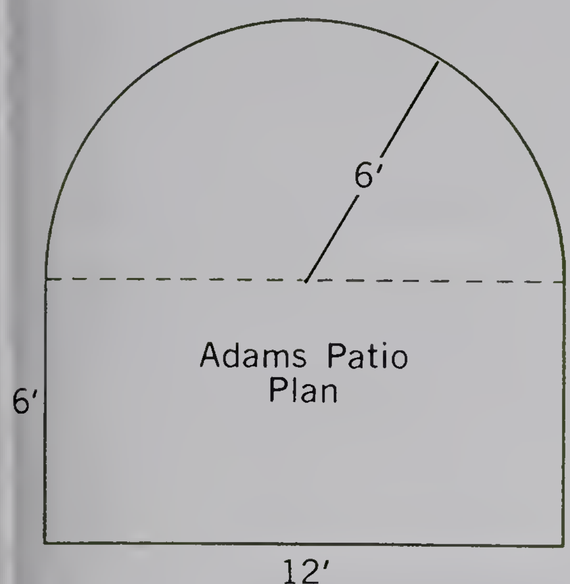
7. How many cubic yards of concrete were needed for 219 feet of foundation in the Williams residence? $16\frac{2}{9}$ cu. yd.

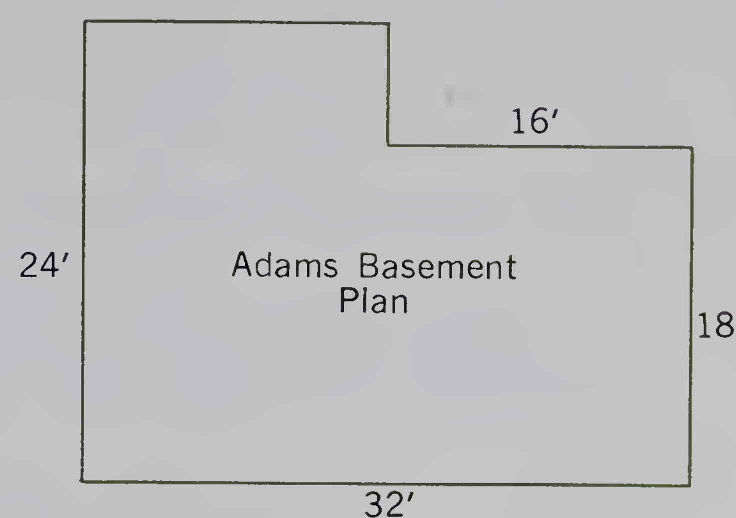
8. At $16.75 per cubic yard, how much would the concrete in Exercise 7 cost? $ 271.72

9. Six inches of gravel fill had to be placed under a driveway 10 feet wide and 36 feet long. What did the gravel cost at $4.50 per cubic yard? $ 30

10. The basement of the Adams house was excavated to a depth of 7 feet. How many cubic yards of dirt were removed? See the figure below on the right. $174\frac{2}{9}$ cu. yd.

11. The cost of a concrete floor is priced at so much per square foot, including concrete and labor. How much would the basement floor in the Adams house cost at \$.38 per square foot? \$ 255.36





12. The Adams family decided to add a concrete patio to their home. The shape and dimensions are shown above. What did the patio cost at \$.48 per square foot? \$ 61.69

13. Mr. Harold installed a concrete garage floor in Mr. Adams' garage. It was 20′ wide by 24′ long. The floor was 4″ thick. He bought the concrete at \$16.80 per cubic yard and paid a man \$.12 a square foot to finish it. What did it cost Mr. Harold to install the garage floor? \$ 157.16

14. What was Mr. Harold's cost per square foot for the garage floor in Exercise 13? (Answer to nearest cent.) \$ .33

15. Mr. Adams paid Mr. Harold 50¢ per square foot for the garage floor. What was Mr. Harold's profit on the job? \$82.84

16. Mr. Adams had a driveway laid from the sidewalk to the garage, a distance of 24 feet. The driveway was 9 feet wide. Before laying the concrete, Mr. Adams excavated the drive to a depth of 1 foot. What was the cost of excavation at \$3 a cubic yard? \$ 24

17. A layer of gravel 8 inches deep was laid in the driveway as a base for the concrete. What did the gravel cost at \$5 a cubic yard? \$ 26.67

18. What did it cost for the cement, 4 inches thick, at 50¢ a square foot? \$108

Painters and plasterers figure their prices according to the area they have to cover. They usually ignore the openings at doors and windows because of the extra time required to trim around these openings.

1. Mr. Jenkins is figuring the cost of painting the walls and ceiling of a room 12 feet wide, 18 feet long, and 8 feet high. How many square feet are there to paint? *696 sq. ft.*

2. If a gallon of paint covers 250 square feet of surface with two coats of paint, to the nearest gallon, how many gallons of paint will Mr. Jenkins need to paint the room in Exercise 1? *3.0 gal.*

3. At \$5.70 per gallon what will the paint in Exercise 2 cost? *\$17.10*

4. Mr. Jordan asked for an estimate for plastering the walls and ceiling of a room 21 feet long, 18 feet wide, and 8 feet high. A contractor quoted \$2.50 per square yard. What was his price? *\$ 278.33*

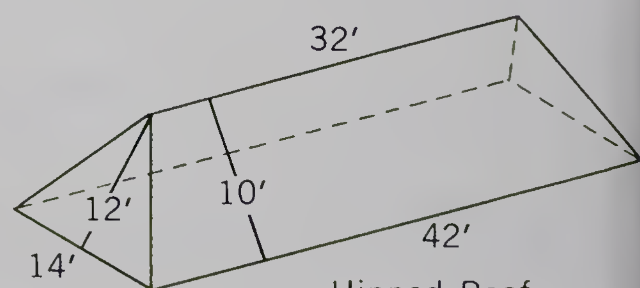
5. Another contractor said he would install plaster board and paint it for Mr. Jordan for \$.25 a square foot. Which is cheaper, plaster board or plaster for this room? How much cheaper? *\$ 27. 83* *plaster board*

6. Roofers usually figure their prices at so much a square (100 sq. ft.). How many squares of roof area are contained in a rectangular roof 60 ft. long and 30 ft. wide? *18*

7. The figure below on the left shows a gabled roof. Notice that a gabled roof consists of two rectangles. What is the area of this roof in square feet? in squares? *1200 sq. ft. ; 12 squares*

8. Find the cost of covering it with shingles at \$32 per square. *\$ 384*

Gabled Roof



Hipped Roof

9. Mr. Jordan is planning to put a new roof on his house. It is a "hipped roof." As you can see, this roof consists of two triangles and two trapezoids. How many squares of roofing will it take to the nearest square? *9*

10. What is the difference in price of covering this roof with shingles at \$28 a square or shakes at \$32 a square? *\$ 36*

Mr. Conley decided the walls on his storage shed needed painting. He wanted two coats of paint. George and Fred Conley agreed to submit a bid on this project to earn some money. They first drew sketches of the shed. In all their calculations they disregarded the doors and windows when figuring the areas.

1. To find the area of the north and south ends, they divided each into a rectangle and triangle. What is the area of each rectangle? *200 sq. ft.*

Mr. Conley's Shed

2. What is the total area of both ends? *520 sq. ft.* of both sides? *620 sq. ft.*
3. What is the area of all the outside walls? *1140 sq. ft.*
4. One gallon of paint covers 250 sq. ft. of siding with one coat. How many gallons will be required to cover the walls with one coat? (Round up to next highest number of gallons. Why?) *5 gal.* *Round up to have enough paint.*
5. The paint for the first coat cost \$3.80 per gallon, and for the final coat \$5.25 per gallon. What is the total cost of the paint? *\$ 45.25*
6. The boys figured they could apply 1 gallon of the first coat in five hours. How long would it take to apply the first coat? *22.8 hr.*
7. They figured they could apply 1 gallon of the second coat in 6 hours. How many hours did they figure for labor for the second coat? *27.36 hr.*
8. At \$1.85 per hour per boy, what should the boys charge for labor? *\$ 92.80*
9. In order to paint the higher parts, they had to rent ladders for \$5. The price for paint brushes was \$7.00. Prepare this statement:

Item	*Cost*
Labor: _?_ (*50.16*) hrs. at \$1.85 per hour	*\$ 92.80*
Paint: First coat _?_ (*5*) gal. at \$3.80 per gal.	*19.00*
Second coat _?_ (*5*) gal. at \$5.25 per gal.	*26.25*
Miscellaneous: Ladders and brushes	*12.00*
TOTAL	*\$ 150.05*

The cost of building a new house depends upon many things. Some sites are more expensive to build on than others. Why? Certainly the size and design of a house will affect the cost. The type of materials used will also make the cost vary.

The Donaldson family had agreed on a basic plan for their new home. Before they were able to get an estimate of the cost, they found it necessary to write up some specifications to show just what they wanted. This included plumbing fixtures, floor coverings, wall coverings, electric fixtures, roofing, and many other items. Since these items may vary considerably in price, they found it necessary to consider comparative costs very carefully. Study the floor plan the Donaldson family decided upon.

1. What is the total length of the house, from the outside wall of the master bedroom to the outside wall of the living room? *54 ft.*

2. What are the dimensions of the child's bedroom? 11' x 13'

3. There is a basement under the house. How can you tell this from the drawing? the stairs

4. What are the dimensions of the living room? 13' x 20'

5. There is a fireplace in the living room with a bookcase alongside it. How wide is this unit? 9 ft.

Here are some questions which the Donaldson family answered when working out their specifications. See if you can do the same. Figure all dimensions to the nearest foot.

1. Mrs. Donaldson chose oak plank flooring in her living room instead of oak strip flooring. This cost her $.53 per square foot. How much did the flooring in the living room cost? $137.80

2. The Donaldsons decided to use tile on their kitchen. They were told this would cost $2.25 per square foot. What is the cost of this tile? $180 (area under counter not tiled); $263.25 (entire floor)

3. The roof area totaled 3900 sq. ft. They decided on cedar shingles for their roof which cost them $28.00 per square (100 sq. ft.). What was the cost of the roof? $1092

4. What is the over-all width of the Donaldson home? 52 ft.

5. The building code in their city says that no house may cover over 72% of the width of the building lot. The Donaldson's lot is 110 ft. wide. What width would the code allow? Would they need to secure a variance to build their house on this lot? 79.2 ft., no

6. In order to prepare the site for the building, dirt had to be removed over an area of 3300 sq. ft. to an average depth of 1′6″. If it cost .85 per cubic yard to excavate this dirt, what was the cost of this item? $155.83

7. Mr. Donaldson had a friend in the lumber business who said he would sell him lumber at a 12% discount. If the regular price of the lumber was $2600, what did the Donaldson family save on this item? $312

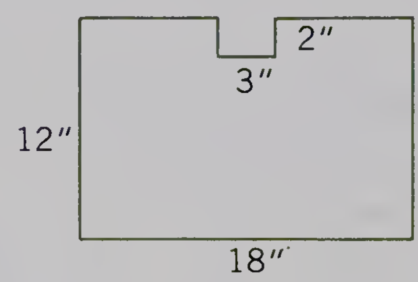
8. At $.38 per sq. ft., what was the cost of the cement floor in the garage? The dimensions are 24′ × 22′. $200.64

9. The Donaldsons wanted a basement for the furnace, water heater, and storage. How much would it cost them to excavate this basement at $4.10 per cubic yard if it was 15 ft. long, 9 ft. wide, and 8 ft. deep? $164

10. The total cost of a house is often estimated in advance at so much per square foot. The total area of the Donaldson house is 1980 sq. ft. The cost is estimated at $14.20 per sq. ft. What is the estimated cost? *$ 28,116*

11. Mr. Jensen, a building contractor, estimated the labor on their home would cost $4.10 per square foot. Using the area in Exercise 10, what will the labor charge be? *$ 8118*

12. The Donaldson house plans have footings with a cross section as shown in the figure below. The total length of the footings is 360 ft. How many cubic yards of concrete (to the nearest yard) will be required for them? *19*



13. At $15.70 per cubic yard, what will be the cost of the concrete in Exercise 12? *$ 298.30*

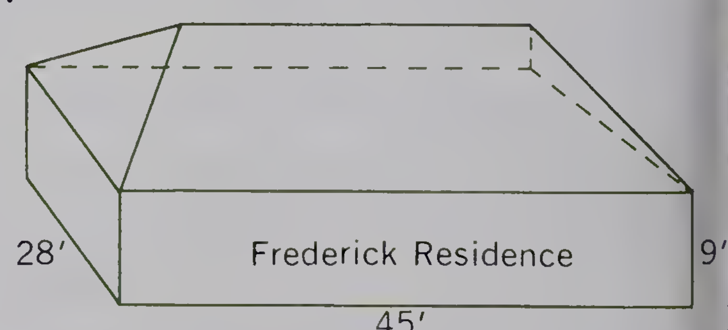
14. Mr. Jensen, a builder, told Mr. Donaldson that building costs were divided up about as follows:

$ 5510.74	Carpentry	19.6%
6972.77	Excavation, concrete, masonry	24.8%
4864.07	Lumber, millwork	17.3%
6775.96	Plumbing, heating, electrical	24.1%
3148.99	Hardware, painting, flooring	11.2%
843.48	Miscellaneous	3.0%
$ 28,116.01		100.0%

Make a circle graph of this cost distribution. *See front.*

15. Using your answer from Exercise 10, how much is estimated for each of the six items in Exercise 14?

16. The Frederick residence is 28 feet wide and 45 feet long. The exterior walls are 9 feet high. What is the area in square feet of the four walls? *1314 sq. ft.*



17. Calculate the cost of siding at 75¢ per square foot. *$ 985.50*

For each problem below, three possible answers are given. Read each problem carefully, and estimate which answer is correct, without using a pencil. Then on a sheet of paper, place the letter (a, b, or c) indicating the correct answer after the number of the problem. If you think none of the given answers is correct, place 0 after the number of the problem. Then work the problem and see how many of your estimates were correct.

1. George receives 20% commission for selling new home owners landscape service for one year. If his commission was $100, how much were his sales?

a. $20 **b.** $500 **c.** $2000

2. At the rate of 35 tiles per hour, how long would it take to place 420 tiles?

a. 12 hours **b.** 5 hours **c.** 24 hours

3. A painter must paint 18,750 sq. feet of wall. At 250 square feet an hour, how long did it take?

a. $7\frac{1}{2}$ hr. **b.** 75 hr. **c.** 38 hr.

4. What is the area of a garage 20 ft. wide and 22 ft. long?

a. 390 sq. ft. **b.** 440 sq. ft. **c.** 44 sq. ft.

5. If 1 gallon of paint covers 300 sq. ft., you will cover 1000 sq. ft. with:

a. 5 gal. **b.** 2 gal. **c.** $3\frac{1}{3}$ gal.

6. A rectangle 3 ft. wide by 6 ft. long contains how many square yards?

a. 2 **b.** 3 **c.** 6

7. A roof area 40 ft. long and 30 ft. wide contains how many squares? (100 sq. ft.)

a. 6 **b.** 8 **c.** 12

8. How long will it take Fred to earn $75.00 at $1.50 per hour?

a. 60 hr. **b.** 40 hr. **c.** 50 hr.

9. If lumber costs $220 per M, what is the cost of one board foot?

a. 2.2¢ **b.** 22¢ **c.** $2.20

10. One piece of 3″ × 4″ × 10′ contains how many board feet?

a. 10 **b.** 20 **c.** 5

11. At $6.00 per foot, what would it cost to put a fence around a square lot 50 ft. on a side?

a. $300 **b.** $600 **c.** $1200

1. A basement 24 ft. long and 18 ft. wide is excavated to a depth of 10 ft. What will it cost at \$6 a cubic yard? *\$ 960*

2. What would 36 pieces of 2″ × 10″ boards, each 12 ft. long, cost at \$96 per M? *\$ 69.12*

3. What is the volume of a cylindrical tank 7 ft. long and 3 ft. in diameter? *49.5 cu. ft.*

4. How many gallons will the tank in Exercise 3 hold if there are $7\frac{1}{2}$ gal. in a cubic foot? *approximately 371*

\$ 7304

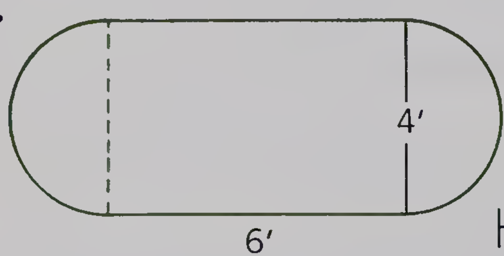
5. If the cost of carpenter labor on a house of 1660 sq. ft. is \$4.40 per square foot, what is the total cost of carpenter labor on the house?

6. Find the cost of five sheets of plywood, each 4′ × 8′, at 27¢ per square foot. *\$ 43.20*

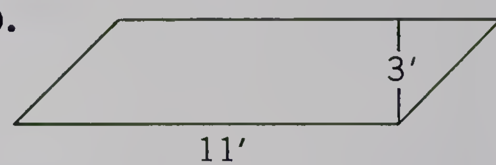
7. How many cubic yards of dirt were removed to form a ditch 3 ft. wide, 6 ft. deep, and $\frac{1}{4}$ mile long? *880 cu. yd.*

8. Find the area of each of the following figures: *a. 36.56 sq. ft. b. 33 sq. ft. c. 21 sq. ft. d. 294 sq. in. e. 154 sq. in.*

a.



b.



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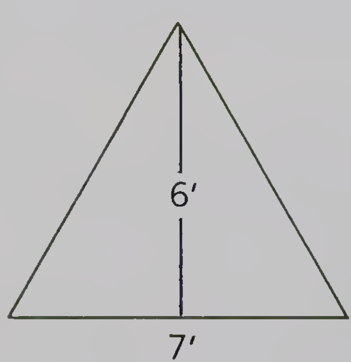

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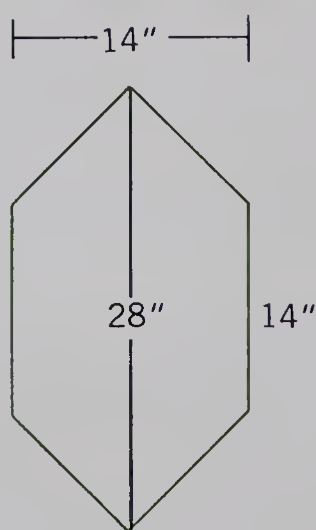

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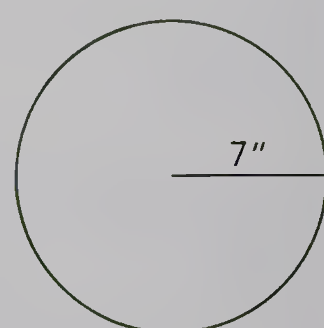


9. What would it cost to put linoleum floors in a room 24′ × 16′ at 48¢ per square foot? *\$ 184.32*

10. A pipeline from Texas to the eastern states is 1344 miles long. How many gallons of oil will it hold when full if its diameter is 24 in.? ($7\frac{1}{2}$ gallons in 1 cubic ft.) *167,270,400 gal.*

You have used formulas in solving problems about areas, perimeters, and volumes. These formulas are valuable because they guide you in using the numbers you are given to find the answer.

Whenever you can state a rule for solving a certain kind of problem, you can usually set up a formula, which is more convenient to use.

1. Write down the formula for the perimeter of a rectangle. Use it to find the perimeter of these rectangles: $P = 2l + 2w$

a. $w = 6''$, $l = 10'$ 21 ft.   **c.** $w = 10'$, $l = 4$ yd. 44 ft.

b. $l = 3'$, $w = 1\frac{1}{2}'$ 9 ft.   **d.** $l = 2$ yd., $w = 18''$ 5 yd.

2. Write down the formula for the area of a trapezoid. Use the formula to find the area of these trapezoids. $A = \frac{1}{2}h(a+b)$

	a	b	h		a	b	h
a.	5″	3″	4″ 16 sq. in.	**c.**	6 rd.	10 rd.	5 rd. 40 sq. rd.
b.	2′	4′	3′ 9 sq. ft.	**d.**	4′	1 yd.	24″ 7 sq. ft.

3. If you know the price of each article, you can find the cost of several by multiplying the price of each by the number you are buying. That is:

$$C = np$$

Here C is total cost, n is number purchased, and p is the price of each. Use the formula to find the cost of 12 sponges at 39¢ each. $4.68

4. Use the formula from Exercise 3 to find total cost in each of the following:

	n	p		n	p
a.	9	60¢ $5.40	**d.**	7	18¢ $1.26
b.	17	8¢ $1.36	**e.**	13	5¢ $.65
c.	12	$1.50 $18.00	**f.**	7	$3.80 $26.60

5. If you know that George purchased eight pencils for 40¢, how would you find the cost of each pencil? $p = \frac{c}{n}$ for six dozen pencils? 72p

6. The formula for finding the cost of each article, if you know how much a certain number of the articles cost, is:

$$p = \frac{C}{n}$$

Use this formula to find the cost of each article in the following:

	C	n		C	n
a.	$1.50	6 $.25	**c.**	$18	9 $2
b.	$7.20	12 $.60	**d.**	$42	14 $3

7. George worked on a road construction crew last summer at $1.80 an hour. He worked 40 hours a week. How much did he earn per week? $ 72.00

8. Let *W* represent wages, *e* represent earnings per hour and *h* represent hours worked. Write a formula that describes the relationship between these three factors. Use the result in Exercise 7 to help you see the relationship. $W = eh$

9. Use the formula to find *W* in the following:

	e	*h*			*e*	*h*	
a.	75¢	12	$ 9.00	**c.**	$1.15	30	$ 34.50
b.	$1.30	20	$ 26.00	**d.**	$1.95	20	$ 39.00

10. Eric earns $1.10 an hour at a supermarket. Last week he earned $38.50. How many hours did he work? 35 hr.

11. Find the relationship that tells you how to work this kind of problem. Write a formula to represent this relationship. $h = \frac{W}{e}$

12. Use the formula to find *h* in the following:

	W	*e*			*W*	*e*	
a.	$18	$1.50	12 hr.	**d.**	$36	.90	40 hr.
b.	$35	$1.40	25 hr.	**e.**	$11.20	.80	14 hr.
c.	$18	.75	24 hr.	**f.**	$19	.95	20 hr.

13. Frank drove from Oakdale to Rosemount, a distance of 120 miles, at 40 m.p.h. How long did it take him? 3 hr.

14. Which relationship tells you how to work this problem? Write it as a formula, using *r* for rate, *d* for distance, and *t* for time. $t = \frac{d}{r}$

15. Use a formula for solving the following for *t*.

	d	*r*			*d*	*r*	
a.	80 mi.	16 m.p.h.	5 hr.	**c.**	500 mi.	125 m.p.h.	4
b.	15 mi.	3 m.p.h.	5 hr.	**d.**	2400 mi.	300 m.p.h.	8

16. Henry walked from Baldwin to Hershey, a distance of 15 miles, in five hours. How many miles did he walk per hour? 3 m.p.h.

17. How long will it take a plane flying 550 miles per hour to travel 2800 miles? $5\frac{1}{11}$ hr.

18. Mr. Hancock bought four new tires at $42 per tire. What did they cost him? $ 168

19. The Blue Cab Company bought a fleet of 15 new taxicabs for $31,500. How much did they pay for each cab? $ 2,100

20. Last week Mr. Gillette worked 38 hours. His wages were $104.50. How much did he earn per hour? $ 2.75

Understand the Problem

Your skill in mathematical problem-solving will improve with practice. You should always give special attention to the first step: Utilize all the available data as stated in the conditions of the problem. These exercises are designed to help you to see how this is done, and how important it is to do it in each problem. If you are not careful, you may

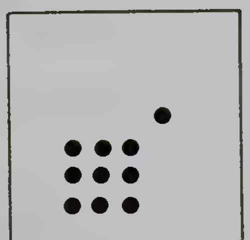
overlook a promising lead because you unintentionally impose restrictions that are not included in the conditions of the problem. For example: Can you arrange these matches so that each touches all the others? Experiment with the possibilities before reading further.

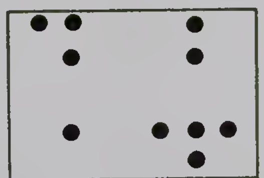
If you try to put all the tips together you find that you are prevented by the thickness of the matches from having each touch the others. But the conditions of the problem do *not* state that the matches must all lie on the same flat surface. Try this: Arrange two matches at an angle,

with the tips touching. Now insert a third between the other two. Each is touching both of the others. Now arrange the other three in the same way, *on top* of the other three.

In each of the following exercises, be careful not to impose imaginary restrictions on your efforts to find a solution.

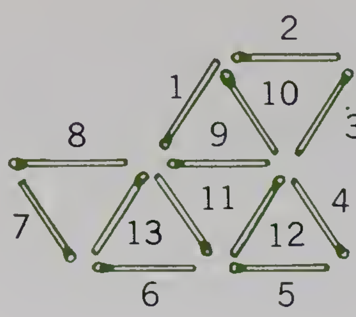
1. Without taking your pencil off the paper, draw four straight lines that pass through each dot in the figure below on the left. *See front.*

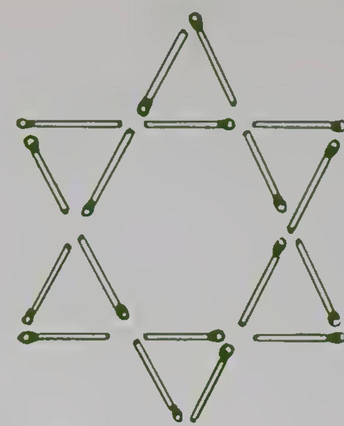




2. Draw four straight lines that are parallel to each other and at equal distances from one another, dividing the rectangle in the figure above on the right into five parts with two black dots in each part. *See front.*

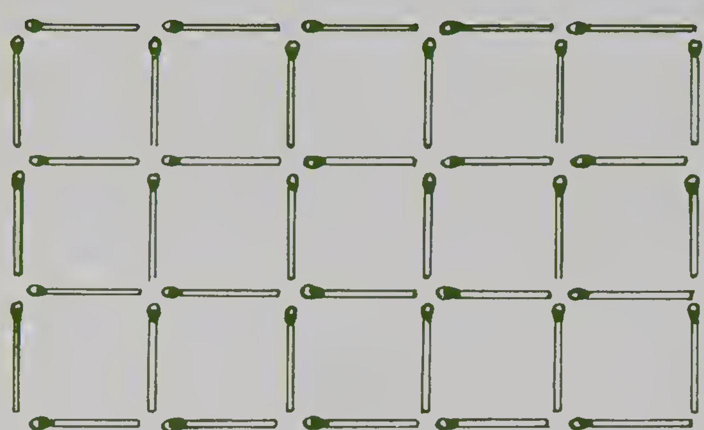
3. By removing three matches from the left figure, you can leave only three triangles. Show how. *Remove matches 9, 11, and 12.*

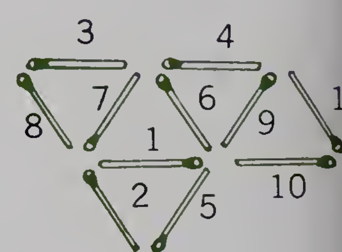




4. Change the position of six matches so as to make six equal four sided figures. *See front.*

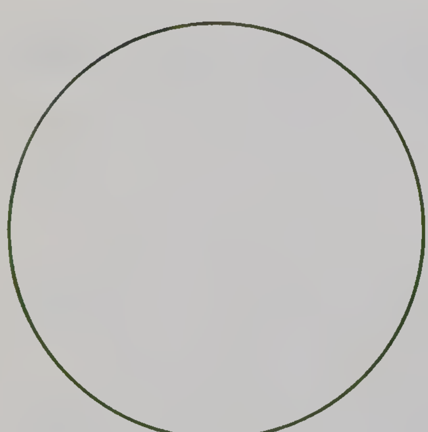
5. By taking 14 matches from the rectangle below, you can leave six equal squares. Show how. *See front.*

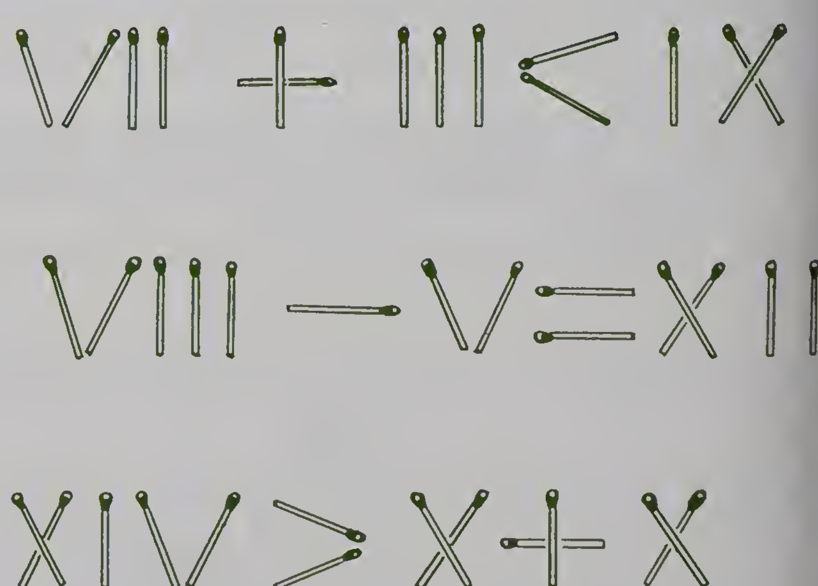




6. By taking away three matches from the figure above on the right, you can leave one trapezoid and one parallelogram. How is it done? *Remove matches 1, 7, 9.*

7. Divide this circle into as many parts as possible with three straight lines. *See front.*





8. Each of these false statements can be changed to a true statement by moving just one match. Show how. *See front.*

Part One

A. Find n:

1. $\frac{3}{8} = \frac{n}{16}$ 6
2. $\frac{3}{8} = \frac{15}{n}$ 40
3. $\frac{n}{6} = \frac{7}{3}$ 14
4. $\frac{6}{n} = \frac{3}{29}$ 58
5. $\frac{8}{n} = \frac{24}{100}$ $33\frac{1}{3}$
6. $\frac{17}{100} = \frac{n}{56}$ 9.52
7. $\frac{36}{n} = \frac{17}{100}$ $211\frac{13}{17}$
8. $\frac{35}{56} = \frac{n}{100}$ 62.5
9. $\frac{13}{50} = \frac{n}{150}$ 39
10. $\frac{17}{102} = \frac{13}{n}$ 78
11. $\frac{5}{19} = \frac{n}{95}$ 25
12. $\frac{18}{n} = \frac{3}{4}$ 24

B. Write as per cents (to the nearest tenth of 1% if there is a remainder):

13. $\frac{11}{15}$ 73.3%
14. $2\frac{3}{4}$ 275%
15. $\frac{1}{300}$ .3%
16. $\frac{11}{60}$ 18.3%
17. $\frac{7}{12}$ 58.3%
18. $\frac{1}{1000}$ .1%

C. Copy each of the following statements.
Then find the value for n:

1. 15% of 80 is n 12
2. 6 is n% of 24 25
3. 35% of \$5000 is n \$1750
4. 87.5% of 160 is n 140
5. 50 is n% of 500 10
6. \$16 is n% of \$96 16.7
7. 120 is n% of 240 50
8. $\frac{1}{4}$% of \$10,000 is n \$25
9. 15 is n% of 120 12.5%
10. 165% of 240 is n 396
11. 120% of 316 is n 379.2
12. $4\frac{1}{2}$ is n% of 30 15

Part Two

A. List the numerals 1 to 10 on your paper. After each numeral, write the word or phrase which is defined by the statement having that number.

1. The unit for measuring lumber is the _?_ board feet
2. A detailed account of what is to be performed by a contractor in return for a specified remuneration is called the _?_ specifications
3. The price a contractor sets for a given job is his _?_ bid
4. The fraction that a line on a drawing is of the length it represents is called the _?_ of the drawing. scale

5. The unit of area used to calculate the cost of roofing is the _?_ square

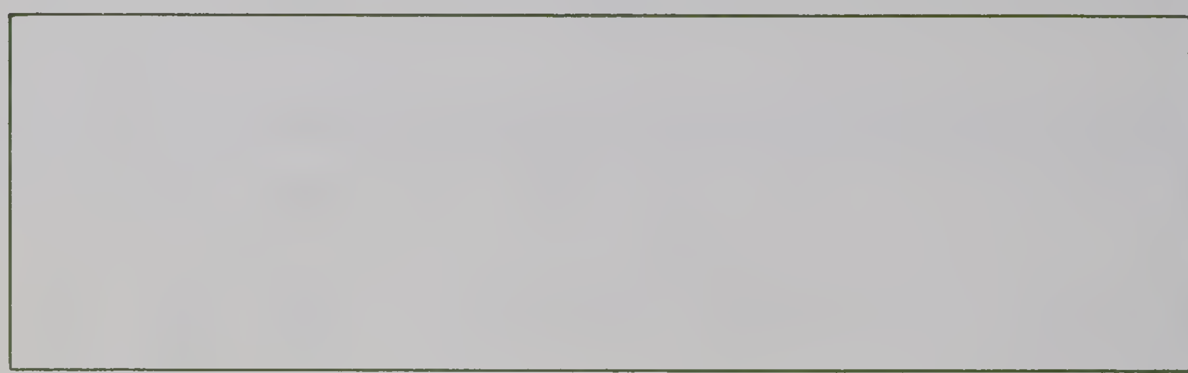
6. A drawing of an object having the same proportions but different dimensions is a _?_ scale drawing

7. A contractor excavated a basement 21 feet wide, 30 feet long, and 9 feet deep. He removed _?_ cubic yards of earth. 210

8. A board $8'' \times 8'' \times 4'$ contains _?_ board feet. $21\frac{1}{3}$ bd. ft.

9. A cubic yard contains _?_ cubic feet. 27

10. A cubic foot contains _?_ cubic inches. 1728



Scale: $\frac{1}{8}$ or 1″ to 8′

B.

1. What is the length of the rectangle represented above? 32 ft.

2. What is its width? 10 ft.

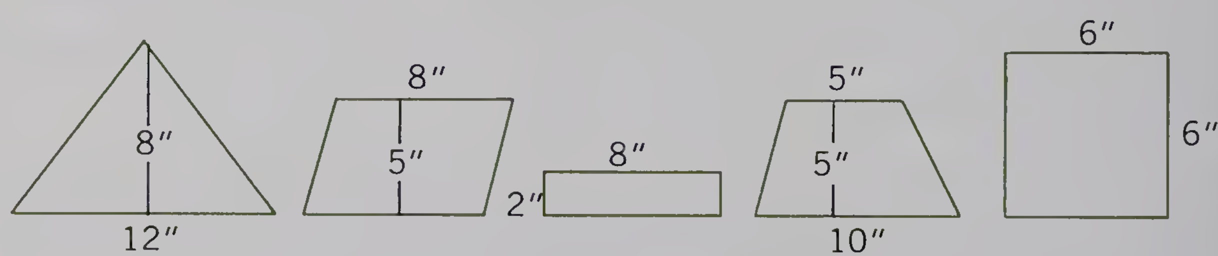
3. What is the ratio of length to width of this rectangle? $\frac{16}{5}$

4. What is the actual ratio of length to width of the drawing? How does this compare to the ratio found in Exercise 3? $\frac{16}{5}$; same

5. What is the area of the rectangle represented in the figure above? 320 sq. ft.

C.

1. Write the name of each figure: triangle ; parallelogram ; rectangle ; trapezoid ; square



$A = \frac{1}{2}bh$; $A = bh$; $A = lw$; $A = \frac{1}{2}h(a+b)$; $A = s^2$

2. Write the formula for finding its area.

3. Calculate the area for each figure.
48 sq. in. ; 40 sq. in. ; 16 sq. in. ; 37.5 sq. in. ; 36 sq. in.

STEPS FOR SOLVING APPLIED PROBLEMS

1. Understand the problem.	2. Note what the problem asks for.	3. Look for hidden questions.
6. Check your answer.	5. Set up and solve the conditional sentence(s).	4. Estimate a reasonable answer.

Part Three

1. Mr. Harold excavated a swimming pool 36′ long, 24′ wide, and 6′ deep for \$480. How much was he paid per cubic yard? *\$2.50*

2. Complete this lumber list:

Number of Pieces	*Dimensions*	*Price per M*	*Cost*
18	2″ × 4″ × 12′	\$92	*\$ 13.25*
14	2″ × 6″ × 18′	\$98	*24.70*
32	1″ × 6″ × 14′	\$112	*25.09*
			TOTAL *\$ 63.04*

3. How much redwood siding is needed to cover a rectangular cottage 28′ long, 18′ wide, with walls 9′ high? (Allow 10% for waste.) *911 bd. ft.*

4. What would be the cost of the redwood in Exercise 3 at \$180 per M? *\$163.98*

5. What is the cost of painting the four walls and ceiling of a room 12′ long, 10′ wide, and 8′ high at 12¢ per square foot? *\$56.64*

6. What would the annual rent on a house worth \$12,800 be if the owner charged 1% of its worth each month? *\$1536*

7. Using a scale of $\frac{1}{8}''$ = 1 ft., copy and complete the following table:

Inches on drawing	$\frac{3}{4}''$	$\frac{1}{2}''$	1″	$\frac{3}{8}''$	$3\frac{1}{2}''$	4″	3″	$1\frac{1}{2}''$
Actual distance	*6′*	*4′*	*8′*	*3′*	28′	32′	24′	12′

8. Draw a floor plan of a room 16 ft. long and 12 ft. wide using a scale of $\frac{1}{4}''$ = 1 ft. *The scale drawing would be 4″ by 3″.*

9. What would be the cost of a new roof at \$28 per square if the roof area is 4600 sq. ft.? *\$1288*

10. How many cubic yards of dirt would be required to fill a hole 18 ft. long, 12 ft. wide, and 6″ deep? *4 cu. yd.*

MATHEMATICS IN THE COMMUNITY

WORDS TO WATCH FOR

assessed value	*exemption*	*property tax*
budget	*income tax*	*real estate*
dependent	*indirect tax*	*revenue*
direct tax	*mill*	*sales tax*
excise tax	*net income*	*tax*

We often take for granted many important services provided by our local, state, and national governments. We have police and fire protection, highways and bridges, schools and libraries, and sanitation systems. Doing without many of these services would make us fully appreciate their importance.

If each family had to provide all of these services for themselves, cost of construction and maintenance of these facilities would be tremendous and the services would probably be inadequate. Thus, these services are provided by the community. Each family pays taxes, directly and indirectly, to support these services.

1. Make a list of at least six other services provided by government at the local, state, or federal level. *Answers will vary.*

2. The total expenditure for the nation's public schools, both elementary and secondary, recently was reported as \$18.2 billion. This amount provided services for 42.9 million pupils in the schools. To the nearest dollar, what was the expenditure per pupil? *\$424*

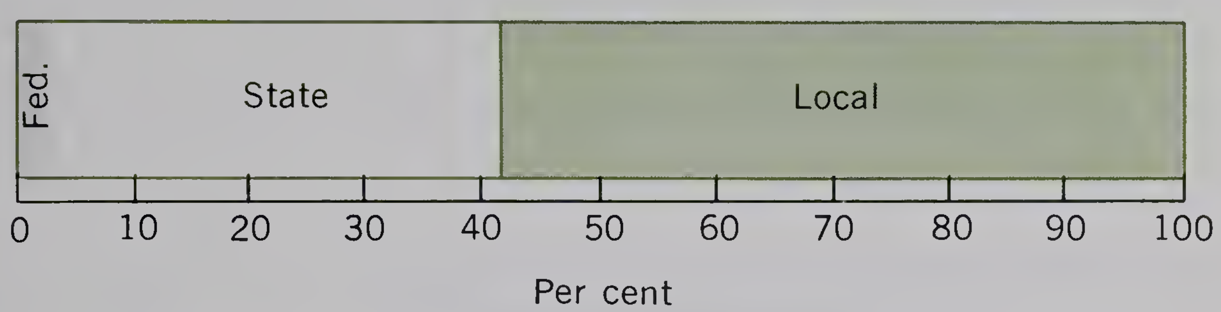
3. The Jensens have three children in school. They pay a total of \$1200 a year in local, state, and federal taxes. This is how much less than the cost of schooling for their children? *\$72*

4. List eight other services that the Jensens are receiving from the community. Suppose the cost of these other services averages about $5000 per family. What is the total value of the services (including education) received by the Jensens? *$ 6272*

5. How much more is this than the taxes paid by the Jensens? *$5072*

6. Some services are provided by one level of government alone, while others are supported by several. The major cost of education is still paid by the local community. The federal government's share of the cost is increasing due to a growing awareness of the importance of education in our society.

COST OF EDUCATION



If the expenditure in the graph totals $21 billion for all three levels of government, how much is spent at each level?
Federal, $ 525 million; State, $ 8.4 billion; Local, $ 12.075 billion

7. Ten years ago the total spent by all three levels was $8 billion. Using the same percentages, find how much was spent at each level?
Federal $200 million; State $ 3.2 billion; Local $ 4.6 billion

8. Some community services are paid for by those who use them. About 6000 communities in the United States have parking meters to regulate parking in business areas. One community installed 450 parking meters, which averaged $8.75 each in revenue per month. What was the annual income from the meters? *$ 47,250*

9. *Direct taxes* are paid directly to the government. The merchant collects *indirect taxes* such as gasoline taxes and other sales and excise taxes. Indirect taxes are often called *hidden* taxes because they are usually included in the selling price of goods purchased by individuals.

The Henderson family paid the following direct taxes in a recent year:

federal income tax, $720
state income tax, $87.50
motor vehicle tax, $25
real estate tax, $400
bicycle tax, $1
dog license, $2

Their family income was $7000. What per cent of the income was paid in direct taxes? *17.7%*

10. What are some examples of other direct taxes? *personal property tax, inheritance tax, gift tax*

You are using community services whenever you visit a state or national park, use the library, attend a public school or state college, or enjoy the protection of the police and fire department. Many hospitals are supported by government too.

Some of these services are provided by the federal government, others by the state, and others by local government (county, school district, and city). You can get some idea of the services provided by each by comparing the following budgets.

1. What is the largest item in the federal budget? *defense*

2. How much was spent on this item? *$ 56.5 billion*

3. What is the largest item in the state budget? *schools*

4. How much was spent on it? *$ 128 million*

5. What is the largest item in the city budgets? *education*

6. How much was spent on it? *$ 640 million*

7. What items appear only in the federal budget? *all except general government*

8. What item appears on both the state and city budget? *education, general government*

9. What per cent of the expenditures in the federal budget was caused by past wars or for national preparedness? *73 %*

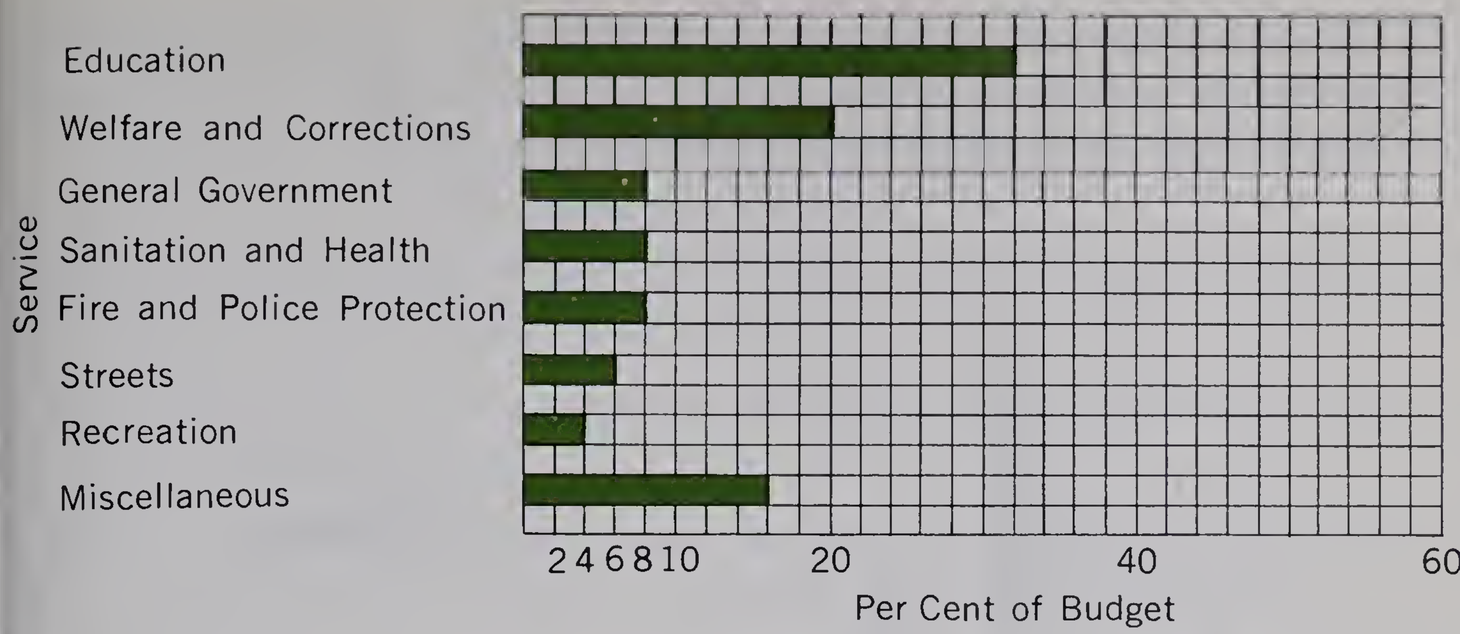
10. Which item in the federal budget would provide for each of these: *See front.*

national parks	interest on savings bonds
foreign aid	veterans' hospitals
farm aid	highway appropriations

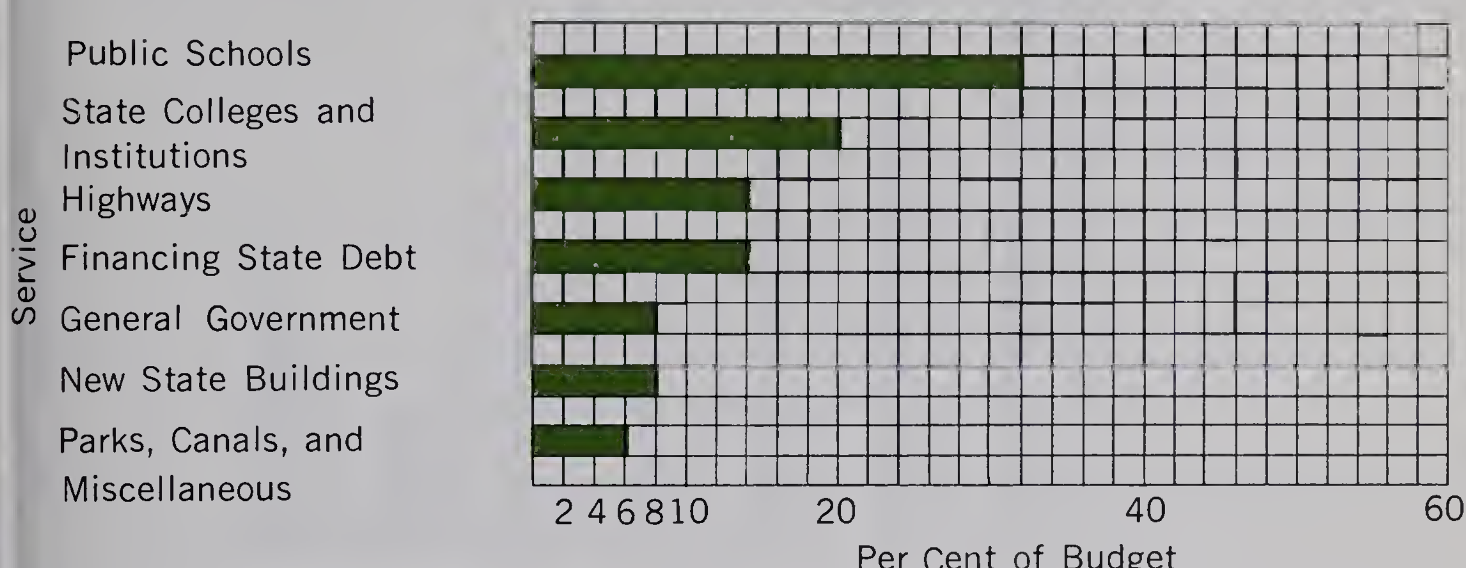
11. With a population of 195 million, what would be the cost per person (to the nearest cent) for federal activities listed in the budget? *$ 512.82*

12. All but two of the following items were paid for by public funds. Pick out these two and then state whether each of the others is paid for by federal, state, or local funds: *(circled)*

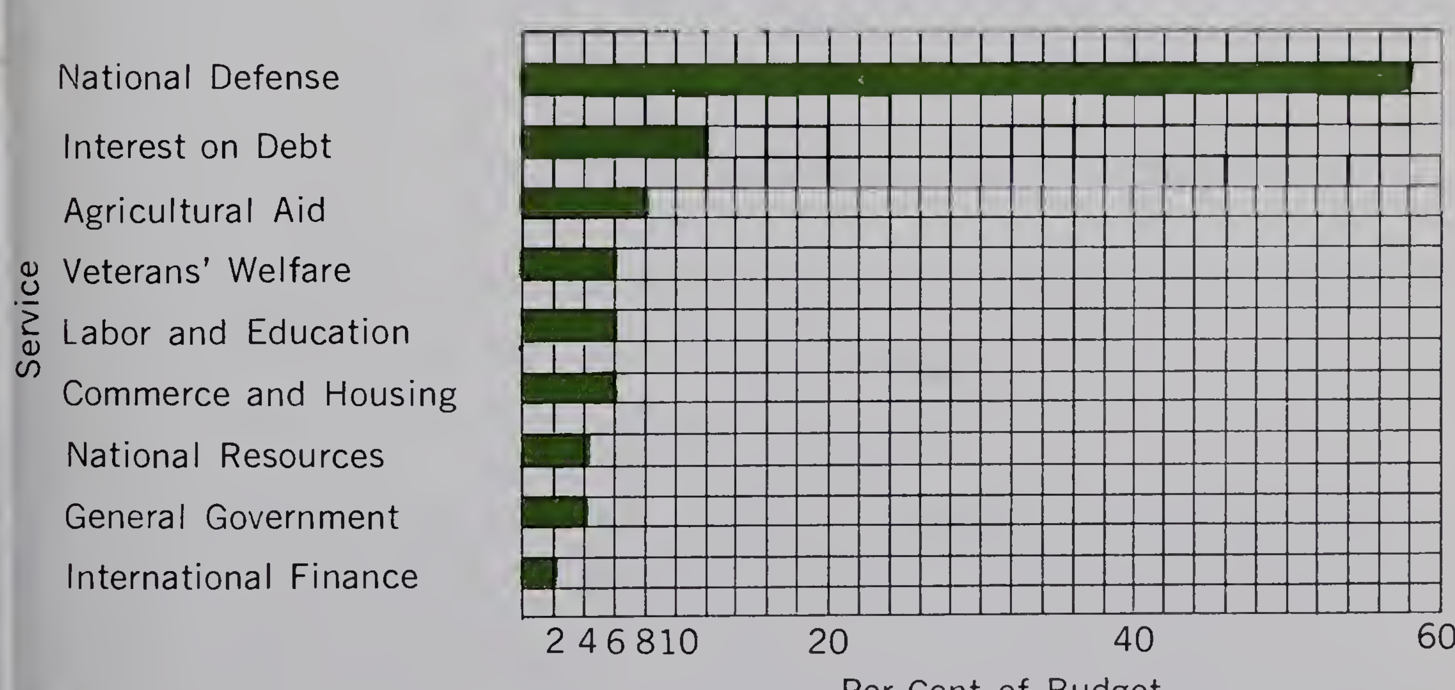
f **a.** An Apollo space shot	*f* **f.** The St. Lawrence waterway
b. A railroad terminal building	*l* **g.** A city playground
l **c.** The court house	*s* **h.** A state office building
f **d.** The post office	*f* **i.** An atomic submarine
f-s **e.** A new freeway	**j.** The Empire State Building



HOW 100 CITIES SPENT $2 BILLION



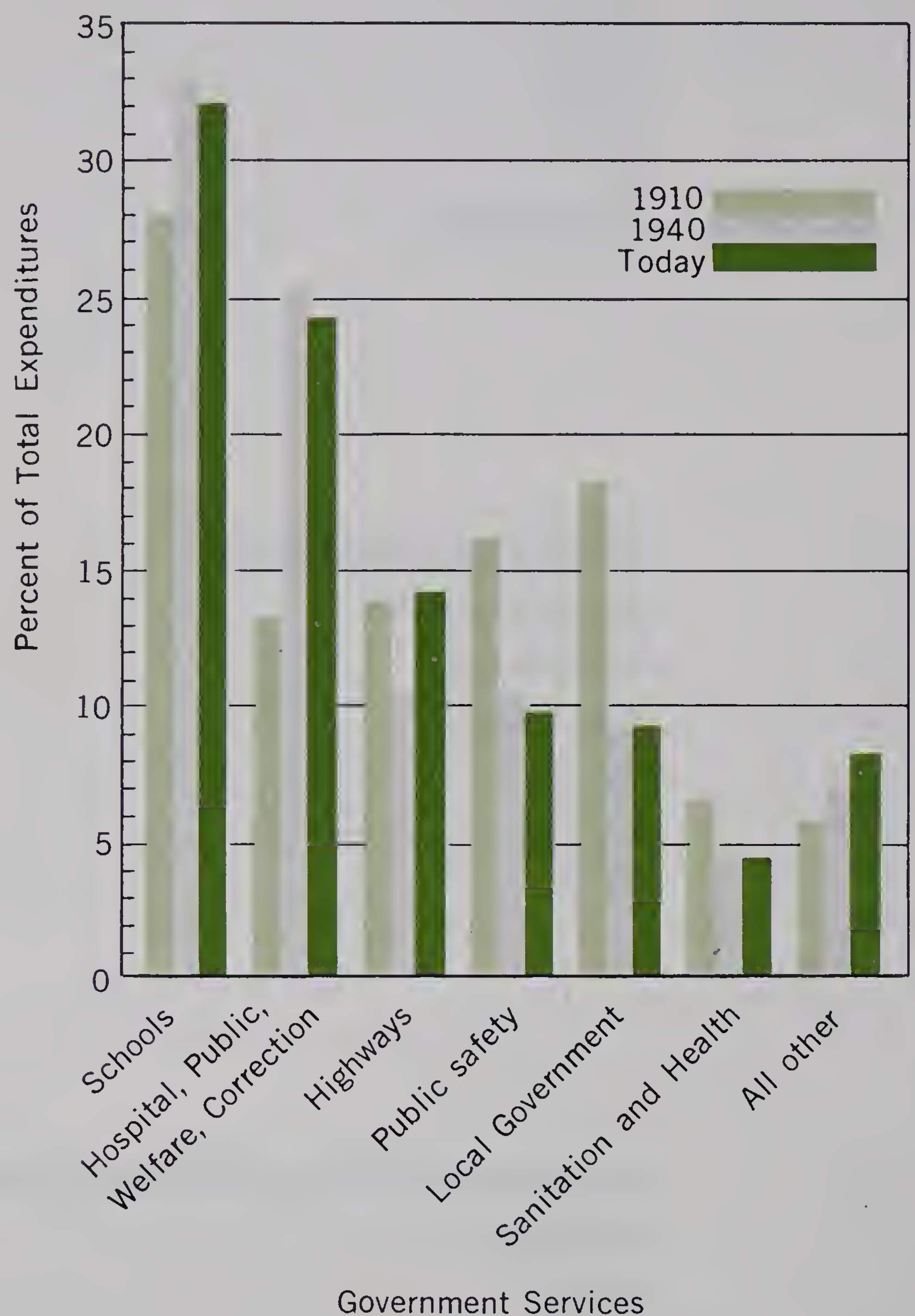
HOW A STATE GOVERNMENT SPENT $400 MILLION



HOW THE FEDERAL GOVERNMENT SPENT $100 MILLION

1. The changing costs of various activities of the state and local governments are portrayed in this graph. Give an example of some public service you are familiar with under each heading.

GOVERNMENT EXPENDITURES



2. Which service receives the largest budget allocation? *schools*

3. Some increases in expenditures are due to the fact that the same service becomes more expensive. In other cases, increases in the amount and quality of service are necessary. Which expenditure has increased more than the rest? Can you explain why?
public welfare ; increase in amount and quality

4. Total expenditures by state and local governments for the three years represented by the graph were: 1910–$1,165 million; 1940–$7,218 million; today–$65 billion. What was the per cent of increase from 1910 to 1940? From 1940 to today? *801 %*
520 %

1910, $ 326.3 million; 1940, $ 2.598 million; today, 21.5 billion

5. What did the schools cost in each of the three years?

6. For comparison with other school systems and for state aid purposes the cost of education in a school system is often reported as cost per pupil. In one city the average cost per pupil is $380 per year. What is the cost per year of a class of 32 pupils? $ 12,160

7. At this rate, what would be the cost per year of maintaining a school with an enrollment of 1200 pupils? $ 456,000

8. If the school is in operation 175 days per year, what is the cost per day? $ 2605.71

9. The major items of expense reported by a city with a population of 322,000 were as follows, rounded to the nearest $100,000:

Education 37.4 %	$14,200,000	Parks, playgrounds	1,300,000 3.4 %
Police, fire 23.4 %	8,900,000	General government	1,200,000 3.2 %
Streets and 21.6 % highways	8,200,200	Miscellaneous	4,200,000 11.1 %

What was the total amount spent during the year? $ 38,000,200

10. Calculate to the nearest tenth of 1% the per cent of total expenditures that went for each purpose.

11. To the nearest dollar, how much did each person in the city pay for education? for police and fire protection? $ 28 $44

12. Prepare a circle graph to show how the expenditures were distributed among the activities. See front.

13. Copy the following list of community services and responsibilities and after each put one or more of the letters F, S, or L to indicate whether it is provided by the federal, state, or local government, by two of them, or by all three.

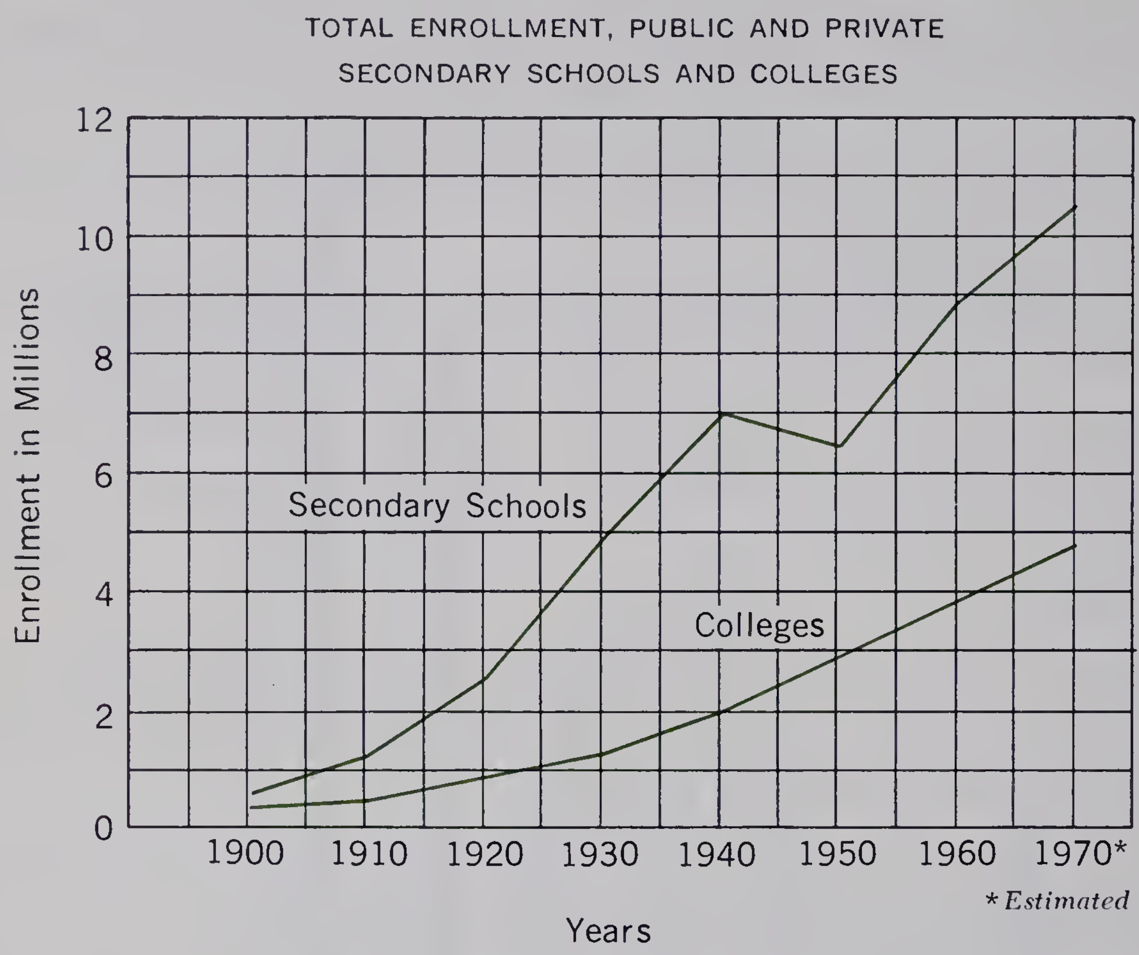
a. Highways F-S
b. Sidewalks L
c. Libraries L
d. Hospitals F-S-L
e. National defense F
f. Parks F-S-L
g. Fire protection L
h. Traffic control L
i. Police protection S-L
j. Elementary schools S-L
k. Secondary schools S-L
l. Colleges S-L

SPECIAL PROJECTS

1. Secure a copy of the local budget. Prepare a table as in Exercise 9; make the same computations called for in Exercises 10 and 11.

2. Secure a copy of your state budget and of the federal budget for the current year. What expenditures have increased or decreased when compared to the graphs on page 325?

One of the reasons for rising costs of education is the rapid increase in school enrollments. High school enrollments have been steadily increasing for fifty years, as is shown in this graph.



1. To the nearest half million, how many pupils were in high school in 1900? in 1930? in 1960? 1900, $\frac{1}{2}$ million; 1930, 5 million; 1960, 9 million
2. What is the estimated high school enrollment for 1970? 10.5 million
3. What was the per cent of increase in high school enrollments during the decade between 1900 and 1910? 100 %
4. During which decade was there a decrease in high school enrollments? 1940 – 1950
5. During which decade did the high school enrollments double? 1920 – 1930
6. During the same period college enrollments have been increasing rapidly. To the nearest half million, how many students were in college in 1900? in 1930? in 1960? $\frac{1}{2}$ million; $1\frac{1}{2}$ million; 4 million
7. How many more students were in college in 1960 than in 1940? 2.0 million
8. The college enrollment was what per cent of the high school enrollment in 1930? in 1960? 44.4% 30%
9. What is the expected increase from 1960 to 1970 in both groups? High School, $1\frac{1}{2}$ million; College, 1 million

Fractional Numbers

A. Write as per cent, to the nearest tenth of 1%.

1. $\frac{3}{4}$ 75.0 %
2. $\frac{1}{2}$ 50.0 %
3. $\frac{7}{16}$ 43.8 %
4. $\frac{3}{5}$ 60.0 %
5. $\frac{7}{10}$ 70.0 %
6. $\frac{5}{8}$ 62.5 %
7. $\frac{7}{12}$ 58.3 %
8. .08 8.0 %
9. 0.6 60.0 %
10. 0.26 26.0 %
11. .065 6.5 %
12. .009 0.9 %
13. 0.18 18.0 %
14. 3.16 316.0 %

B. Find the value for n:

1. 2% of 176 is n 3.52
2. 150% of 38 is n 57
3. n is 116% of 45 52.2
4. 37.5% of 56 is n 21
5. 0.4% of 80 is n 0.32
6. n is 175% of 80 140

C. Find the value for n:

1. 3 is n% of 60 5
2. n% of 248 is 62 25
3. n% of 45 is 54 120
4. n% of 38 is 76 200
5. 12.5 is n% of 2500 0.5
6. n% of 800 is 4 0.5

D. Find the value for n:

1. 8 is 10% of n 80
2. 130% of n is 65 50
3. 48 is 5% of n 960
4. 42% of n is 126 300
5. 36 is 6% of n 600
6. 54 is 120% of n 45

E. Find the value for n:

1. 40 decreased by 10% is n 36
2. n is 20% less than 85 68
3. 50% more than 200 is n 300
4. n is 6000 increased by 8% 6480
5. 720 is n% more than 600 20
6. n% less than 300 is 240 20

If you need more practice, turn to the Practice Exercises on page 498. If not, you may work on the following page.

Cryptanalysis

Cryptography is the art of secret writing. It has been practiced in one form or other since writing itself was invented. In general, cryptography has two forms: the *code*, in which substitutions are made for words or sounds, and the *cipher*, in which letters, digits, or symbols are substituted for the letters in a word.

To decipher a code requires a code book. Often a dictionary is chosen because it is such a common book. Thus, 125–2–17 might mean: on page 125, second column, the 17th word. Naturally the recipient of a message must have the code book in order to decipher the message. This type of code is used for diplomatic messages for the most part, and the recipient has a code book readily available at his office.

Ciphers used for military purposes, however, can be intercepted by the enemy when sent by telegraph or radio. In this case the recipient usually memorizes the key to the cipher so that no code book or apparatus is required for deciphering. Even though the cipher can be broken by mathematical means, it is still used so that the enemy will not get the information immediately. By the time the intercepted message is deciphered, the information may be of no use to the interceptor.

Ciphers come in many varieties. Julius Caesar is said to have used a cipher in which the third letter following was substituted for any given letter in the message. Thus, the first line in his *Commentaries* which reads, *Gallia est omnis divisa in partes tres,* would be written in his cipher:

KDOOMD HXY RPQMX GMZMXD MQ SDVYHX YVHX

Clear	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*k*	*l*	*m*	*n*	*o*	*p*	*q*	*r*	*s*	*t*	*v*	*x*	*y*	*z*
Cipher	D	E	F	G	H	I	K	L	M	N	O	P	Q	R	S	T	V	X	Y	Z	A	B	C

Note that there is no J in the Latin alphabet.

Caesar's cipher is called an *informal substitution* cipher. In this system the word spacings are retained, and one letter is substituted for another. It is called informal because the substitutions are not complicated ones. Let us see how mathematics can be used to break this type of cipher. Here is an interesting message which we can use to illustrate the processes.

NALO RUXZGV WUUV NAFN OALVGO EUP MO FRFLV;
AUJ UEN AGPGFENGP JLXX OAG JFH FVZ JFVG,
AUJ UEN AGPGFENGP, PLOLVR, OGFPTA EUP MO
NAPUMRA NAGOG OFWG IPFVTAGO—
FVZ EUP UVG LV KFLV. UWFP YAFCCFW

In breaking the cipher, we follow a systematic set of procedures. At the same time we feel free to make guesses and check on them.

Step 1. The first procedure is to make a count of the letter frequencies. We set up a table to show how many times each letter is used:

A – 14	E – 7	I – 1	M – 4	Q – 0	U – 13	Y – 1
B – 0	F – 13	J – 5	N – 9	R – 3	V – 12	Z – 3
C – 3	G – 16	K – 1	O – 10	S – 0	W – 4	
D – 0	H – 1	L – 8	P – 10	T – 2	X – 3	

In our language the most frequently occurring letters, in order of their frequency, are: *e t o a n i r s h*. If our cipher were long enough we might identify most of the letters by their frequency. As it is, we can see a striking difference in their frequencies, and we can say that *probably* the letters above are represented in the cipher by A F G L E O P U and V, although we cannot say which letter in the cipher represents which letter in the clear.

Step 2. Now copy the cipher above, leaving ample space between the lines so that you can insert the letters of the clear as you identify them. The *clear* is the deciphered message. Also write the letters of the alphabet across the page so that you can write over each letter, the letter of the clear it represents.

First let us *guess* that G, since it occurs with the highest frequency, represents *e*. We may be wrong, but if so, we will soon find out from the impossible letter combinations that we create. So in our message wherever G occurs, we write *e* above it. Next Step 2, examine the *digraphs*, that is, two-letter combinations. These may be double letters, which in order of frequency of occurrence are *ss, ee, tt, ff, ll* and *oo*. In the first line we see the double UU in the middle of a four-letter word so U is probably a double vowel. Since we have made a choice for *e*, let us guess it is *o*. So U = *o*.

Another useful digraph is the *diphthong*, which is a two-letter combination with one speech-sound. In order of frequency these are *th, sh,* and *ng* (which would most likely occur at the end of a word). Note that *th* consists of high frequency letters. We find NA occurring four times. It is worth guessing that this represents *th*. If so, N = *t*, A = *h*.

There are only two two-letter words, MO consisting of low-frequency letters, and LV consisting of high-frequency letters. The most common two-letter words, in order of frequency, are: *of, to, in, it, is, be, as, at*. Since we have tentatively identified U = *o*, LV cannot be *of* or *to*. Let us try *in*; if so, L = *i*, V = *n*.

Step 3. We have enough letters of the clear tentatively identified to begin to examine word patterns, trying to fill in those that are incomplete. This will also reveal mistaken guesses that must be done over again. Your cipher should now look like this:

```
thi   o  en   oon th  t  hine    o          in
NALO RUXZGV WUUV NAFN OALVGO EUP MO FRFLV;

ho  o  t he e   te   i     he       n     ne
AUJ UEN AGPGFENGP JLXX OAG JFH FVZ JFVG,

ho  o  t he e   te   i in    e    h  o
AUJ UEN AGPGFENGP, PLOLVR, OGFPTA EUP MO

th  o    h the  e      e    n he
NAPUMRA NAGOG OFWG IPFVTAGO—

 n   o  one in   in              o       h
FVZ EUP UVG LV KFLV.            UWFP YAFCCFW
```

Clear:	h						e					i		t							o	n				
Cipher:	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z

It appears that our guesses were all reasonable—this is unusual. It is easy now to fill in missing letters to make complete words. Each time we find a new letter, we insert it wherever it occurs, thus helping to find other new letters. Thus NALO must be *this*, so let O = *s*. NAFN must be *that*, so F = *a*. AUJ must be *how*, so J = *w*. See if you can complete the decipherment. See front.

Using the steps illustrated above, decipher each of these messages.

1. APIL KEPLM STR KMGMT CMSLK SNP PIL ASJZMLK HLPINZJ APLJZ IQPT JZYK EPTJYTMTJ S TMF TSJYPT, EPTEMYGMR YT VYHMLJC STR RMRYESJMR JP JZM QLPQPKYJYPT JZSJ SVV UMT SLM ELMSJMR MOISV.

SHLSZSU VYTEPVT

2. QCW USOQDY KGINWSOWP KFLJNQSGIW JR FSQCWFSQKIO KG JTN QKFW FSBWO KQ SDD QCW FJNW GWIWOOSNY QCSQ WSIC JR TO OCJTDP BGJV OJFWQCKGE JR QCW GSQTNW SGP TOWO JR FSQCWFSQKIO. VCKDW QCW PWWLWN SGP FJNW IJFLDWX TOWO SNW GJQ SDVSYO TGPWNOQJJP, QCW WOOWGQKSD GSQTNW JR QCW OTHAWIQ KO NWSPKDY SLLNWIKSQWP.

FJNNKO BDWKG

See front.

3. A cryptographer is always looking for a *probable word.* If he can identify a word by guessing, it gives him several letters and helps him to guess several more words, and perhaps unravel the cipher. If his guess fails, he can always go back to the clues in the frequency count.

Take a look at the appearance of the words in this cipher. You see a single-letter word O. Probably O = ? The message begins with two two-letter words with the same initial letter. What are the words likely to be? Notice a two-letter word beginning with the final letter of the first word. What probably is this word? This gives you the letter that appears twice as a double. What probably is the three-letter word beginning with O? This should give you enough letters to unravel the word patterns. Here is the cipher:

DQ DP OFVOYP O KJJN DARO QJ IDUR OAUDCR OHA DQ DP WOQOF QJ IDUR IJJA OAUDCR.

JPCON VDFAR

4. Y RNFGIVG EYG KYT OPMGH MG QUV NAYR KMQU LOVVRMGH GATV FZAG KUMBU UV UYR DYOOVG, KUVG Y ZMH ZYTTVR QUYQ KYP. "PAF KYOOAK DYMNOP KVOO," TYMR QUV ZMH. "LFQ, EP DMGV DVOOAK, PAF UYXV EFBU QA OVYNG YLAFQ NAAQMGH."

YELNATV LMVNBV

5. BQU UFIGEBUF VJIBQULOUL QEV OJ IVU DJL EO L UTGUYB EB BQU CUNKOOKON JD E XJLF.

VEAIUM MEONQJLOU GMUAUOV

6. Select a key word for preparing a cipher. It should contain no repeated letters, and some of the letters should be from the latter part of the alphabet. *Vocabulist* is a good example of a key word:

Clear:	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*l*	*m*	*n*	*o*	*p*	*q*	*r*	*s*	*t*	*u*	*v*	*w*	*x*	*y*	*z*
Cipher:	V	O	C	A	B	U	L	I	S	T	D	E	F	G	H	J	K	M	N	P	Q	R	W	X	Y	Z

Thus to encipher *come here,* we have: CHFB IBMB.

Other good key words are NORWAY and INCOME TAX. Prepare a cipher message of at least 40 words. Then see if your classmates can decipher it.

This work on cryptanalysis may serve as an introduction to a fascinating activity for you. Many daily newspapers and monthly periodicals provide messages or famous quotations to decipher. Many readers enjoy this activity. See if you can find any sources in your area that provide this type of section. In the next chapter, you will find another section devoted to this topic.

The funds required to pay for community services are raised mainly through taxes. Without taxation power the government could not provide the many services necessary for the general welfare of the people. Everyone, including you, pays taxes. The following exercises illustrate the variety of taxes we pay. After you have worked the exercises, make a list of the taxes you have paid in the past month and estimate how much they amounted to.

1. Over half of the *revenue* (or income) of the states comes from the sales tax. In some states food is exempt. In this problem consider a 3% sales tax on all articles except food, which is not subject to the sales tax. Mrs. Jones made the following purchases, how much tax did she pay? 65¢

a. 1 dress	\$17.50	**d.** 1 pair gloves	\$1.75
b. 1 pair hose	2.25	**e.** 10 lb. potatoes	1.55
c. 2 loaves bread	.54	**f.** 1 dozen oranges	.75

2. In a recent year 62 billion gallons of gasoline were consumed by cars, trucks, and buses in the United states. At that time the federal tax on gasoline was 4¢ per gallon, and the state tax averaged 5¢ per gallon. What was the total income from taxes on gasoline? \$ 5.58 billion

3. Mr. Clark averages 14.8 miles per gallon in driving his automobile. Last year he traveled 13,246 miles in his car. State and federal taxes were a total of 9¢ per gallon. How much did he pay in gasoline taxes? \$ 80.55

4. Mr. Adams pays a property tax of \$450 on his home, which he values at \$20,000. The tax is what per cent of the value of his home? 2.25%

5. The excise tax on long distance telephone calls is currently 10% of the cost of the long distance call. What would be the tax on a \$7.50 call? What would be the total charge for this call if there was an additional 5% city tax on such calls? 75¢; \$8.63

6. The city of Brownsville requires all dog owners to pay a license fee of \$3.50. This income helps finance the Animal Shelter. Last year \$12,390 was collected in this way. How many dogs were licensed? 3540

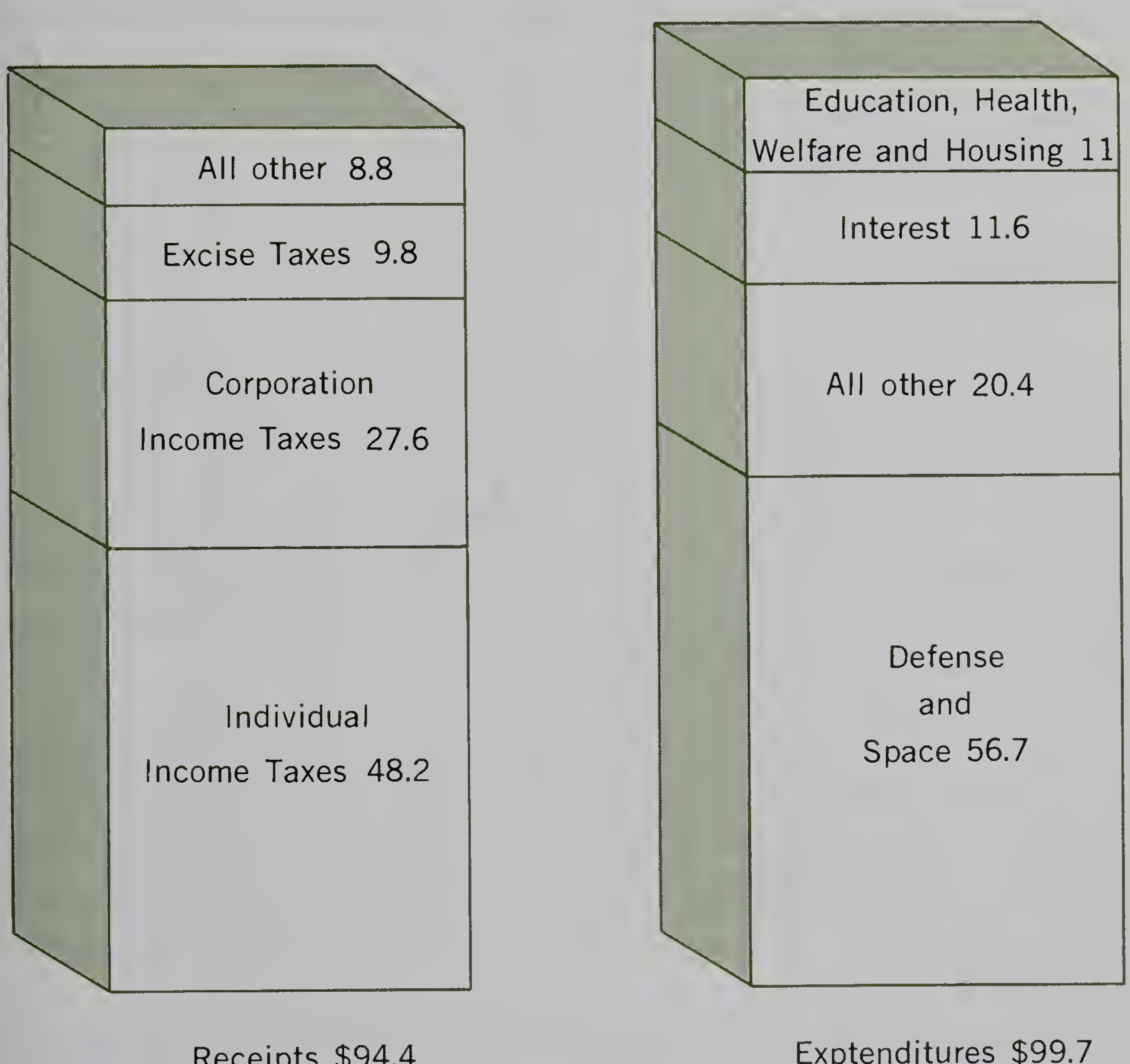
7. The state motor vehicle tax is one of the main sources of revenue for maintenance of state highways. The rates vary among the states. Mr. Ericson bought a new car for \$2200. His motor vehicle tax was \$44. The tax was what per cent of the price of the car? 2%

8. The tax on theater tickets is 10% of the price of admission. The price of the ticket to a play, including tax, was \$4.95. How much was the tax? 45¢

Though the majority of public funds are raised through taxation, some revenues are also collected through other means. Ships that pass through the Panama Canal pay tolls, for example. Fines and penalties, intended primarily to discourage illegal activities, are added to tax revenues to support government activities. You pay postage for postal services, although sometimes there is a deficit when expenses are greater than income. The remainder of the needed revenue then comes from taxation.

In order to provide an estimate of the money needed for the coming year, the proper officials of each unit of government prepare in advance a *budget* of proposed expenditures. They also propose a *tax levy* to provide funds needed to support the expenditures. Let us examine a typical situation of proposed expenditures and tax levies for the federal government as shown in this graph.

A FEDERAL BUDGET IN BILLIONS OF DOLLARS



1. This total budget is for $99.7 billion. What is allocated for *Defense and Space Programs*? $56.5 billion

2. What activities are included under defense and space programs?
ormy, navy, air service, space shots

3. All levels of government set aside part of their yearly budget to pay back money borrowed and to pay interest. How much is provided for interest on the national debt? $11.6 billion

4. How much is provided for education, health, housing and welfare programs? $11 billion

5. *All Other* includes veterans aid, farm programs, highways, foreign aid, etc. How much is budgeted for these items? $ 20.3 billion

6. How many sources of federal income are shown on the graph? 4

7. How many dollars are to be raised through individual income taxes?
$ 45.5 billion

8. How many dollars are to be raised through excise taxes? $ 9.3 billion

9. The excess of expenditures over income may be met through borrowing. How many dollars have to be borrowed? $ 5.3 billion

10. The federal budget varies from year to year. Secure a copy of the current federal budget and compare the items with those shown in the graph above. Which items have increased? Which have decreased? Is there a change in the proportion of income secured from any source? Answers will vary.

The Federal Income Tax

The federal government and most of the state governments raise a large proportion of their funds through taxes on individual incomes. The state tax regulations are similar to those of the federal government, although the rates are lower. The provisions of the federal income tax laws are changed from time to time. In general they are intended to provide an equitable, economical, and adequate system of taxation based on adjusting the tax burden to ability to pay.

1. The federal law provides for *exemptions:* a married taxpayer may claim exemptions for himself, for his wife, and for dependents, as defined in the regulations. For each exemption the taxpayer is entitled to subtract $600 from his reported taxable income. Mr. Brown has a wife and two children. How many exemptions may he claim? 4

If the family income is over $10,000 a detailed return is required, using Form 1040. If it is under $10,000 the use of the "short form," Form 1040A, is permitted. If a person's income is under $5,000, he may use the tax table, shown on the next page, to calculate his tax.

TAX TABLES FOR PERSONS WITH INCOMES UNDER $5,000 WHO DO NOT ITEMIZE DEDUCTIONS ON THEIR RETURNS

If you checked as your filing status on page 1, Form 1040 ....
Line 1a use——————→TAX TABLE A—For Single Persons
Line 1b, 1d, or 1e use———→TAX TABLE B—For Married Persons Filing Joint Returns and Unmarried Head of Household
Line 1c use———————→TAX TABLE C—For Married Persons Filing Separate Returns

Tables A and B reflect the lowest tax after considering both the 10 percent standard deduction and the minimum standard deduction. Table C shows the tax based on either the 10 percent or the minimum standard deduction.

1966 TAX TABLE A—FOR SINGLE PERSONS

Read down the income columns below until you find the line covering the total income (page 1, line 9, Form 1040). Then read across to the appropriate column headed by the number corresponding to the number of your exemptions, this is your tax.

If your total income is—		And the number of exemptions is—		
At least	But less than	1	2	3 (If 4 or more there is no tax)
		Your tax is—		
$0	$900	$0	$0	$0
900	925	2	0	0
925	950	5	0	0
950	975	9	0	0
975	1,000	12	0	0
1,000	1,025	16	0	0
1,025	1,050	19	0	0
1,050	1,075	23	0	0
1,075	1,100	26	0	0
1,100	1,125	30	0	0
1,125	1,150	33	0	0
1,150	1,175	37	0	0
1,175	1,200	40	0	0
1,200	1,225	44	0	0
1,225	1,250	47	0	0
1,250	1,275	51	0	0
1,275	1,300	54	0	0
1,300	1,325	58	0	0
1,325	1,350	61	0	0
1,350	1,375	65	0	0
1,375	1,400	68	0	0
1,400	1,425	72	0	0
1,425	1,450	76	0	0
1,450	1,475	79	0	0
1,475	1,500	83	0	0
1,500	1,525	87	0	0
1,525	1,550	91	0	0
1,550	1,575	94	0	0
1,575	1,600	98	0	0
1,600	1,625	102	2	0
1,625	1,650	106	5	0
1,650	1,675	109	9	0
1,675	1,700	113	12	0
1,700	1,725	117	16	0
1,725	1,750	121	19	0
1,750	1,775	124	23	0
1,775	1,800	128	26	0
1,800	1,825	132	30	0
1,825	1,850	136	33	0
1,850	1,875	139	37	0
1,875	1,900	143	40	0
1,900	1,925	147	44	0
1,925	1,950	151	47	0
1,950	1,975	155	51	0
1,975	2,000	159	54	0
2,000	2,025	163	58	0
2,025	2,050	167	61	0
2,050	2,075	171	65	0
2,075	2,100	175	68	0
2,100	2,125	179	72	0
2,125	2,150	183	76	0
2,150	2,175	187	79	0
2,175	2,200	191	83	0
2,200	2,225	195	87	0
2,225	2,250	199	91	0
2,250	2,275	203	94	0
2,275	2,300	207	98	0
2,300	2,325	211	102	2
2,325	2,350	215	106	5
2,350	2,375	219	109	9
2,375	2,400	223	113	12
2,400	2,425	227	117	16
2,425	2,450	231	121	19

If your total income is—		And the number of exemptions is—					
At least	But less than	1	2	3	4	5	6 (If 7 or more there is no tax)
		Your tax is—					
$2,450	$2,475	$236	$124	$23	$0	$0	$0
2,475	2,500	240	128	26	0	0	0
2,500	2,525	244	132	30	0	0	0
2,525	2,550	248	136	33	0	0	0
2,550	2,575	253	139	37	0	0	0
2,575	2,600	257	143	40	0	0	0
2,600	2,625	261	147	44	0	0	0
2,625	2,650	265	151	47	0	0	0
2,650	2,675	270	155	51	0	0	0
2,675	2,700	274	159	54	0	0	0
2,700	2,725	278	163	58	0	0	0
2,725	2,750	282	167	61	0	0	0
2,750	2,775	287	171	65	0	0	0
2,775	2,800	291	175	68	0	0	0
2,800	2,825	295	179	72	0	0	0
2,825	2,850	299	183	76	0	0	0
2,850	2,875	304	187	79	0	0	0
2,875	2,900	308	191	83	0	0	0
2,900	2,925	312	195	87	0	0	0
2,925	2,950	317	199	91	0	0	0
2,950	2,975	322	203	94	0	0	0
2,975	3,000	327	207	98	0	0	0
3,000	3,050	333	213	104	4	0	0
3,050	3,100	342	221	111	11	0	0
3,100	3,150	350	229	119	18	0	0
3,150	3,200	359	238	126	25	0	0
3,200	3,250	367	246	134	32	0	0
3,250	3,300	376	255	141	39	0	0
3,300	3,350	385	263	149	46	0	0
3,350	3,400	393	272	157	53	0	0
3,400	3,450	402	280	165	60	0	0
3,450	3,500	410	289	173	67	0	0
3,500	3,550	419	297	181	74	0	0
3,550	3,600	427	306	189	81	0	0
3,600	3,650	436	315	197	89	0	0
3,650	3,700	444	324	205	96	0	0
3,700	3,750	453	334	213	104	4	0
3,750	3,800	462	343	221	111	11	0
3,800	3,850	470	353	229	119	18	0
3,850	3,900	479	362	238	126	25	0
3,900	3,950	487	372	246	134	32	0
3,950	4,000	496	381	255	141	39	0
4,000	4,050	504	390	263	149	46	0
4,050	4,100	513	399	272	157	53	0
4,100	4,150	521	407	280	165	60	0
4,150	4,200	530	416	289	173	67	0
4,200	4,250	538	424	297	181	74	0
4,250	4,300	547	433	306	189	81	0
4,300	4,350	556	442	315	197	89	0
4,350	4,400	564	450	324	205	96	0
4,400	4,450	573	459	334	213	104	4
4,450	4,500	581	467	343	221	111	11
4,500	4,550	590	476	353	229	119	18
4,550	4,600	598	484	362	238	126	25
4,600	4,650	607	493	372	246	134	32
4,650	4,700	615	501	381	255	141	39
4,700	4,750	624	510	391	263	149	46
4,750	4,800	633	519	400	272	157	53
4,800	4,850	641	527	410	280	165	60
4,850	4,900	650	536	419	289	173	67
4,900	4,950	658	544	429	297	181	74
4,950	5,000	667	553	438	306	189	81

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2. Use the table to find the tax of a single person with an income of:
a. $4800 $641 **b.** $3960 $496 **c.** $4750 $633

3. At what income level does a single person with two exemptions begin to pay income tax? with three exemptions? $2300
$1600

4. Mike Jones is a carpenter's helper, earning $90 a week, 52 weeks a year. He claims his mother as an exemption. How much income tax should he pay? $ 501

5. After Jim graduated from high school he got a job in a factory at $250 a month. Since he is supporting only himself, he has one exemption. Use the table to calculate his tax. $ 333

6. Mary is a secretary in a doctor's office at a salary of $70 a week. She claims one exemption. How much is her tax? $ 436

7. Martin earned $3400 last year. How much tax should he pay? $ 402

8. The amount left after income tax is paid is called *income after taxes.* What is the income after taxes of a person with one exemption whose earnings per year are:
   a. $4200 $ 3662 **b.** $4975 $ 4308 **c.** $4750 $ 4117

9. The tax on $4000 with one exemption is how many times the tax on $1000 with one exemption? 31.5

10. What is the income after taxes of a person with two exemptions and an annual income of:
    a. $4440 $ 3981 **b.** $3600 $ 3285 **c.** $4200 $ 3776

QUESTIONS FOR RESEARCH AND REPORT

1. If the taxpayer has an income over $5000, he may use a "long form," Form 1040, on which he makes detailed deductions. The deductions he lists are expenditures that in effect lower his income. In the calculation of tax on Form 1040A standard deductions have already been included. On Form 1040 they may include, for example, interest paid on debts, and property taxes paid to local governments. Do you think these deductions should be made from income before income taxes are calculated?

2. In general the purpose for allowing deductions is to adjust the person's taxes to his ability to pay. Obtain a copy of Form 1040, with directions for filling it out, and find out what expenses are deductible from income. State in each case whether you think the deduction is justifiable, and why you think as you do.

3. Under certain conditions medical expenses and costs of prescriptions are deductible. Find out what the conditions are. Can you justify the conditions?

4. Do you think of any expenses that should be deductible that are not? Explain why you think as you do.

5. Explain the difference between a deduction and an exemption.

If you are earning wages or a salary, your employer is required to withhold a certain amount from your pay each payday and turn it over to the government. This is called *withholding tax*; it actually is payment of your income tax. At the end of the year you receive a report (Form W-2) showing your total wages and amount of tax withheld. When you make out your tax return, if your income tax is more than the amount withheld, you pay the difference. If the amount withheld is more than your income tax, you claim a refund from the government.

Currently the withholding tax rate begins at 14%. If you have exemptions, the withholding tax is calculated on 90% of the amount left. The 10% is allowed for a standard deduction.

1. The amount allowed for an exemption depends on the payroll period. If paid daily the amount allowed for an exemption is $1.80 per day. If paid weekly the amount allowed for an exemption is $13.00 per week. If paid monthly the amount allowed for an exemption is $56.00 per month. Mike earns $40 per week at a Farmer's Market. He claims one exemption, himself. Thus, his employer should subtract $13 from Mike's $40 and calculate the withholding tax on 90% of the rest. How much tax should be withheld per week? $ 3.40

2. Mike receives the remainder of his salary after the tax is deducted. How much does he receive per week? $36.60

3. Mike worked for 40 weeks during the year. His annual earnings, before taxes, are reported on his W-2 form. What are his total earnings? $1600

4. How much withholding tax should appear on his W-2 form? $136

5. Jim has a job at the filling station, where he works 2 hours each afternoon on school days, and 6 hours each Saturday, at $1.25 an hour. How much does he earn each week? $ 20

6. Jim can claim an exemption, even though his father claims him as a dependent, because he is a full-time student. The withholding tax is therefore 14% of his earnings after the exemption and standard deduction. How much should be withheld from Jim's weekly wages? $ .88

7. Mildred has a job after school as a cashier at a supermarket. Her salary is $25 a week. She claims one exemption. How much is her withholding tax per week? $ 1.51

8. Mildred worked at the supermarket for 30 weeks last year. How much is her withholding tax that is shown on the W-2 form? $45.30

1. Most merchandise imported from a foreign country is subject to a customs duty. When the Stewarts were in Europe they bought a Renault car in France for \$1100. When they returned to this country they had to pay *customs duty* on the car. This was a tax of $8\frac{1}{2}\%$ of its value. How much was the duty? \$ 93.50

2. A dealer imports cameras from Germany and pays duty of 25% of their cost. What is the duty on a camera for which he pays \$15? \$ 3.75

3. The dealer counts the duty as part of the cost. What must he sell the camera for, if his margin is to be 25% of the selling price? \$ 25

4. A dealer in foreign cars imports Volkswagens from Germany. He pays a wholesale price of \$950 each. The duty is 10%. Transportation is \$200. The retail price of the car is \$1600. What is the margin? \$ 355

5. The purchaser of the car also pays a federal excise tax of 10% of the retail price on new cars, and a state sales tax of 3%. Counting the import duty, how much tax does the purchaser pay? \$ 303

Congress changes the duties from time to time. Here are some examples from a schedule. Where a per cent is listed, it is understood as referring to the purchase price of the article.

Article	*Duty*
Automobiles	New 10%, used $8\frac{1}{2}\%$
Bicycles	$7\frac{1}{2}\%$
Cameras	25%
Chinaware	10¢ per dozen plus 45%
Glassware (table)	30%
Silk	Woven 35%, Raw Free

6. Use the rates in the table to find the duty on the following:
a. Woven silk worth \$637 and 500 lb. of raw silk. \$ 222.95
b. 14 doz. glasses valued at 30 cents each. \$ 15.12
c. 10 doz. plates valued at 75 cents a plate. \$ 41.50
d. 10 bicycles costing \$30 each. \$ 22.50

7. A dealer can buy a set of table glassware from Belgium at \$3 and pay 60¢ shipping charges plus the duty, or he can buy it in this country for \$5 a set. Which way is cheaper, and how much?
Buy in Belgium and save \$ .50.

8. If the dealer is to sell the glassware at a profit, he must charge 35% more than it cost him altogether. How much must he sell it for if he imports it? How much would he charge if there were no duty?
with duty, \$ 6.08; without duty, \$ 4.86

You have seen that several different kinds of taxes are collected by the federal government. This is also true of the state and local governments.

On many articles you purchase, taxes have been paid by the manufacturer, and you pay them as part of the selling price without your knowing it. These are *indirect taxes, collected from the consumer, and paid to the government by the businessman. Federal taxes on entertainment, luxuries, gasoline, transportation, long distance telephone calls,* and so on, within the country, are called *excise taxes.*

Direct taxes, such as income taxes and property taxes, are *paid directly to the government* by the wage earner or property owner.

1. The federal, state, and local governments use a variety of taxes:

	Collected by		
Kinds of Taxes	*Federal*	*State*	*Local*
Direct			
Tax on personal incomes	Yes	Yes	No
Tax on corporation incomes	Yes	Yes	No
Property tax	No	Yes	Yes
Motor vehicle tax	No	Yes	No
Indirect			
Retail sales	No	Yes	Yes
Gasoline	Yes	Yes	No
Tobacco	Yes	Yes	No
Excise	Yes	No	No

Which of the three divisions of government collects revenue from the greatest variety of sources? state

2. Recently, the federal government collected $80 billion in taxes. Of this, $21.5 billion dollars was from taxes on corporation incomes and $46.6 billion from taxes on personal incomes. What per cent came from each source? corporation, 26.9 %; 58.3 % personal, 58.3 %

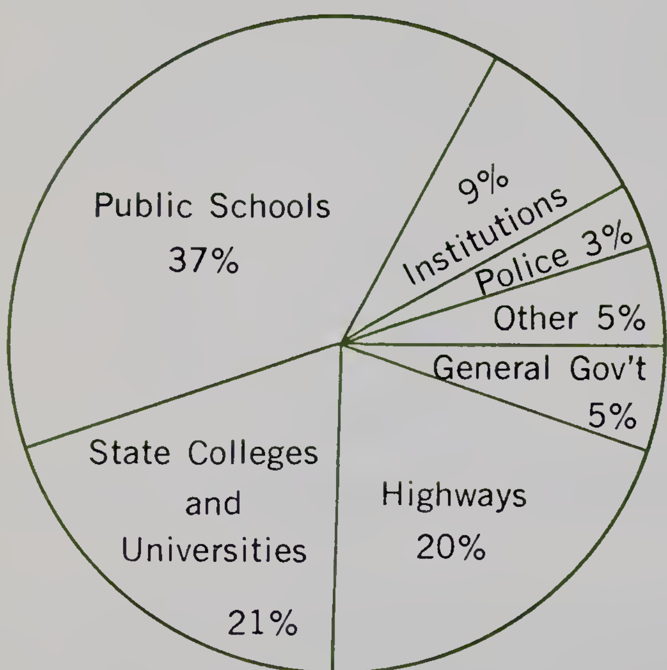
3. In the same period, the federal government collected $3.6 billion in excise taxes on alcohol and tobacco. What per cent was this of the total amount of taxes collected? 4.5 %

4. City and county taxes recently amounted to about 8 billion dollars. Of this, 60% was raised by property taxes. About how much did property taxes amount to? $ 4.8 billion

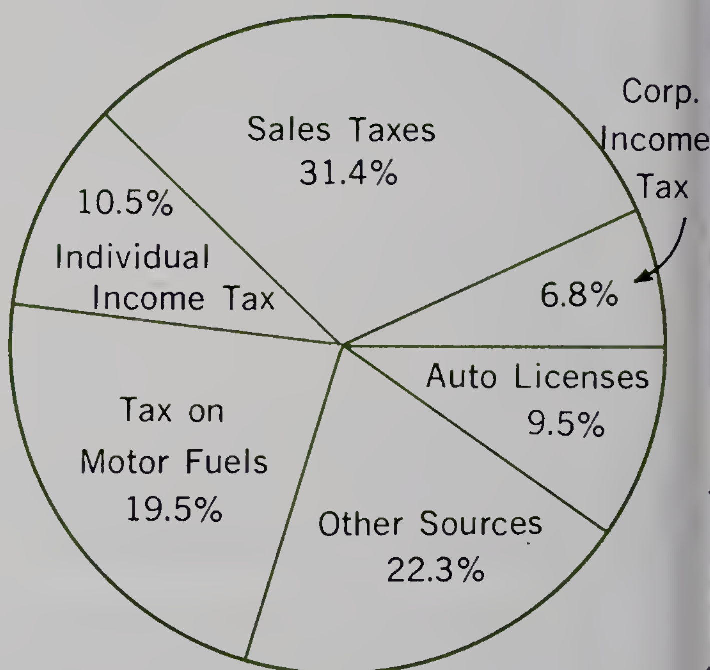
The services provided by a state government are somewhat different from those of the national government. This can be seen from the graph below showing how one state, for example, spent about $410 million in a year.

1. The item for *Public Schools* provides aid to local elementary and secondary school systems so that the quality of education will not suffer in districts less prosperous. How much was spent on this item? $ 151.7 million
2. How much was spent on *State Universities and Colleges*? $86.1million
3. Prepare a table showing the amount spent on each of the other items. See front.
4. The population of the state was 4,500,000. What was the state expenditure per person to the nearest cent? $ 91.11
5. What services are provided by both the state and federal government? Which are provided only by the state? See front.

HOW A STATE BUDGET OF
$410 MILLION WAS DISTRIBUTED



SOURCES OF STATE
INCOME $16,000,000,000



6. The graph on the right shows the sources of state revenue. What is the major one? How much is raised from this source? Sales tax, $ 5,024 million
7. Calculate how much revenue was raised from each of the other sources. See front.
8. The graph on the next page shows a number of states that levy sales taxes at various rates. In a recent year state and local governments collected $7.36 billion in sales taxes. What is the most common rate of sales tax? 3 %

9. The sales tax is calculated to the nearest cent. In a state where the tax is 3%, what is the sales tax on each of the following purchases?

a. \$7.50 23 ¢ **c.** \$15.80 47 ¢ **e.** \$16.90 51 ¢ **g.** \$38.20 \$1.15
b. \$2.55 8 ¢ **d.** \$25.45 76 ¢ **f.** \$22.87 69 ¢ **h.** \$14.35 43 ¢

10. Find the sales tax to the nearest cent on each of the following purchases if made in the state of Illinois:

a. \$2.60 9 ¢ **c.** \$9.40 33 ¢ **e.** \$8.40 29 ¢ **g.** \$11.80 41 ¢
b. \$4.50 16 ¢ **d.** \$7.10 25 ¢ **f.** \$9.90 35 ¢ **h.** \$4.75 17 ¢

11. A family of four in the state of Pennsylvania spends \$40 a month for clothing. How much sales tax does the family pay annually on purchases of clothing? \$24

12. Mary lives in Columbus, Ohio. She works as a sales clerk in a variety store. To save time in calculating the 3% sales tax, she set up a table like this:

Amount of Purchase	*Tax*
Up to 15 cents	None
15 cents to 49 cents	1 cent

See front.

Check her calculations; continue the table to include sales to \$5.

13. The Jones family has an income of \$450 a month. They have budgeted \$42 a month for clothing. The sales tax in their state is 2%. How much is a year's tax on clothing purchases? \$10.08

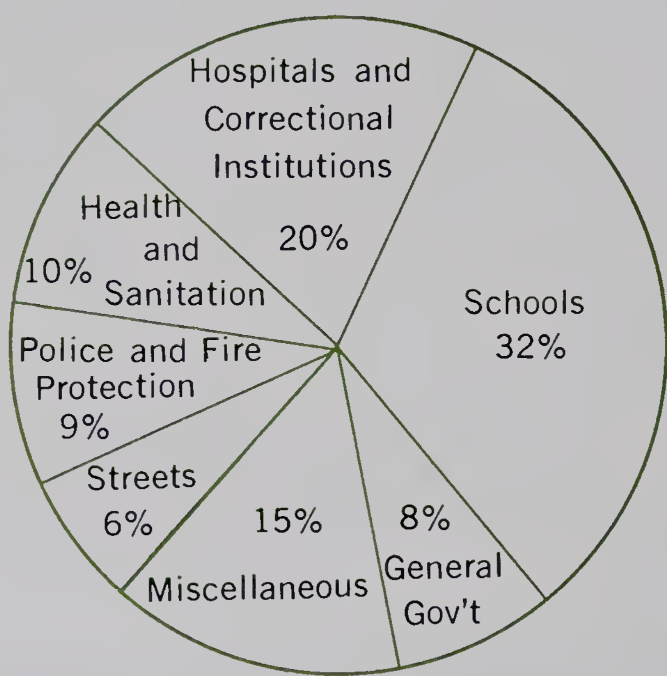
The total amount needed for government activities makes up the *budget*. Most local governments raise money to meet the budget through *property tax* on real estate. In a recent year \$21.3 billion were collected by state and local governments in property taxes. The total value of the property in the governmental unit is called its *assessed valuation*. Each owner pays taxes according to the value of his property.

See front.

1. The Elmtown budget for last year was \$1,200,000. It was divided up as shown below. How much was each item in the budget?

2. The assessed value is often a fixed per cent of its true value. In Elmtown the assessed value is 60% of the true value. Mr. Brown's property has a true value of \$20,000. What is the assessed value of his property? \$12,000

ELMTOWN BUDGET



3. The tax on Mr. Brown's property is 5% of its assessed value. How much is his tax? \$ 600

4. Mr. Brown also owns a plumbing shop in Elmtown. Its assessed value is \$30,000. How much is the tax? \$1500

5. The tax rate may be stated in each of four ways:

In per cent	In dollars per \$1000
In dollars and cents per \$100	In mills per dollar.

(*Remember*: 1 mill = \$.001, which is $\frac{1}{10}$ of a cent.) Thus the tax rate in Elmtown is 5%, or 50 mills per dollar, or \$5 per \$100, or \$50 per \$1000.

In Springdale the rate is 5.5%. Express the rate in mills per dollar; in dollars per \$1000; in dollars and cents per \$100.

55 mills per dollar; \$ 55 per \$1000; \$ 5.50 per \$ 100

6. If Mr. Brown's shop had been located in Springdale instead of Elmtown, what would his tax have been? *$1650*

Funds to meet the Elmtown budget are raised through a property tax. To find the tax rate, the budget is divided by the assessed valuation of the property. We may write a formula to find the tax rate:

$$r = \frac{b}{a}$$

where r is the tax rate, b is the budget, and a is the assessed valuation. The Example shows how to use this formula in finding the tax rate in Elmtown.

EXAMPLE

The total assessed valuation of property in Elmtown is \$24,000,000. The tax rate is the ratio of the budget (the taxes to be raised) to the assessed value. This may be expressed as per cent.

$$\frac{\$1,200,000}{\$24,000,000} = .05 \text{ or } 5\% \qquad \frac{b}{a} = r$$

Therefore the tax rate is 5% of the assessed value. Check the calculation. Is it correct? *Yes*

7. In Evansville the budget is \$24,000,000. The assessed valuation is \$400,000,000. Use the formula to find the tax rate. *6 %*

If you know the budget and tax rate, you can find the assessed valuation by solving the formula for a.

$$a = \frac{b}{r}$$

EXAMPLE

The budget in Ellendale is \$880,000. The tax rate is \$55 per \$1000. What is the assessed valuation?
Expressed as a decimal the tax rate is:

$$r = .055.$$

Using the numbers for letters in the formula:

$$a = \frac{b}{r} \qquad a = \frac{\$880,000}{.055} \text{ or } \$16,000,000.$$

8. Describe the procedure for finding the assessed valuation if you know the budget and tax rate. *Divide the budget by the tax rate expressed as a decimal.*

Use the formulas to find the missing figure in each of the following:

	Assessed Valuation	Budget	Tax Rate
9.	$ 10,000,000	$ 550,000	55 ? mills per dollar
10.	$? $20,000,000	$ 1,000,000	50 mills per dollar
11.	$1,185,000,000	$99,540,000	84 ? mills per dollar
12.	$? $2,938,158	$ 111,650	3.8%
13.	$? $200,000,000	$ 9,500,000	$47.50 per $1000
14.	$ 20,000,000	$ 1,000,000	5 ? %
15.	$ 32,000,000	$ 800,000	$2.50 ? dollars per $100
16.	$? $60,000,000	$ 4,620,000	$77 per $1000

17. Which tax rate in Exercises 9–16 is the smallest? 15 largest? 11

Other Direct Taxes

A wide variety of direct taxes are levied to raise funds. These exercises illustrate some of them. Make a list of others you know about.

1. The city council of Centerville plans to build a bicycle shelter at the community center. To help finance it, they decided to levy a $1.25 tax on each bicycle. If 1564 bicycle owners paid the tax, how much money was collected? $ 1955

2. The average value of the bicycles in Centerville was $25. What was the average rate of tax on each bicycle? 5 %

3. One of the main sources of revenue for building and maintaining state highways is the state motor vehicle tax. The rates vary among the states. Mr. Jensen bought a new car for $3250. His motor vehicle tax was $65. This was what per cent of the cost of the car? 2 %

4. In some states the motor vehicle tax is based on the weight of the car. Mr. Smith's car weighs 4100 lb. and the tax is $1\frac{1}{2}$¢ per lb. How much is the tax? $ 61.50

5. About 6000 towns and cities use parking meters on downtown streets as a way of collecting revenue and controlling parking. Elmtown has installed 600 parking meters, each of which averages $9.25 per month revenue. What is the yearly income from the parking meters? $ 66,600

Part One

A. Change to per cent:

1. .18 18 %	**3.** .003 0.3 %	**5.** .256 25.6 %	**7.** 1.35 135 %
2. 5.7 570 %	**4.** .406 40.6 %	**6.** .002 0.2%	**8.** .025 2.5 %

B. Change to per cent, to the nearest tenth of 1% if there is a remainder:

1. $\frac{5}{6}$ 83.3 %	**3.** $\frac{7}{8}$ 87.5 %	**5.** $\frac{5}{11}$ 45.5 %	**7.** $\frac{5}{3}$ 166.7 %
2. $\frac{7}{13}$ 53.8 %	**4.** $\frac{5}{16}$ 31.3%	**6.** $\frac{9}{17}$ 52.9 %	**8.** $\frac{3}{5}$ 60 %

C. Find the value for n:

1. 5% of 96 is n 4.8	**11.** 15 is n% of 75 20
2. 12 is n% of 40 30	**12.** 40% more than 80 is n 112
3. 20% of n is 30 150	**13.** 22% of 56 is n 12.32
4. 20% less than 40 is n 32	**14.** 32 is 80% of n 40
5. 140% of n is 35 25	**15.** 16 is n% of 40 40
6. 35% more than 160 is n 216	**16.** 0.25% of 800 is n 2
7. 16% of n is 96 600	**17.** 36 increased by 25% is n 45
8. 48 is 20% more than n 40	**18.** 64 is n% more than 40 60
9. 45 is n% less than 50 10	**19.** 80 increased by n% is 96 20
10. 35% of n is 175 500	**20.** n increased by 30% is 78 60

D. Find the interest:

	Principal	*Rate*	*Time*
1.	$200	4%	60 days $ 1.33
2.	$175	6%	90 days $ 2.63
3.	$600	$4\frac{1}{2}$%	6 months $ 13.50
4.	$425.50	5%	80 days $ 4.73
5.	$197.85	$3\frac{1}{2}$%	3 months $ 1.73
6.	$215.40	$4\frac{1}{2}$%	120 days $ 3.23

E. State the tax rate in four ways: 7.4 % ; .74 mills per $1

1. $17,500,000 *Assessed value* $1,295,000 *Budget*
$7.40 per $100 ; $ 74 per $1,000

8.3 % ; 83 mills per $1 ; $ 8.30 per $100 ; $ 83 per $ 1,000

2. $84,750,000 *Assessed value* $7,034,250 *Budget*

3. $29,416,000 *Assessed value* $1,706,128 *Budget*
5.8 % ; 58 mills per $1 ; $ 5.80 per $100 ; $ 58 per $ 1,000

Part Two

A. On a sheet of paper list the numerals 1 through 10. After each numeral list one or more of the words *national, state, local,* or *private industry,* to show which most commonly provides each service or facility listed below. If the facility receives support from several sources, list them.

1. Post office national
2. Filling stations private
3. Library local
4. Sidewalks local
5. Fire protection local
6. Water supply local
7. National defense national
8. Bus depot private
9. Highways national, state
10. Airport national

B. On your paper write the numerals from 1 through 10. Find which word (or words) on the right is defined by each statement on the left and write the word next to the corresponding number on your paper.

1. Federal tax on jewelry, admissions, luggage, etc. excise
2. The amount by which the expenditures of the government exceed the revenue. deficit
3. Income tax deducted by employer from employee's pay, and turned over to the government. withholding tax
4. A certain per cent of the true value of property that is used for taxing purposes. assessed value
5. One tenth of a cent. mill
6. Tax figured as a per cent of an individual's net yearly income, minus exemptions and deductions. income tax
7. Tax collected by seller from purchaser of merchandise. sales tax
8. Local tax on real estate and personal possessions. property tax
9. Person supported by another person. dependent
10. The amount by which the revenues of the government exceed the expenditures. surplus

assessed value
budget
deficit
dependent
excise
exemption
income tax
mill
property tax
revenue
sales tax
surplus
tax
tax rate
withholding tax

STEPS FOR SOLVING APPLIED PROBLEMS

1. Understand the problem.
2. Note what the problem asks for.
3. Look for hidden questions.
4. Estimate a reasonable answer.
5. Set up and solve the conditional sentence(s).
6. Check your answer.

Part Three

1. Last year Mr. Anderson drove his car 9600 miles. He averaged 15 miles to a gallon of gasoline. How many gallons did he use during the year? *640 gal.*

2. Mr. Anderson paid 34¢ per gallon for gasoline. What was the total cost of gasoline? *\$ 217.60*

3. The state gasoline tax was 7¢ per gallon. How much did Mr. Anderson pay in state gasoline taxes? *\$44.80*

4. The federal tax on gasoline was 4¢. How much did Mr. Anderson pay in federal gasoline taxes? *\$25.60*

5. What per cent of the total cost of gasoline was state tax? federal tax? *state, 20.6 % ; federal, 11.8 %*

6. Mr. Anderson paid \$120 for a set of tires for his car. There was a 4% state sales tax. How much was the tax on the tires? *\$ 4.80*

7. In addition to the gasoline taxes (Exercises 3 and 4) and the sales tax on the tires (Exercise 6), Mr. Anderson paid his license fee of \$35 and \$2 for a driver's license. How much did he pay in taxes during the year, to drive his car? *\$112.20*

8. The assessed valuation of property in Woodside is \$60,000,000. The budget is \$3,600,000. State the tax rate in four ways.
6 % ; 60 mills per \$1 ; \$6 per \$100 ; \$60 per \$1,000.

9. Mr. Jenkins has a house in Woodside worth \$20,000. Property is assessed at 25% of its true value. What is the assessed value of Mr. Jenkins' house? *\$ 5000*

10. What is Mr. Jenkins' property tax? *\$300*

11. What is the true value of property in Woodside?
(HINT: \$60,000,000 is what per cent of the true value?) *\$240 million*

12. The Jenkins family has an income of \$5600 annually. They spend \$50 per month on clothing. Clothing is subject to a 3% sales tax. How much per year will their sales tax amount to on clothing? *\$ 18*

Part One

Find the value of n to make each of these a true statement.

A. **1.** $n + 18 = 37$ 19

2. $45 - n = 29$ 16

3. $56 - n = 37$ 19

4. $17n = 289$ 17

5. $\frac{n}{2} = 39$ 78

6. $n \div 14 = 6$ 84

7. $13n = 104$ 8

8. $42 + n = 91$ 49

9. $132 \div n = 12$ 11

10. $12n = 96$ 8

B. **1.** $\frac{n}{7} = \frac{18}{63}$ 2

2. $\frac{17}{119} = \frac{5}{n}$ 35

3. $\frac{21}{n} = \frac{6}{2}$ 7

4. $\frac{n}{96} = \frac{15}{90}$ 16

5. $\frac{7}{12} = \frac{n}{60}$ 35

6. $\frac{18}{24} = \frac{n}{100}$ 75

C. **1.** 16 is $n\%$ of 40 40

2. 125% of n is 65 52

3. 32 is $n\%$ of 80 40

4. 112.5% of n is 144 128

5. 125 is $n\%$ of 1000 12.5

6. n is 45% of 80 36

7. 0.5% of 940 is n 4.7

8. 45% of n is 108 240

9. 0.75% of n is 24 3,200

10. 137.5% of 99 is n 136.125

Part Two

Solve and check.

A. **1.** $x + 14 = 37$ 23

2. $\frac{x}{2} + 3 = 33$ 60

3. $\frac{x}{12} = \frac{1}{6}$ 2

4. $2x + 5 = 93$ 44

5. $\frac{x}{63} = \frac{2}{3}$ 42

6. $\frac{2x}{3} + 14 = 204$ 285

7. $3x = 16 - 5x$ 2

8. $4x + 3 = 655$ 163

9. $5x - 67 = 2x - 1$ 22

10. $3x + 4 = 71 + 2x$ 67

B. Solve for x and y by any method and check.

1. $2x + 3y = 7$ $^{-}1, 3$
   $2x + y = 1$

2. $x - y = 2$ $3, 1$
   $2x + 3y = 9$

3. $2x + y = 8$ $14, {}^{-}20$
   $3x + 2y = 2$

4. $2x - 3y = {}^{-}3$ $0, 1$
   $5x + 2y = 2$

5. $x - 2y = 3$ ${}^{-}15, {}^{-}9$
   $3x - 5y = 0$

6. $4x + 9y = 7$ $\frac{-1}{2}, 1$
   $2x - 2y = {}^{-}3$

Part Three

1. The Anderson's house and garage cost $30,000 to build. The house cost 9 times as much as the garage. What was the cost of each? $27,000; $3000
2. Jim and Mary Brown have monthly allowances totaling $4.50. Mary is the older and her allowance is twice that of Jim. How much is the allowance of each? Mary $3.00; Jim $1.50
3. The perimeter of a rectangle is 240 rods. The length is three times the width. What are its dimensions? 30 rd. by 90 rd.
4. Miss Olsen's mathematics class has 27 pupils. There are 5 more girls than boys in the class. How many of each are there? 11 boys and 16 girls
5. Mike and his father made a trip of 720 miles in their car. The distance Mike drove was $\frac{5}{4}$ the distance his father drove. How far did each drive? Mike, 400 mi.; father, 320 mi.
6. The length of a rectangle is 6 inches greater than its width. Its perimeter is 108 inches. What are its dimensions? 24 in. by 30 in.

Part Four

1. How many cubic inches of steel are removed in drilling a hole 2.5 in. in diameter through a 4 inch steel plate? $19\frac{5}{8}$ cu. in.
2. Mr. Erickson sold a rectangular piece of land 60 rd. long and 80 rd. wide at $125 an acre. How much did he receive for it? (160 sq. rd. = 1 acre.) $3750
3. What will it cost to excavate the basement for Mr. Adams's house 32 ft. by 24 ft. to a depth of 9 ft., at $4.50 a cubic yard? $1152
4. A cylindrical water tank in Elmtown is 30 ft. high and 28 ft. in diameter. How many gallons of water will it hold? 138,600 gal.
5. Steel weighs .29 lb. per cubic inch. What is the weight of a steel ball bearing 2″ in diameter? 1.2 lb.

6. How much will a steel rod weigh that is 4″ in diameter and 6 ft. long? 262.3 lb.

7. A spherical water tank has a diameter of 16 feet. Water weighs about 62.4 pounds per cubic foot. How many tons will the water in the tank weigh when the tank is full? approximotely .67 tons

8. Aluminum weighs 2.7 times as much as an equal volume of water, and steel is 7.87 times as heavy as water. A beam is 2″ × 8″ × 12′. What will this beam weigh if it is made of aluminum? of steel? 224.6 lb.; 654.8 lb.

9. If a rectangular storage bin is 18 ft. long, 9 ft. wide, and 7 ft. high, find the number of cubic yards of storage space in this bin. 42 cu.yd.

Part Five

1. If a number is doubled and this result decreased by 5, the result is 51. What is the number? 28

2. Divide 42 into two parts so that one part exceeds the other by 16. 13; 29

3. The length of a rectangle is 5 times its width. Its perimeter is 216′. What are its dimensions? 18 ft. by 90 ft.

4. If three times a certain number is increased by 9, the result is 48. What is the number? 13

5. A chain 615 feet long is cut into two parts so that the first section is 14 times as long as the second. What is the length of each part? 41 ft., 574 ft.

6. Juan and Terry bought a secondhand car for $136. Juan paid 3 times as much as Terry. How much did each pay? Terry, $34; Juon, $102

7. If the price of an $84 tape recorder is decreased $\frac{1}{6}$, how much will it sell for? $70

8. Twice a certain number decreased by 17 is the same as increasing that number by 14. What is the number? 31

9. The sum of two numbers is 271. One number is 5 less than 3 times the other. Find the numbers. 69; 202

10. There are 68 more boys than girls enrolled in MHS. The total enrollment is 1286. How many girls are enrolled? 609 girls

11. Seven times a certain number decreased by 29 is equal to twice the number increased by 76. What is the number? 21

12. How many cubic feet are in $14\frac{1}{3}$ cubic yards? 387

13. What is the total surface of a cylindrical storage tank whose length is 14′ and diameter is 6′? 320 sq. ft.

CHAPTER TEN

MORE ABOUT ALGEBRA

WORDS TO WATCH FOR

binomial	*cryptography*	*graphic*	*quadratic*
cipher	*elimination*	*hyperbola*	*substitution*
concentration	*ellipse*	*parabola*	*systems of equations*
conic section	*focal point*	*paraboloid*	*trinomial*

Nearly all industries and governmental agencies give qualifying tests to job applicants. These tests include questions that require problem-solving skill. As you build up a background of mathematical relationships, your understanding of such questions will increase. One purpose of this chapter is to expose you to these relationships so that you can develop and increase your skill in problem solving.

Algebra can be a powerful instrument for solving verbal problems. A problem that is complicated to solve by arithmetic methods becomes simple to solve by algebraic analysis. In this chapter we will study a few problem "types" that yield interesting results and reveal basic relationships. You will have an opportunity to summarize what you have learned about the uses of algebra and, in the process, to develop some new concepts and approaches to problem solving.

One of the most important features of algebra is the conditional equation. Let us begin by solving some equations. If you have difficulty with these exercises, review page 235.

Solve and check:

1. $3x - 2 = 16 + x$ 9

2. $4 - 2s = 7s + 16$ $^{-}1\frac{1}{3}$

3. $\dfrac{3a}{2} + 5 = a + 11$ 12

4. $5r + 11 = 7r + 15$ $^{-}2$

5. $2x - 5 = 5x - 11$ 2

6. $2(3x - 5) = 3(x + 7)$ $10\frac{1}{3}$

7. $5a - 11 = 2 - (3a - 2)$ $1\frac{7}{8}$

8. $2s - 4 = 6s - 12$ 2

9. $2(3r + 5) = 5 + 2(r - 4)$ $^{-}3\frac{1}{4}$

10. $\frac{3p}{5} - \frac{1}{4} = \frac{p}{2} + \frac{2}{3}$ $9\frac{1}{6}$

11. $\frac{2p}{5} + 9 = p + \frac{3}{4}$ $13\frac{3}{4}$

12. $6r + 17 = 3r - 12$ $^{-}9\frac{2}{3}$

13. $4(5 - 2x) = 2(3 - 5x)$ $^{-}7$

14. $2x - 9 = 5 - 5x$ 2

15. $3r - 5 = 2r + 1$ 6

16. $8y + 7 = {}^{-}16y + 7$ 0

17. $4 - 3r = 18 - 10r$ 2

18. $2t - 6 = \frac{t}{2} + 4$ $6\frac{2}{3}$

19. $\frac{7}{15} = \frac{x + 2}{45}$ 19

20. $2 - \frac{x}{3} = \frac{10}{9}$ $2\frac{2}{3}$

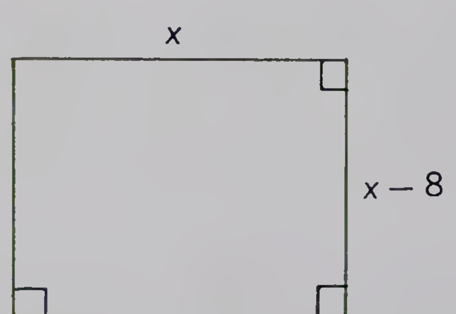
Algebraic Language

If you have difficulty in translating English words to algebraic phrases, use some simple examples to guide your thinking.

EXAMPLES

1. Represent 12 less than a number.
How would we write 12 less than 14? Using numbers may provide a clue. (12–14) would give us ($^{-}2$) and you know this is not right.
So we would not write ($12-x$). What about (14–12)?
This would give us $^{+}2$ so ($x-12$) would be the representation called for.
More numerical examples of the same type can help you see what is being requested. Try 14 less than a number; 2 more than a number.
Did you get $x-14$? $n+2$?

2. The length of a field is x feet and the width is 8 less than the length. Represent the width, perimeter, and area in terms of x. We will use a diagram to help us visualize the information.



length is x
width is $(x - 8)$
perimeter is $(4x - 16)$
area $= lw = x(x - 8) = x^2 - 8x$

1. Using x to represent a real number, write a mathematical phrase for:

a. 7 less than a number $x-7$

b. 13 more than a number $x+13$

c. 5 times a number $5x$

d. $\frac{3}{8}$ of a number $\frac{3x}{8}$

e. the square of a number x^2

f. the square root of a number $\sqrt{x}$

g. 25 divided by a number $\frac{25}{x}$

h. 5 more than twice a number $2x+5$

i. 8 less than $\frac{2}{3}$ of a number $\frac{2}{3}x-8$

j. 4 times a number, decreased by 16 $4x-16$

k. 15 times a number, increased by 18 $15x+18$

l. 3 less than 4 times a number $4x-3$

2. A field is x rods in length. The width is $\frac{2}{3}$ the length. Express the width and perimeter in terms of x. What is the area? $W=\frac{2x}{3}$; $P=2x+\frac{4x}{3}$; $A=x(\frac{2x}{3})$

3. A field is x rods in width. Its length is 48 rods greater than its width. Express its length and perimeter and area in terms of x. $l=x+48$; $P=2(x+x+48)$; $A=x(x+48)$

4. Henry earned $\frac{2}{3}$ as much as Jim, who earned x dollars. How much did both earn together? $x+\frac{2x}{3}$

5. Mike is x years old. In terms of x, how old will he be in 6 years? $x+6$

6. Helen earned x dollars and spent $\frac{2}{3}$ of it to purchase a camera. In terms of x, how much does she have left? $\frac{x}{3}$

7. A truck driver travels for 8 hours at the rate of x miles per hour. In terms of x, how far has he traveled? $8x$

8. A Boy Scout troop can hike x miles an hour. How far can the boys hike in $\frac{1}{2}$ hour? $\frac{x}{2}$ in 30 minutes? $\frac{x}{2}$ in 10 minutes? $\frac{x}{6}$ in y minutes? $\frac{x(y)}{60}$

9. It takes Henry x hours to mow the lawn. What part of it can he do in 1 hour? $\frac{1}{x}$ in 3 hours? $\frac{3}{x}$ in h hours? $\frac{h}{x}$

10. The distance from Baldwin to Clear Lake is x miles. How long will it take to drive the distance at 50 miles per hour? $\frac{x}{50}$

11. Helen can type a certain manuscript in 7 hours. What fraction of it can she do in 1 hour? $\frac{1}{7}$ in 2 hours? $\frac{2}{7}$ in x hours? $\frac{x}{7}$

12. A car is traveling at the rate of 48 miles per hour. How far will it travel in x hours? $48x$

13. From Los Angeles to New York is 2550 miles by air. How long does it take a plane flying non-stop at a rate of x miles an hour? $\frac{2550}{x}$

The same algebraic phrase might be used to describe many relationships. If we know from a problem that John earned x dollars and Henry earned \$7 less than $\frac{2}{5}$ as much, $\left(\frac{2x}{5} - 7\right)$ would describe the amount of money Henry earned. The same phrase might be used to describe the width of the field that is 7 rods less than $\frac{2}{5}$ of its length of x rods.

Write a familiar situation or relationship which each of these phrases might describe. *Answers will vary.*

1. $\left(\frac{1}{2}x + 25\right)$	**5.** $\left(\frac{x}{3} + 13\right)$	**9.** $\left(\frac{2x}{3} - 5\right)$
2. $\left(\frac{1}{2}x - 9\right)$	**6.** $\left(\frac{x}{5} + 11\right)$	**10.** $\left(2x - \frac{x}{2}\right)$
3. $(2x + 3)$	**7.** $\left(\frac{3x}{7} - \frac{5}{2}\right)$	**11.** $(17 - 11x)$
4. (x^2)	**8.** $\left(x + \frac{x}{2}\right)$	**12.** $\left(18 - \frac{x}{2}\right)$

Write an equation for each of the following exercises, using x to represent "a certain number." Then solve for x in each exercise.

1. The sum of a certain number and 7 is 19. $x + 7 = 19$; 12

2. Twice a certain number decreased by 3 equals 21. $2x - 3 = 21$; 12

3. If twice a certain number is increased by 3, the result is the same as if the same number is increased by 9. $2x + 3 = x + 9$; 6

4. If three times a certain number is increased by 1, the sum is equal to twice the same number increased by 4. $3x + 1 = 2x + 4$; 3

5. Twice a certain number is equal to $\frac{1}{3}$ of the same number increased by 20. $2x = \frac{x}{3} + 20$; 12

6. If you take $\frac{3}{4}$ of a certain number, the result is equal to half the same number increased by 1. $\frac{3x}{4} = \frac{x}{2} + 1$; 4

7. If you take $\frac{3}{5}$ of a certain number increased by 2, the result is equal to half the same number increased by $2\frac{1}{2}$. $\frac{3x}{5} + 2 = \frac{x}{2} + 2\frac{1}{2}$; 5

8. If you take $\frac{3}{4}$ of a certain number decreased by 1, the result is equal to $\frac{1}{2}$ of the same number. $\frac{3x}{4} - 1 = \frac{x}{2}$; 4

9. If 18 is divided by a certain number, the quotient is 3. $\frac{18}{x} = 3$; 6

10. If a certain number is subtracted from 15, the difference is four times the same number. $15 - x = 4x$; 3

Set up an equation for each exercise. Remember to specify what the variable represents. Then solve the equation and check your answer in the original problem.

1. Jim and Harry split \$38 so that Jim gets \$19 more than Harry. How much does each receive? Jim \$ 28.50, Harry \$ 9.50

2. One number is twice as great as a second number. If 2 is added to the smaller number and 16 is subtracted from the greater, the two numbers will be equal. What are the numbers? 36, 18

3. The perimeter of a triangle is 24 inches. The longest side is 4 inches longer than the shortest, and the third side is two inches longer than the shortest. Find the length of each side. 6 in., 10 in., 8 in.

4. The measures of the angles of a triangle total 180°. In a certain triangle the measure of the greatest angle is three times that of the smallest, and the measure of the third angle is twice that of the smallest. Find the measure of each angle. 30°, 90°, 60°

5. A rectangular field is twice as long as it is wide. The perimeter is 480 yards. Find the dimensions of the field. 80 yd., 160 yd.

6. A farmer measures a rectangular field and finds that he can enclose it with 1400 feet of fencing. The length of the field is 200 feet greater than the width. Find the dimensions of the field. 250 ft., 450 ft.

7. If you decrease twice a certain number by 7, then the result is the same as increasing the original number by 8. What is the number? 15

8. **a.** Mr. Harvey is twice as old as his son Bill. If Bill's present age is x years old, represent Mr. Harvey's age in terms of x. $2x$
   b. Represent Mr. Harvey's age five years from now. Do the same for Bill. Mr. Harvey, $2x+5$; Bill, $x+5$
   c. Five years from now, the sum of their ages will be 70 years. What is the present age of each? 20 yr., 40 yr.

9. Find three consecutive numbers such that twice the second added to seven times the first is 168 more than the third. 21, 22, 23
   HINT: 3, 4, 5 are consecutive numbers. How would you represent the next consecutive number after x? $x+1$

10. Separate 91 into two parts so that the larger is six times the smaller. 13; 78

11. In a school election one candidate received 117 more votes than the other. The number of votes cast was 719. How many votes did each candidate receive? 418; 301

Signed Numbers and Equations

A. *Add:*

1. $^{+}5 + (^{+}9)$ $+14$
2. $^{+}18 + (^{-}16)$ $+2$
3. $^{-}15 + (^{-}22)$ -37
4. $^{-}27 + (^{+}63)$ $+36$
5. $^{-}27 + (^{-}63)$ -90
6. $^{+}16 + (^{-}32)$ -16
7. $^{+}27 + (^{-}27)$ 0
8. $^{-}18 + (^{+}15)$ -3
9. $^{+}22 + (^{+}36)$ $+58$
10. $^{-}35 + (^{+}35)$ 0
11. $^{-}16 + (^{-}24)$ -40
12. $^{-}27 + (^{-}56)$ -83

B. *Subtract:*

1. $^{+}16 - (^{+}25)$ -9
2. $^{-}27 - (^{-}40)$ $+13$
3. $^{+}35 - (^{-}29)$ $+64$
4. $^{-}30 - (^{-}35)$ $+5$
5. $^{-}36 - (^{-}25)$ -11
6. $^{+}27 - (^{+}15)$ $+12$
7. $^{-}29 - (^{-}19)$ -10
8. $^{+}49 - (^{-}21)$ $+70$
9. $^{-}31 - (^{-}30)$ -1
10. $^{+}48 - (^{+}16)$ $+32$
11. $^{-}50 - (^{-}19)$ -31
12. $^{+}45 - (^{+}30)$ $+15$

C. *Multiply:*

1. $^{+}16 \cdot (^{+}8)$ $+128$
2. $^{+}14 \cdot (^{-}5)$ -70
3. $^{-}9 \cdot (^{-}14)$ $+126$
4. $^{-}7 \cdot (^{+}16)$ -112
5. $^{-}8 \cdot (^{-}18)$ $+144$
6. $^{+}15 \cdot (^{-}6)$ -90
7. $^{+}19 \cdot (^{+}6)$ $+114$
8. $^{-}12 \cdot (^{-}12)$ $+144$
9. $^{-}11 \cdot (^{+}9)$ -99
10. $^{+}18 \cdot (^{-}8)$ -144
11. $^{-}12 \cdot (^{-}8)$ $+96$
12. $^{+}7 \cdot (^{+}15)$ $+105$

D. *Divide:*

1. $^{+}24 \div (^{+}8)$ $+3$
2. $^{+}66 \div (^{-}3)$ -22
3. $^{-}85 \div (^{+}17)$ -5
4. $^{-}49 \div (^{-}7)$ $+7$
5. $^{+}56 \div (^{-}7)$ -8
6. $^{+}36 \div (^{+}9)$ $+4$
7. $^{-}144 \div (^{-}8)$ $+18$
8. $^{-}48 \div (^{+}3)$ -16
9. $^{-}108 \div (^{-}9)$ $+12$
10. $^{-}96 \div (^{+}6)$ -16
11. $^{+}72 \div (^{+}12)$ $+6$
12. $^{+}64 \div (^{-}8)$ -8

E. Solve and check.

9 **1.** $x - 6 = 12 - x$

$1\frac{2}{5}$ **2.** $\frac{5x}{6} + 2 = 3\frac{1}{6}$

5 **3.** $7x + 3 = 5x + 13$

24 **4.** $\frac{2x}{3} - 10 = \frac{x}{3} - 2$

3 **5.** $2x - 5 = x - 2$

3 **6.** $\frac{3x}{4} - 1\frac{1}{4} = \frac{x}{4} + \frac{1}{4}$

If your performance on the test was satisfactory, you may work in the Experts' Corner on the following page. If you need practice on one or more of the parts in the test, turn to the Practice Exercises on page 491.

Cryptography: Short Messages

When we encounter a fairly long message in cipher, as we learned in an earlier chapter, and the letter or symbol count shows that it is a substitution cipher, we can decipher it by depending largely on letter frequencies. Letters of high frequency—*e*, *t*, *o*, *a*, and *n*—are identified by a little experimentation and those of next highest frequency—*i*, *r*, *s*, and *h*—are also not difficult to recognize.

When we have only a sentence to work on, however, the letter count is not reliable. It is useful to show that we have a substitution cipher to deal with and also to show which letters are probably among those of high frequency. But it does not lead to any reliable conclusions.

If the word grouping is retained in a cipher message, there are several weak spots that can be worked on to help decipher the message. While a cipher intended for serious use seldom retains word groupings, we will retain them to illustrate the various approaches to decipherment.

Two procedures you should use in the process of decipherment are:

1. Copy the cipher message with spaces between the lines so you can insert the letters in the clear as you identify them.
2. Write the alphabet with space over the letters to insert the cipher representing each letter as you discover it, like this:

   Cipher: _ _ _ _ _ _ ···
   Clear: *a b c d e f* ···

In so doing, you may discover the key to the cipher and save yourself a lot of time.

Here is the cipher message:

DXPGZ RK ADO XJO DXPGZ RK PBQD HXFDQ X HXJ EDXGREZ UDXGREZ XJO UBQD

It is a good plan to proceed in systematic steps:

Step 1. Make the letter count and identify those with highest frequency. The count is:

A – 1	F – 1	K – 2	P – 3	U – 2
B – 2	G – 4	L – 0	Q – 3	V – 0
C – 0	H – 2	M – 0	R – 4	W – 0
D – 8	I – 0	N – 0	S – 0	X – 9
E – 3	J – 3	O – 3	T – 0	Z – 4

From this tabulation we can say only that from among those of highest frequency—X, D, G, R, and Z—we will probably find letters representing *e, t, o, a, n, i, r, s,* or *h,* although not all of them. We will, however, give these letters of highest frequency our first attention.

Step 2. Check for one-letter words. The most common one-letter word is *a,* with *i* and *o* a distant second and third. Try X = *a*

Step 3. Examine the two-letter combinations, first the two-letter words. The most common are: *of, to, in, it, is, be, as,* and *at.* RK has a high-frequency initial letter. Probably it is *to.* Try R = *t,* K = *o.* Since R = *t,* and X = *a,* probably D = *e,* as a high-frequency letter.

Now examine the message for the diphthong *th.* We find two cases of RE. This may be *th.* Try E = *h.*

Next we look for double letters. If there were any, they would be *ss, ee, tt, ff, ll, mm,* or *oo,* in order of frequency. However, we do not find any double letters in this cipher.

Step 4. Examine the three-letter words. The most common are: *the, and, for, are, but,* and *not.* We see that XJO is a three-letter word beginning with *a.* Try *and,* J = *n,* O = *d.*

Step 5. Word patterns. If you have inserted the letters you discovered above over the corresponding letters in the cipher, you will find a number of words with only one letter missing. A little trial-and-error will reveal the missing letter. A little practice will make this easier for you. For example: X HXJ is *a* □ *an.*

This might be *a man* or *a can*; more likely *a man.* Try H = *m.* EDXGREZ is *hea*□*th*□; probably *healthy.* If so, G = *l,* Z = *y.* Then UDXGREZ is □*ealthy.* U = *w.* Early to bed and early to rise makes a man healthy wealthy and wise.
Finish the decipherment.

Several other techniques that we did not make use of are:

1. The use of the probable word. If a sentence begins with a three-letter word, it is very likely to be *the.* Frequently you can spot *is* near the beginning of a sentence. Other possibilities will occur to you as you look over a ciphered message and try to picture the words.

2. The meaning of the sentence. Often the message will begin to make sense when it is only partially deciphered.

3. The key to the cipher. By inserting the letters you have deciphered over the letters of the alphabet—since D = *e,* you write D over *e*—you may break the cipher by discovering that the key is a word written over the beginning or the end of the alphabet, with the rest of the letters written in order over the remainder.

In the present case, the cipher is:

Cipher: X A N O D I T E B C F G H J K L M P Q R S W U V Z Y
Clear: *a b c d e f g h i j k l m n o p q r s t u v w x y z*

The final letters of the alphabet were jumbled to avoid having the cipher and the clear identical, although this does not matter.

It should also be noted that all our guesses were correct as we deciphered the message. Here we omitted consideration of the wrong guesses to save time and space. Some of your guesses will be wrong—perhaps most of them. These will be revealed to you by words ending in *i*, *u*, or *q* or *v*, two-letter words beginning with *e* or *r*, or impossible letter combinations in a word. Don't let it bother you. You will have learned what does *not* work.

Here are some cipher messages to practice on. Be careful to use the steps. See how much easier the decipherment becomes with experience. See front for exercises 1 through 13.

1. SCB GRH YCJ AQ RLORAU SJ GRDB R GAQSRDB PRH RPCABZB FASSFB.

2. VQ VP JG HJ SPE QJ NMISE UVQA QAE VHETVQNODE; UAEH UVHQEM RJFEP KSQ JH YJSM JTEMRJNQ.

3. E GCPPGT GTEMICIR CN E AEIRTMJSN PBCLR, QSP IJIT EP EGG CN OEPEG.

4. PHU JGEW PHBGS XJMOU PHVG FVDBGS V FBOPVDU BO GJP PMWBGS VP VEE.

5. MHAMOBLPN QV BCARE XBLV, CQAMOBLPN QV BCARE FPBLV BZG FONLEAMOBLPN QV BCARE EOBLV.

6. ME MY HARDALEE MO X UXIIMKXS NELE X TIMOL XIW ORAT?

7. NZ KNB NPG ABM YBGH NVG NZPR BCZE GBDZHNVAU NPG AB NZPR HB YBGZ.

8. ABC DCEA FGHIJKJLHAJMO MK H NPMNBCA JE AM BHQC H RMMS TCTMPU.

9. AZP EVPTBTGAPBA VTMNZAPC XB TA AZP PJEPGBP FL FMC PGPUXPB.

10. V MI AHMLSZ OG AHJZQPOMHJ WGD M IMH GY WGHGQ KGASJ OMTZ M HZDPFMFZQ VH WVP WMHJP DVOWGAO M PWAJJZQ GY JVPXAPO.

11. NOJJVOZW, VB PVYW, VL PVRW O UKWP VB MXW NVULM DY O QOMMPW.

12. FUR TRADER FY FLHR ORTDNDIR DA SREULSA FUR BERLFRAF PRLFCER KUDNU TDAFDIBCDAURA ORI PEYO LIDOLGA.

13. PTOBZ QVKNSYG MV GVKQ YEMMYO AVG
BWC AOBM SEX ISOW SO PWOOFOP
SO VWYG CVOP EM MV BWWVG
AODBKPO SO ZWVIP EM MOBPOP.

Analysis of work problems shows an important relationship. If John takes 20 minutes to rake the lawn and Joe takes 40 minutes to do the same task, would it take 30 minutes to do the task together? This may seem logical, but you may notice that John could do the entire lawn in only 20 minutes. Let's see how we solve this type of problem.

> John can do a certain task alone in 6 hours. Bill can do the same task in 8 hours. How long will the task take if the two boys work together?

It is evident that working together each will do a certain fraction (part) of the work. We need to find what this fraction is. It is also clear that when the task is done, the individual fractional parts will add to 1. That is, if one person does $\frac{2}{5}$ of the job, the other person has done $\frac{3}{5}$.

The fractional portions that all participants do in completing a certain task will have a sum of 1.

We will now determine what fraction of the task each one does.

If John does the task in 6 hours, he does $\frac{1}{6}$ of it in 1 hour.

If Bill can do the task in 8 hours, he does $\frac{1}{8}$ of it in 1 hour.

If they work together for 2 hours, the fractions are $\frac{2}{6}$ and $\frac{2}{8}$.

$\frac{3}{6}, \frac{3}{8}$; $\frac{4}{6}, \frac{4}{8}$; $\frac{x}{6}, \frac{x}{8}$

What are the fractions if they work together 3 hours? 4 hours? x hours?

If the whole task takes T hours, then John does $\frac{T}{6}$ of the task and Bill does $\frac{T}{8}$ of the task. How do you know $\frac{T}{6} + \frac{T}{8} = 1$? *One is the whole task.*

Now we can analyze the problem as follows:

Let T = the amount of time taken together to do the task.

Then $\frac{T}{6} + \frac{T}{8} = 1$	The fractional parts have a sum of 1.
$24\left(\frac{T}{6} + \frac{T}{8}\right) = 24 \cdot 1$	We multiply by the common denominator to obtain whole numbers.
So $4T + 3T = 24$	
$7T = 24$	$f_1 f_2 = p$
$T = \frac{24}{7} = 3\frac{3}{7}$.	It will take the boys $3\frac{3}{7}$ hours together.

EXAMPLE

George can paint a garage alone in 10 hours. Alone, Martin can paint the same garage in 6 hours. How many hours will it take them working together?

Let T = the amount of time taken together

In T hours George can do $\frac{T}{10}$ of the job; Martin can do $\frac{T}{6}$ of it.

Then $\frac{T}{10} + \frac{T}{6} = 1.$

$$30\left(\frac{T}{10} + \frac{T}{6}\right) = 30 \cdot 1$$

We multiply both sides of the equation by the common denominator 30.

$$3T + 5T = 30$$

$$8T = 30$$

$f_1 f_2 = p$

$$T = 3\tfrac{3}{4}.$$

The time taken together is $3\frac{3}{4}$ hours.

ANALYSIS OF WORK PROBLEMS

one task

$A = \frac{T}{r_1}$	$B = \frac{T}{r_2}$

$$\frac{T}{r_1} + \frac{T}{r_2} = 1 \text{ OR}$$

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{T}$$

By using a similar analysis, the problem at the top of the previous page in the introduction has an answer of $13\frac{1}{3}$ minutes.

Solve. Follow the procedures outlined. Be sure to specify what the variable represents.

1. Harry can paint the outside of his house in 6 days. His younger brother would take 8 days to do the same task. How long would it take the two boys if they worked together? $3\frac{3}{7}$ *days*

2. Bill can spade a flower garden in 6 hours. Harry would take 4 hours. Their father would take 3 hours. If all three worked together, how long would it take? $1\frac{1}{3}$ *hr.*

3. Mary can dust and clean the house in 80 minutes. Her mother would take 1 hour. How long will it take if they work together? $34\frac{2}{7}$ *min.*

4. The cold water tap can fill the bathtub in 6 minutes. The hot water tap will fill it in 12 minutes. If both taps are open, how long will it take to fill the tub? *4 min.*

5. John can wax the car by himself in 4 hours. His younger brother would take 8 hours. How long would it take if they worked together?

6. Larry can mow and clip the lawn in 40 minutes and his brother would take 50 minutes. His father had mowed $\frac{1}{2}$ the lawn when he was called away on business. Larry and his brother working together finished the job. How long did it take to finish the job? HINT: Let the sum of the fractional portions equal $\frac{1}{2}$. *$11\frac{1}{9}$ min.*

7. Mary can make enough sandwiches for a party in 30 minutes. Helen would take 40 minutes, while it would take Dorothy one hour. How long would it take if all three girls worked together? *$13\frac{1}{3}$ min.*

8. Suppose the cold water tap fills a tub in ten minutes, and the hot water tap takes 12 minutes. The drain will empty the tub in 15 minutes. How long would it take to fill the tub if both taps are running and the drain is open? HINT: Since the drain is doing the opposite operation of the taps, the fractional part done by the drain should be negative in sign. *$8\frac{4}{7}$ min.*

9. An open tank can be filled by one pipe in 4 hours and drained by another in five hours. If by mistake both pipes were left open, how long would it take to fill the tank to overflowing? *20 hr.*

10. Bill can do a certain job in 21 hours. Three hours after he started, Fred arrived to help him finish. Together they finished the job in 5 hours. How long would it have taken Fred alone to do the job? HINT: What part did the boys do together? *$8\frac{1}{13}$ hr.*

11. Can you see any relationship between the given numbers and the answer obtained in exercises 1, 3, 4, and 5? also in 2 and 7? *See front.*

12. Machine A can print 5000 posters in 2 hours and machine B can do the same job in 40 minutes. How long would it take to print 5000 posters using both machines together? *30 min.*

13. John can make 30 cabinets in an eight hour day while Tim can make 30 cabinets in five hours. How long would it take them to produce 30 cabinets working together? *$3\frac{1}{13}$ hr.*

14. Claude can rake the lawn in an hour and a half while his brother takes an hour. How many minutes will it take to rake the lawn working together? *36 min.*

15. Machine A can produce a given number of units in 5 hours, machine B does the same task in 4 hours, and machine C takes 8 hours. How long would it take the three machines working together to produce the given number of units? *$1\frac{17}{23}$ hr.*

Let's examine relationships that can be found in problems involving distance, rate, and time. You are familiar with the formula $d = rt$, and its equivalent forms $r = \frac{d}{t}$, and $t = \frac{d}{r}$. What do the variables represent? How is each form used? Filling out a chart and making a diagram that conforms to the conditions of the problem, as shown below, will be helpful in organizing the data and writing a descriptive equation.

EXAMPLES

1. Fred and Bill leave the house at the same time to take bicycle trips. They are going in opposite directions. Fred travels at a rate of 12 miles per hour and Bill travels at 15 miles per hour. In how many hours will they be 108 miles apart? Let x = number of hours traveled.

	d	$= r$	$\times t$
Fred	$12x$	12	x
Bill	$15x$	15	x

$D_F + D_B = D_{Total}$

What do the subscripts describe?

$$12x + 15x = 108 \quad \text{Why?}$$
$$27x = 108$$
$$x = \frac{108}{27} = 4 \text{ hours}$$

2. A freight train traveling east at 30 mph left Hillsdale. Three hours later a passenger train traveling 50 mph left Hillsdale on a parallel eastbound track. How long will it take the passenger train to overtake the freight train? How far from Hillsdale will the event take place?
Let t represent the time the passenger train travels.

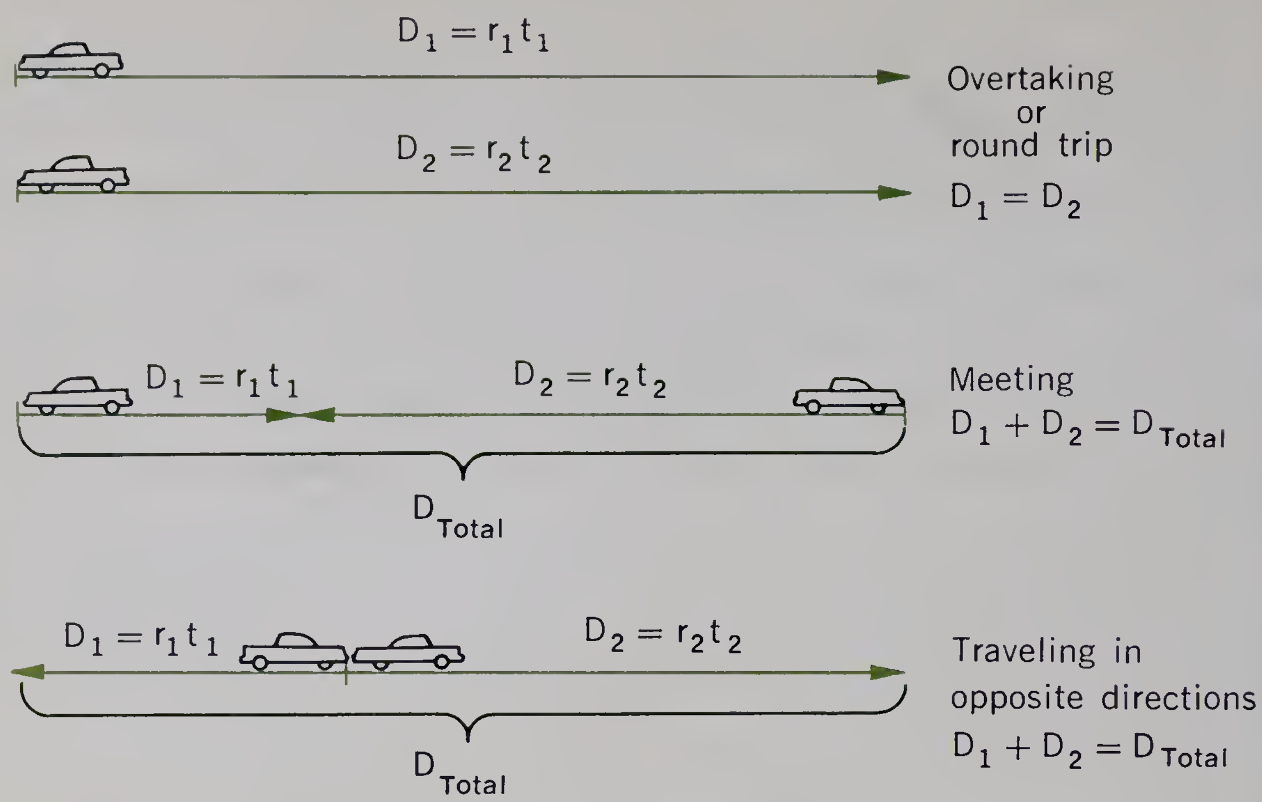
	d	$= r$	$\times$ t
freight	$30(t + 3)$	30	$t + 3$
passenger	$50t$	50	t

$$\xrightarrow{\quad 30(t+3) \quad} D_f$$
$$\xrightarrow{\quad 50(t) \quad} D_p$$

In this problem the distances traveled will be equal. Why?

$$50t = 30(t + 3) \qquad D_p = D_f$$
$$50t = 30t + 90 \qquad \text{Distributive property}$$
$$20t = 90 \qquad \text{Check: } D_p = 50\left(4\tfrac{1}{2}\right) = 225$$
$$t = \frac{90}{20} = 4\frac{1}{2} = 4\frac{1}{2} \text{ hours} \qquad D_f = 30\left(7\tfrac{1}{2}\right) = 225$$
$$\text{Distance} = 50 \cdot 4\frac{1}{2} = 225 \text{ miles}$$

DISTANCE PROBLEM ANALYSIS



Read the following problems carefully. Solve and check your solution against the conditions of the problems.

1. A passenger train and a freight train 450 miles apart are traveling toward each other at rates of 60 m.p.h. and 30 m.p.h. respectively. How long will it be before they pass each other? *5 hr.*

2. Mary walks at a rate of 4 m.p.h. She left her house and two hours later her brother, Joe, went after her on a bicycle. If he traveled at a speed of 16 m.p.h., how long did it take Joe to catch up to Mary? *40 min.*

3. If they traveled along a straight road, how far did Joe travel to catch up to Mary? *$10\frac{2}{3}$ mi.*

4. An airplane heading east leaves the Denver airport and flies at a speed of 450 miles per hour. An hour later a second plane heading west leaves Denver and flies at a speed of 300 miles per hour. How long after the second plane left will they be 2000 miles apart? *2 hr. 4 min.*

5. Frank left camp traveling on a bicycle at a speed of 20 m.p.h. Two hours later Bill left camp in an automobile along the same route as Frank. What speed must Bill maintain to overtake Frank in two hours? *40 m.p.h.*

6. One car traveling at a rate of 60 miles per hour is 5 miles behind another car traveling in the same direction at 40 miles per hour. How many minutes will it take for the faster car to overtake the slower car? *15 min.*

Farmers, chemists, merchants, manufacturers, pharmacists, dietitians, and many other groups of people need to know how to solve mixture problems to do their work properly. Developing a good sense about quantities needed to obtain a certain mixture has many practical applications.

Before we look at mixture problems in detail, let us consider a few problems of representation.

1. If a mixture of two substances weighs 25 pounds, what is the weight of the second part if the first weighs 18 pounds? 12 pounds? 9 pounds? x pounds? 7 lb.; 13 lb. 16 lb.; (25−X) lb.
2. If you have 19 pounds of a substance and add a second substance to form a mixture, what is the weight of the mixture if you add 2 pounds? 5 pounds? 7 pounds? x pounds? 21 lb.; 24 lb.; 26 lb.; (19+X) lb.

EXAMPLE

A garden supply store plans to blend two kinds of grass seed into a mixture for lawns. If the first kind is worth \$1.30 per pound and the second, \$1.50 per pound, how much of each should be used to obtain 100 pounds of a mixture worth \$1.45 per pound?

Let x = weight of \$1.30 seed

	Amount	$\times$ *Price in Cents*	= *Value in Cents*
part_1	x	130	$130x$
part_2	$100-x$	150	$150(100-x)$
Mixture	100	145	$(145)\cdot(100)$

The sum of the values of the ingredients will always equal the total value of the mixture. Therefore the equation is:

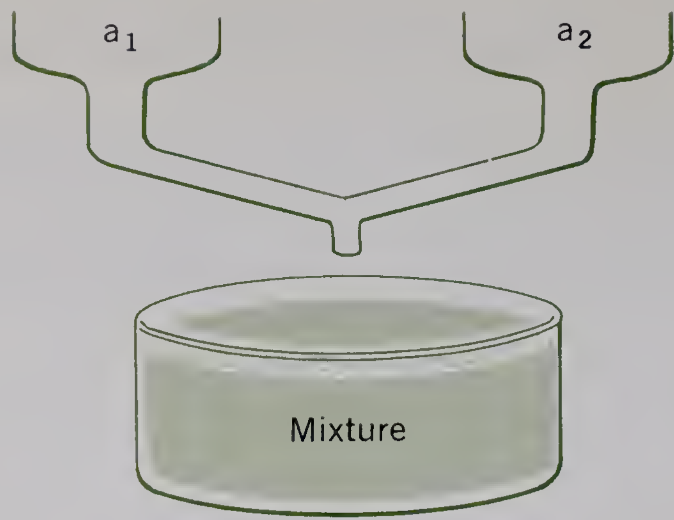
$$130x + 150(100-x) = (145)\cdot(100) \qquad V_1 + V_2 = V_{\text{total}}$$

$$130x + 15000 - 150x = 14500$$

$$-20x = -500$$

$$x = \frac{-500}{-20} = 25 \text{ pounds}$$

The solution is: 25 lbs. of \$1.30 seed
75 lbs. of \$1.50 seed



$V_A = (a_1)(\text{cost}_1)$

$V_B = (a_2)(\text{cost}_2)$

$V_M = (a_1 + a_2)(\text{cost}_M)$

V = Value
a = amount
c = cost

Use a chart similar to that in the example and solve the following problems:

1. A merchant has one type of coffee worth 60¢ per pound and another worth 90¢ per pound. How much of each should he use to get 60 pounds of a mixture worth 70¢ per pound? *40 lb. of 60¢; 20 lb. of 90¢*

2. A candy merchant wants to mix two types of candy. The first type sells for 90¢ per pound, and the second type sells for \$1.40 per pound. How much of each type should he use to obtain a 100-pound mixture which will sell for \$1.00 per pound? *20 lb. of \$1.40; 80 lb. of 90¢*

3. A farmer wants to feed his chickens with 30¢ per pound feed. If he mixes two kinds of feed, 15¢ per pound and 40¢ per pound, how much of each would he use to mix a 100-lb. batch? *60 lb. of 40¢; 40 lb. of 15¢*

4. A wholesaler wants to package a mixture of nuts worth 65¢ per pound. If one type costs 80¢ per pound and the second costs 60¢ per pound, how much of each type would he use to obtain a mix of 20 pounds? *5 lb. of 80¢; 15 lb. of 60¢*

5. A certain blend of tea is made from \$2.40-per-pound and \$3.00-per-pound teas. How much of each type of tea would be used to obtain 200 pounds of this blend which is worth \$2.50 per pound?

6. A candy store owner used two kinds of candy to form a 50-pound mixture. He used 20 pounds of candy worth 60¢ a pound and 30 pounds of candy worth 90¢ per pound. How much is the mixture worth per pound? *78¢* *$33\frac{1}{3}$ lb. of \$3; $166\frac{2}{3}$ lb. of \$2.40*

7. If two items a_1 and a_2 are mixed together in equal quantities and their costs are different, what would be a short method for computing the cost of the mixture? HINT: What is the cost per pound of a mixture of coffee, half of which cost 80¢ per pound and the other half cost 90¢ per pound? *half the sum of the costs*

Another type of mixture problem which is quite common and which occurs in many jobs and professions is the problem dealing with the concentration of a mixture. The *concentration* of a mixture can be defined as the ratio of the amount of the pure substance compared to the total amount of the mixture. This may be expressed as a ratio, decimal, or per cent.

$$\text{concentration} = \frac{\text{amount of pure substance}}{\text{total amount of mixture}}$$

The ratio is commonly expressed as a per cent. If the concentration of an anti-freeze solution is 15%, this means that 15% of the total mixture is pure anti-freeze.

Notice in the following examples that the above definition of concentration is used to form an equation that will enable you to solve the problem.

EXAMPLES

1. A nurse is told to use water to dilute 10 ounces of a certain medicine which is 80% pure so the resulting solution will have a concentration of 50%. How many ounces of water should she add? 1

The amount of pure medicine present at the start is 80% of 10 ounces, or 8 ounces. Why? Since no more medicine is added, the final amount of the pure substance remains at 8 ounces. Since the total amount of the mixture is increased by the amount of water added, we represent the total amount of mixture as $(10 + x)$.

Let x = amount of water added

$$\begin{aligned} .50 &= \frac{8}{10 + x} \\ .5(10 + x) &= 8 \\ 5 + .5x &= 8 \\ .5x &= 3 \\ x &= \frac{3}{.5} = 6 \text{ ounces} \end{aligned} \qquad \text{or} \qquad \begin{aligned} \frac{50}{100} &= \frac{8}{10 + x} \\ 50(10 + x) &= 800 \\ 500 + 50x &= 800 \\ 50x &= 300 \\ x &= 6 \text{ ounces} \end{aligned}$$

2. How much pure anti-freeze must be added to 10 gallons of a 15% anti-freeze solution to raise the concentration to 20%? 2

Let x = amount of pure anti-freeze added

See if you can set up and solve this problem before turning the page.

$$.20 = \frac{\text{amount of pure anti-freeze}}{\text{total amount of mixture}} = \frac{.15 \times 10 + x}{10 + x}$$

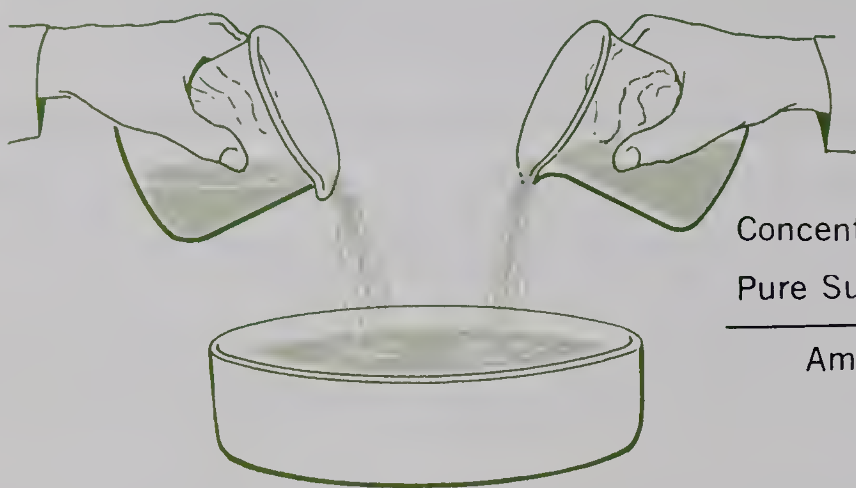
$$\frac{.20}{1} = \frac{1.5 + x}{10 + x}$$

$$.2(10 + x) = 1.5 + x$$

$$2.0 + .2x = 1.5 + x \qquad \text{Note: } x = 1x = 1.0x$$

$$-.8x = -.5 \quad \text{or } .5 = .8x$$

$$x = \frac{-.5}{-.8} = .625 \text{ gallons}$$



$$\text{Concentration of Mixture} = \frac{\text{Pure Substance of } a_1 \text{ and } a_2}{\text{Amount of Mixture}}$$

Solve the following exercises, using the procedure illustrated.

1. How much water should a chemist add to a quart of pure sulfuric acid to form an 80% solution? $\frac{1}{4}$ qt.

2. Mrs. Jones bought 10 ounces of an 80% solution of boric acid. The *Medical Guide* recommended a 30% solution for a certain treatment. How much water should she add to the 10 ounces to dilute the solution to a 30% solution? 16 $\frac{2}{3}$ oz.

$\frac{6}{7}$ qt.

3. Mr. Williams had 6 quarts of a 20% anti-freeze solution in his car radiator. He planned to take a trip into colder climate which required a 30% solution for protection against freezing. How much pure anti-freeze did he need to add to bring the concentration up to 30%? (Assume that the radiator can hold the additional liquid.)

4. How much pure butterfat should be added to 10 gallons of milk testing at 2% butterfat to bring the mixture to test at 2.5% butterfat? .05 gal.

5. How much water should be boiled out of 3 gallons of a 10% salt solution to raise the concentration of the resulting solution to 15%? HINT: Is the amount of liquid increasing or decreasing? 1 gal.

6. Mr. Harvey is using a solution of liquid fertilizer with a concentration of 20%. How much pure liquid fertilizer should he add to 5 gallons of the 20% solution to raise the concentration to 25%? $\frac{1}{3}$ gal.

STEPS FOR SOLVING APPLIED PROBLEMS

1. Understand the problem.
2. Note what the problem asks for.
3. Look for hidden questions.
4. Estimate a reasonable answer.
5. Set up and solve the conditional sentence(s).
6. Check your answer.

1. A private plane heading due north at a speed of 140 m.p.h. left the Dallas airport. A second plane traveling at 210 m.p.h. left 2 hours later in pursuit of the first plane. How long will it be before the first plane is overtaken and how far will it be from Dallas? *6 hr.; 840 mi.*
2. Working alone Bill can paint a fence in 6 hours. Fred can paint the same fence in 8 hours. How long would it take the boys to paint the fence if they worked together? *$3\frac{3}{7}$ hr.*
3. Mr. Richards mixed walnuts worth 90¢ per pound with pecans worth $1.10 per pound. How much of each did he use to obtain 50 pounds worth 96¢ per pound? *35 lb. of 90¢; 15 lb. of $1.10*
4. Bill started walking south on a road at 4 m.p.h. At the same time and place, George left on his bicycle, going north on the same road at 12 m.p.h. How far apart will they be in 3 hours? *48 mi.*
5. Mr. Michael mixed 2 pints of water into 4 quarts of a 20% salt solution. What was the concentration of the resulting solution? *16 %*
6. Suppose that Mr. Michael wanted to use water to dilute 4 quarts of 20% salt solution to 15%. How much water should he add? *$1\frac{1}{3}$ qt.*
7. It takes 14 hours to fill Mr. Johnson's swimming pool and 28 hours to drain it. How long will it take the pool to overflow if Mr. Johnson accidentally leaves the drain open when he turns on the water? *28 hr.*
8. Bill, riding his bicycle at the rate of 12 miles per hour, left his house ahead of his brother. If his brother leaves 2 hours later, at what speed must he ride in order to catch Bill in 4 hours? *18 m.p.h.*
9. If Mary can do a certain task alone in 4 hours and her sister can do it alone in 6 hours, how long will it take the two girls together? *2 hr. 24 min.*
10. Mr. Hardy mixed candy worth 75¢ per pound with candy worth 90¢ per pound. How much of each kind did he mix to obtain 30 pounds worth 80¢? *10 lb. of 90¢; 20 lb. of 75¢*

In Chapter Six you learned to solve problems by graphing equations with two variables. Let's review the method.

EXAMPLE

The sum of two numbers is 14. If their difference is 4, find the two numbers.

Let x represent the first number.
Let y represent the second number.

I $x + y = 14$

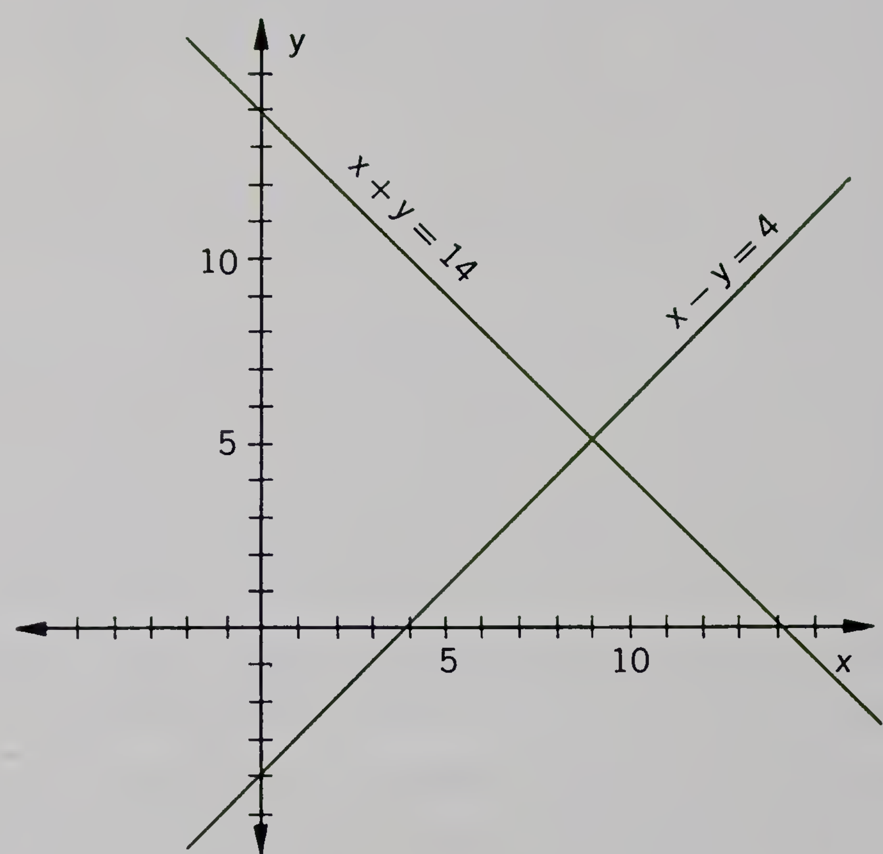
II $x - y = 4$

I

x	-2	0	2	4
y	16	14	12	10

II

x	-2	0	2	4
y	-6	-4	-2	0



Since the two graphs intersect at the point whose coordinates are $x = 9$ and $y = 5$, the two numbers are 9 and 5.

The graphical method for solving a system of equations has some disadvantages. If the answers are fractional, it may be very difficult to read the answers exactly from the graph. If the lines do not intersect on the graph, the solution set cannot be determined. There may be no solution, or it may be necessary to draw the graph to another scale.

Let us consider some non-graphic methods that can be used to solve a system.

Let's consider the system of equations used for the graphical example.

We have already used the mathematical property, "If equals are added to equals, the sums will be equal." If we add $(2 + 3) = 5$ to $7 = (3 + 4)$, we obtain the sums $(2 + 3) + 7 = 5 + (3 + 4)$ which are equal. It becomes obvious when you examine the system of equations

$$\begin{array}{ll} \text{I} & x + y = 14 \\ \text{II} & x - y = 4 \end{array}$$

that the sum of the left-hand members will be $(x + y) + (x - y)$ or $2x$. The y variable will be eliminated from the system. We can then solve this new equation for x, substitute this value in either equation to find the corresponding value of y.

$$\begin{array}{rrl} \text{I} & x + y &= 14 \\ \text{II} & x - y &= 4 \\ \hline & 2x &= 18 \\ & x &= 9 \end{array}$$

Substituting in I $\quad 9 + y = 14$

$y = 14 - 9 = 5$

Checking in II $\quad 9 - 5 = 4$

The solution set of the system is $x = 9$, $y = 5$.

EXAMPLE

Solve:

$$\begin{aligned} 3x + y &= 7 \\ 3x - 2y &= {}^{-}5 \end{aligned}$$

If we add these equations, neither variable will be eliminated from the system.

However, if we subtract the second equation from the first, you will notice that the x terms will be eliminated.

NOTE: To subtract we change the signs of the subtrahend and add.

$$\begin{array}{rrl} \text{I} & 3x + y &= 7 \\ \text{II} & {}^{-}3x + 2y &= {}^{+}5 \\ \hline & 3y &= 12 \end{array}$$

Is this equivalent to multiplying each term in equation II by ${}^{-}1$?

$$y = \frac{12}{3} = 4$$

Substituting $y = 4$ in equation I,

$$\begin{aligned} 3x + 4 &= 7, \\ 3x &= 7 - 4 = 3, \\ 3x &= 3, \\ x &= 1 \end{aligned}$$

Checking in equation II, $3(1) - 2(4) = 3 - 8 = {}^{-}5$.

The solution set of the system is $x = 1$, $y = 4$.

Solve the following systems by elimination of a variable. Be sure to check your solution set in both equations.

1. $2x + y = 10$
$3x - y = 5$ (3, 4)

2. $a + b = {}^{-}7$
$a - b = 1$ ($^{-}3$, $^{-}4$)

3. $m + 2n = {}^{-}5$
$m - 2n = 15$ (5, $^{-}5$)

4. $3x + 3y = 7$
$3x - 3y = {}^{-}1$ $(1, 1\frac{1}{3})$

5. $3r - s = 14$
$2r + s = 6$ (4, $^{-}2$)

6. $x - 2y = 0$
$x + y = 9$ (6, 3)

7. $2p + q = 7$
$2p - q = 1$ (2, 3)

8. $a + 5b = 21$
$2a - 5b = 0$ (7, 2.8)

9. $r = 2s + 7$
$r + 2s = 3$ (5, $^{-}1$)

10. $y = 2x - 7$
$2x + y = {}^{-}1$ (1.5, $^{-}4$)

Coefficients Differ

There will be many systems of equations in which the coefficients of the variables are not identical. Therefore, if these equations are added or subtracted, neither variable will be eliminated. Study the following examples to see what we can do about this situation.

EXAMPLES

1.
$$\begin{array}{ll} \text{I} & 4x - 3y = 17 \\ \text{II} & x + 2y = {}^{-}4 \end{array}$$

Multiply the equation II by $^{-}4$ which will give $^{-}4x - 8y = {}^{+}16$.
Then the two equations will be:

$$\begin{array}{lrl} \text{I} & 4x - 3y &= 17 \\ \text{II} & {}^{-}4x - 8y &= 16 \\ \hline \text{Adding} & {}^{-}11y &= 33 \\ & y &= {}^{-}3 \end{array}$$

NOTE: We could multiply equation I by 2 and equation II by 3 if we wish to eliminate y from the system.

Substituting $y = {}^{-}3$ in equation I

$$\begin{aligned} 4x - 3({}^{-}3) &= 17 \\ 4x - {}^{-}9 &= 17 \\ 4x &= 17 + {}^{-}9 \\ 4x &= 8 \\ x &= \frac{8}{4} = 2 \end{aligned}$$

The solution set is $x = 2$ and $y = {}^{-}3$.

Check to see if these values satisfy both equations. You might check both of these examples by the graphical method for review.

2. I $3c + 8d = 3$
II $^-6c - 15d = {}^-3$

NOTE: We could multiply equation I by 15 and equation II by 8 to eliminate d from the system.

Multiply equation I by 2.

$$\begin{aligned} \text{I} \quad 6c + 16d &= 6 \\ \text{II} \quad {}^-6c - 15d &= {}^-3 \\ \hline \end{aligned}$$

Adding $d = 3$

Substituting $d = 3$ in equation I

$$3c + 8 \cdot 3 = 3$$
$$3c + 24 = 3$$
$$3c = {}^-21 \text{ and } c = {}^-7$$

The solution set is $c = {}^-7$ and $d = 3$. Check these answers in both equations.

The choice of variable to eliminate is arbitrary, as you can see in the examples. We eliminated a variable from the system by making one set the negative of the other. Solve the following examples by elimination of a variable.

1. $8m - n = 0$
$4m + 2n = 15$ $(\frac{3}{4}, 6)$

2. $2a + 3b = 8$
$3a + 2b = {}^-3$ $({}^-5, 6)$

3. $8r - 9s = {}^-2$
$6r + 3s = 5$ $(\frac{1}{2}, \frac{2}{3})$

4. $12x - 5y = 14$
$4x + 10y = 0$ $(1, \frac{^-2}{5})$

5. $z + w = 2$
$4z + 7w = {}^-1$ $({}^-3, 5)$

6. $x + 4y = 1$
$3x - 8y = 33$ $(7, {}^-1\frac{1}{2})$

7. $10t + 8w = 1$
$4t - 12w = {}^-11$ $(\frac{^-1}{2}, \frac{3}{4})$

8. $9c + d = 6$
$3c - 2d = {}^-26$ $(\frac{^-2}{3}, 12)$

9. $3r - 8s = {}^-13$
$^-2r + 12s = 7$ $({}^-5, \frac{^-1}{4})$

10. $5x = 3y$
$2x - y = 1$ $(3, 5)$

Solve the following problems:

11. There are two numbers c and d, such that twice the first increased by three times the second is 28. The first number decreased by the second is 4. What are the two numbers? 8 and 4

12. Bill and Frank together have \$17. Bill has \$5 more than Frank. How much does each boy have? Frank, \$6, Bill, \$11

13. The length of a rectangular field is 10 rods more than the width. The perimeter is 420 rods. Find the dimensions of the field. 100 rd. by 110 rd.

14. Find two numbers whose sum is 45 and whose quotient is 14. 42 and 3
HINT: If $\frac{x}{y} = 7$, is $x = 7y$ equivalent? Yes

15. Twice a certain number n decreased by four times a second number c equals 58. The sum of the two numbers is 39. Find the numbers.
$n, 35\frac{2}{3}$; $c, 3\frac{1}{3}$

Let us consider the following problem.

EXAMPLE

$$\begin{aligned} \text{I} \quad 2a + b &= 8 \\ \text{II} \quad 3a - 2b &= {}^{-}2 \end{aligned}$$

Solve for b in the first equation.

I $\quad b = (8 - 2a)$

Replace b in the second equation by $(8 - 2a)$

II $\quad 3a - 2(8 - 2a) = {}^{-}2$

Solve for a.

$$\begin{aligned} 3a - 16 + 4a &= {}^{-}2 \\ 7a &= {}^{-}2 + 16 = 14 \\ a &= \frac{14}{7} = 2 \end{aligned}$$

In equation I, $b = (8 - 2 \cdot 2) = 4$

Therefore, the solution set is $a = 2$ and $b = 4$. Check to see if these answers satisfy both original equations.

Solve the following exercises by substitution.

1. $x = y - 1$
   $2x - y = 4$ $(5, 6)$
2. $3c + d = 0$
   $4c - \frac{1}{3}d = 9$ $(\frac{9}{5}, \frac{{}^{-}27}{5})$
3. $y - 9x = 4$
   $15x - y = 0$ $(\frac{2}{3}, 10)$
4. $7x + y + 1 = 0$
   $y + 10x = 2$ $(1, {}^{-}8)$
5. $y + x = 5$
   $2y + 3x = 2$ $({}^{-}8, 13)$
6. $5a + b = 1$
   $a + b = 5$ $({}^{-}1, 6)$
7. $r - s = 2$
   $3r = 2s + 1$ $({}^{-}3, {}^{-}5)$
8. $a - b = 4$
   ${}^{-}2a + 5b = 1$ $(7, 3)$
9. $x = 2y + 16$
   $4x + 3y = {}^{-}2$ $(4, {}^{-}6)$
10. $a - b = 6$
    $\frac{2}{3}a - \frac{3}{5}b = 3$ $({}^{-}9, {}^{-}15)$

Solve the following problems:

11. The sum of two numbers is 43. Their difference is 15. Find the numbers. Let x = larger number and y = smaller number. Thus, $x + y = 43$ and $x - y = 15$. $x, 29; y, 14$
    Copy and complete the solution.
12. Separate 104 into two parts, such that the larger exceeds the smaller by 26. Let x = smaller and y = larger number. $x, 39; y, 65$

STEPS FOR SOLVING APPLIED PROBLEMS

1. Understand the problem.	2. Note what the problem asks for.	3. Look for hidden questions.
6. Check your answer.	5. Set up and solve the conditional sentence(s).	4. Estimate a reasonable answer.

1. Harry sold his bicycle for \$39. This was $\frac{3}{4}$ of what he paid for it. How much did he pay for his bicycle? \$ 52

2. The length of a rectangular lot is 6 feet less than three times its width. Its perimeter is 480 feet. What are its dimensions? $61\frac{1}{2}$ ft., $178\frac{1}{2}$ ft.

3. A 16-foot board is sawed into two pieces so that one piece is 5 feet longer than the other. How long is each piece? $5\frac{1}{2}$ ft., $10\frac{1}{2}$ ft.

4. Jim, Harry, and Mike all worked in a factory last summer. Jim earned $\frac{4}{5}$ as much as Harry, and Mike earned \$80 less than Harry. Their total earnings were \$760. How much did each earn? Jim, \$240; Harry, \$300; Mike \$220

5. A farmer divided 1200 acres of land among his sons so that the oldest received $1\frac{1}{2}$ times as much as the second son, while the youngest received 300 acres less than what the other two received together. How many acres did each receive? oldest, 450 A.; second, 300 A.; youngest, 450 A.

6. An airplane flew for five hours with a tail wind. The return trip heading into the wind took six hours. If the speed of the wind was 25 miles per hour, what was the speed of the plane in still air? HINT: If x is the speed of the plane, then $(x + 25)$ and $(x - 25)$ are the ground speeds with and against the wind. What do you know about the distances traveled? 275 m.p.h.

7. Mr. Allen drove a distance of 40 miles in one hour 20 minutes. What was his average speed in miles per hour? 30 m.p.h.

8. Find two numbers such that eight times the first number divided by three times the second number yields a quotient of 4 and the first number increased by twice the second number gives 7. 3, 2

9. At a school play 450 tickets were sold for a total of \$600. If student tickets cost \$1 and adult tickets cost \$2, how many of each were sold? adult, 150; student, 300

10. Three pounds of butter and two pounds of meat cost \$3.85. Two pounds of butter and three pounds of meat cost \$3.90. What was the price per pound of each? butter, 75 ¢ per lb.; meat, 80 ¢ per lb.

In many types of problems it is impossible to make an equation or two equations in which the variable or variables are of the first degree, that is, have exponents of 1 and form linear equations. However, it may be possible to form a quadratic equation in which the variable is raised to the second power. The standard form of a quadratic equation is $ax^2 + bx + c = 0$ where $a \neq 0$.

EXAMPLE

A rectangular garden has an area of 80 square yards. The length is 11 yards more than the width. Find the dimensions.

Let x = length of field in yards
Let $(x - 11)$ = width of field in yards

Since the area of a rectangle is found by multiplying length and width, we obtain the following equation.

$$\begin{aligned} x(x - 11) &= 80 \\ x \cdot x - 11 \cdot x &= 80 \\ x^2 - 11x &= 80 \\ x^2 - 11x - 80 &= 0 \end{aligned}$$

Can you find a value for x that will satisfy this equation?

Try $x = 16$. Also try $x = {}^-5$. Since x stands for the length of a rectangle, it must be positive; ${}^-5$ cannot be a solution to the problem even though it satisfies the equation. The length of the garden is 16 yards and the width is $x - 11$ or 5 yards.

We have said that an equation of the form $ax^2 + bx + c = 0$, with a, b, and c numerical coefficients, and a is not zero, is called a quadratic equation.

If $2x^2 - x - 6 = 0$, what is a? 2 b? -1 c? -6

See if 2, 1, 0, ${}^-1$, ${}^-\frac{3}{2}$, ${}^-2$ are in the solution set. How can we solve such an equation? It is obvious that we cannot depend upon guesswork.

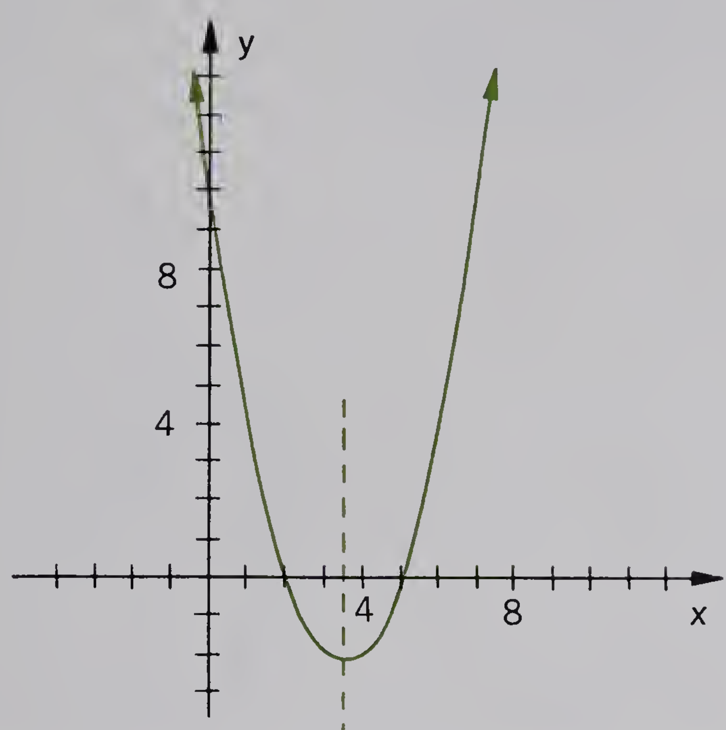
Let us determine the solution set of a quadratic equation by graphing. Find the values of x which will satisfy the equation $x^2 - 7x + 10 = 0$. First we shall make the equation read as follows by changing 0 to y.

$$x^2 - 7x + 10 = y$$

Determine several ordered pairs (values of x and y) which will satisfy this equation. Substituting different values for x, you can obtain y values as indicated on the next page.

x	-3	-2	-1	0	1	2	3	4	5	6	7
y	40	28	18	10	4	0	-2	-2	0	4	10

Plot these points on the graph and draw a smooth curve through each of the points as shown in the figure. We can see that the graph is a curved line. We can also see that the curve is symmetric with respect to the dashed vertical line, $x = +\frac{7}{2} = \frac{-b}{2a}$. What are the two values of x where $y = 0$? Notice on the graph that $x = 2$ and $x = 5$ are the two values of x that will satisfy the equation $x^2 - 7x + 10 = 0$.



The curve illustrated in the previous example is one of a family of curves called a *parabola*. The parabola has many applications in optics, engineering, astronomy, ballistics, and other branches of science. It also has many uses in our home and everyday life. You will be able to find considerable material about the parabola in your school library.

Solve the two values of the variable which will satisfy the following quadratic equations by the graphical method.

1. $x^2 + 7x + 6 = 0$ $^-6, ^-1$

2. $x^2 + x - 6 = 0$ $2, ^-3$

3. $x^2 - 10x + 24 = 0$ $4, 6$

4. $x^2 - 2x - 35 = 0$ $7, ^-5$

5. $x^2 - 2x - 3 = 0$ $^-1, 3$

6. $x^2 + 3x + 2 = 0$ $^-1, ^-2$

7. $x^2 - 5x + 4 = 0$ $4, 1$

8. $x^2 - x - 2 = 0$ $2, ^-1$

9. $x^2 + 3x - 10 = 0$ $^-5, 2$

10. $x^2 - 4x - 21 = 0$ $^-3, 7$

11. $x^2 - 3x - 10 = 0$ $^-2, 5$

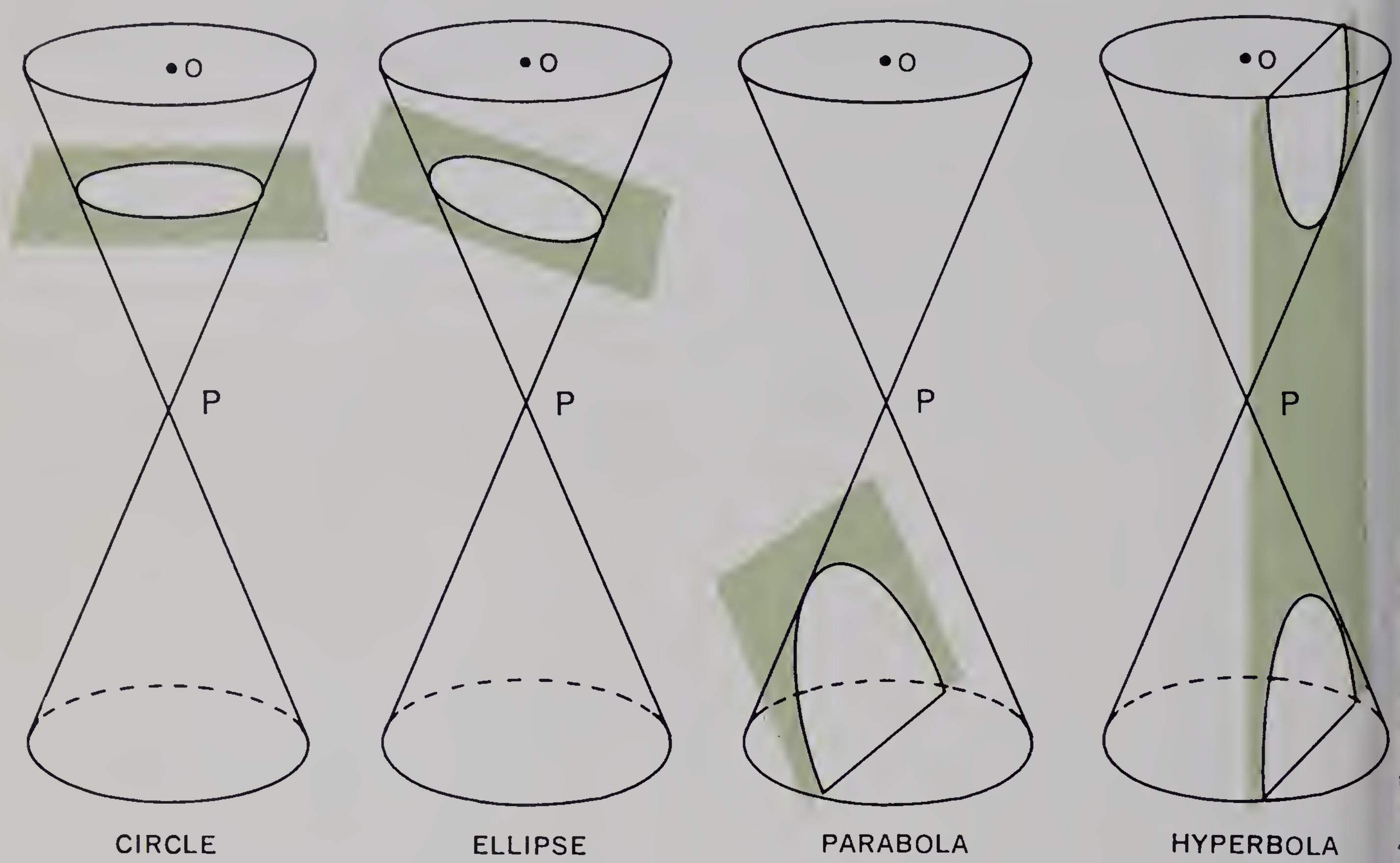
12. $x^2 - 10x + 21 = 0$ $3, 7$

13. $x^2 - 5x + 6 = 0$ $2, 3$

14. $x^2 - x - 12 = 0$ $^-3, 4$

Let us place two cones tip to tip, as shown in this figure, and visualize planes through them in various ways. The intersection of these planes with the surface of the cones will give us four differently shaped curves.

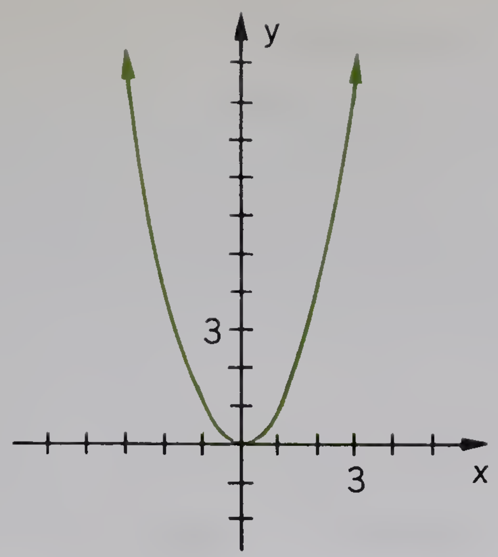
CONIC SECTIONS



These four curves are commonly called *conic sections.* Each has certain mathematical properties and must be drawn following certain rules. Many of the basic facts about conic sections have been known for over 2000 years. One of the first books published about them was written by a famous Greek mathematician, named Euclid, who did his work about 300 B. C. (Do you know of any other very well-known works by Euclid?) Many other great men of science and mathematics have devoted considerable study to this family of curves because of the very important and interesting properties they possess. As you read about some of these properties, see if you can understand how they have affected your environment.

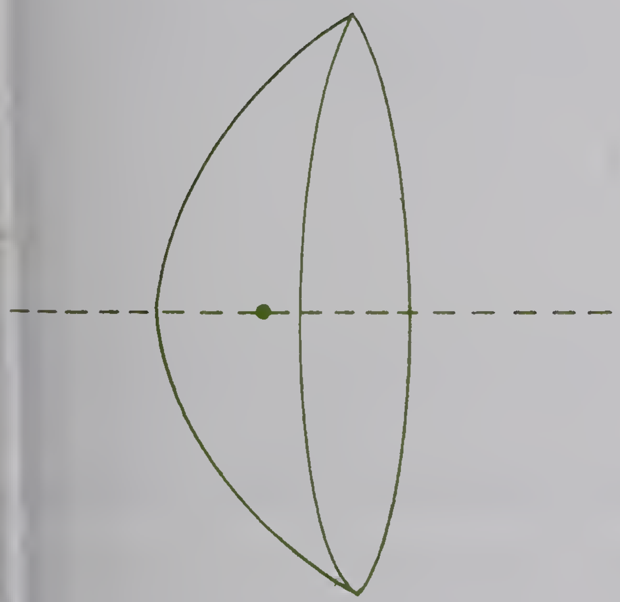
The Parabola

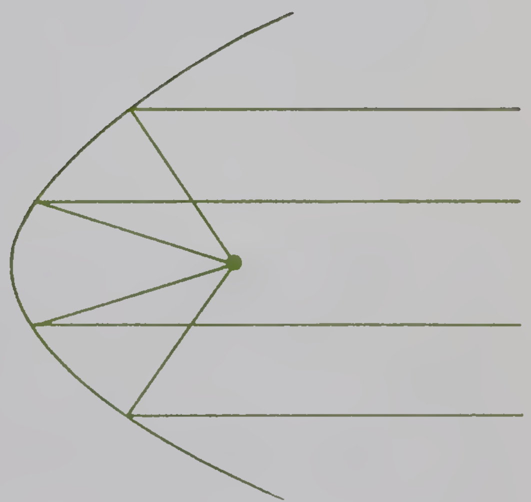
If we graph an equation of the type $y = x^2$ by plotting ordered pairs which satisfy the equation and join these points with a smooth curve, the curve will be a parabola.



You will notice that the graph of this parabola is symmetric with respect to the y-axis. For a curve to be symmetrical with respect to an axis, the portion of the curve on one side of the axis exactly fits the curve on the other side.

If we revolve the parabola about its axis, through space, we will generate a surface called a *paraboloid*. One of the very interesting properties of a paraboloid is that it will gather light or sound or other transmission from far distances and reflect them all to one point called its *focal point*. Because of this property a paraboloid is used for sound pickup in microphones, radar receivers, reflecting mirrors for telescopes, and other similar devices. If an intense source of light is placed at the focal point and the paraboloid is used as a reflecting surface, the light will be reflected in a parallel beam. Thus, reflectors for searchlights, auto headlights, and so forth are shaped as paraboloids.





When an object is released from a moving vehicle, the path followed by the falling object (neglecting wind and air resistance) will be parabolic. Bombsights are designed with the properties of a parabola in mind.

Projectiles or missiles fired from the earth with a given initial speed will follow a parabolic path neglecting wind and air resistance.

Solve the following exercises.

1. Find the ordered pairs suggested for the equation $y = x^2 + 4$.

x	0	1	$^-1$	2	$^-2$	3	$^-3$	4	$^-4$	5	$^-5$
y	4	5	5	8	8	13	13	20	20	29	29

2. Draw the graph of $y = x^2 + 4$ by plotting the ordered pairs found in Exercise 1 and drawing a smooth curve through them. *See front.*

3. What is the axis of symmetry of the graph? *y axis*

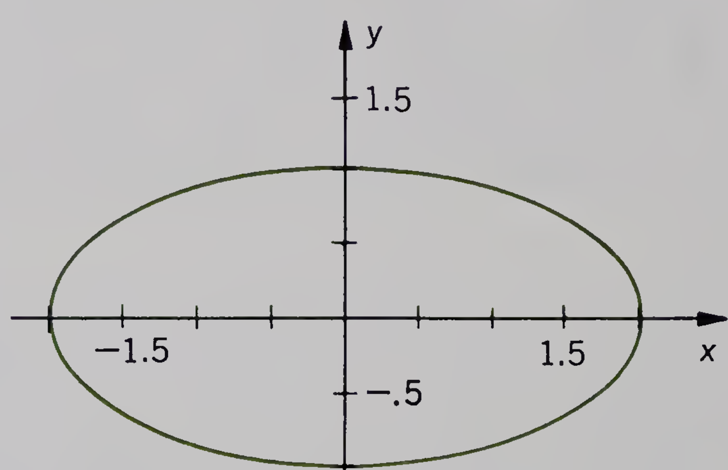
4. What is the smallest value of y on the graph? *4*

5. Why are there no negative values for y? *x^2 is always positive*

6. Graph the parabola determined by the equation $x = y^2 - 5$ as in Exercises 1 and 2. Notice that x cannot be less than $^-5$. *See front.*

The Ellipse

One of the many equations whose graph will be an ellipse is $x^2 + 4y^2 = 4$. Notice that if $y^2 = 1$, $y = {}^+1$ and $y = {}^-1$ are members of the solution set.



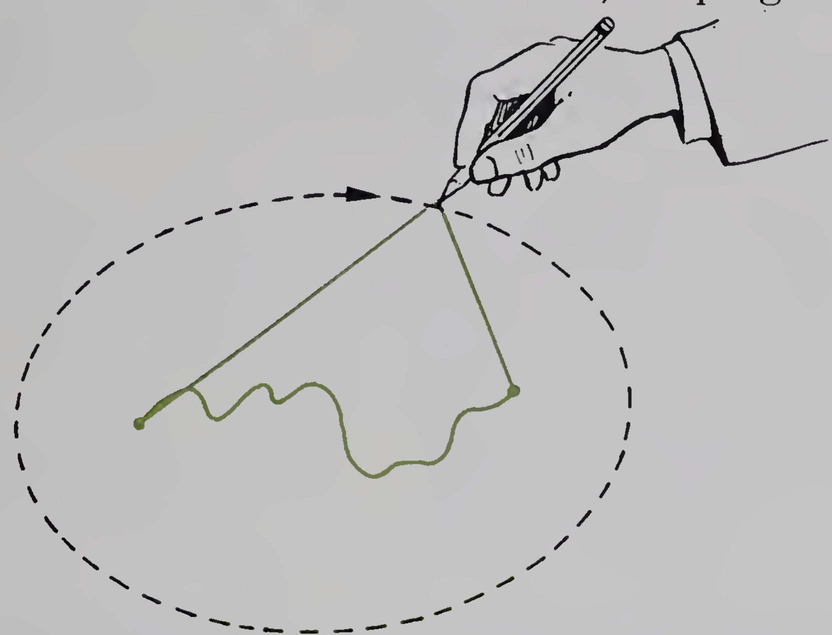
We see that this member of the family of conic sections has two axes of symmetry. In the above figure they are the x-axis and the y-axis.

The ellipse is very important to astronomers and space scientists, as well as to others. The planets of our solar system do not revolve around the sun in a circular orbit, but in an elliptical one. That fact was first observed in 1609 by a German scientist and astronomer named Johann Kepler. Each of the satellites now in orbit around the earth also moves in an elliptical path. By using high-speed computers and applying the properties of an ellipse space mathematicians are able to determine accurately and quickly the location and height of any satellite at any time.

An *ellipse* is the set of all points in a plane such that the sum of the distances from any point in the set to two fixed points of the plane is always the same value.

Using this property, draw an ellipse by following these directions.

1. Fix a piece of string to a drawing board with two thumbtacks, giving the string sufficient slack (as shown below).
2. Move a pencil against the string until the string is taut.
3. Move the pencil around the thumbtacks, keeping the string taut.



1. Find the ordered pairs for the equation $2x^2 + y^2 = 16$. NOTE: You will need square root tables.

x	0	0	1	1	2	2	2.8	$^-2.8$	2.4	$^-2.4$
y	4	$^-4$	3.7	$^-3.7$	2.8	$^-2.8$	0	0	2	2

2. Graph the ellipse, using the table of Exercise 1.

3. What are the two axes of symmetry? *x and y axes*

4. Where does the curve cross the x-axis and the y-axis? *$x = \pm 2.8, y = \pm 4$*

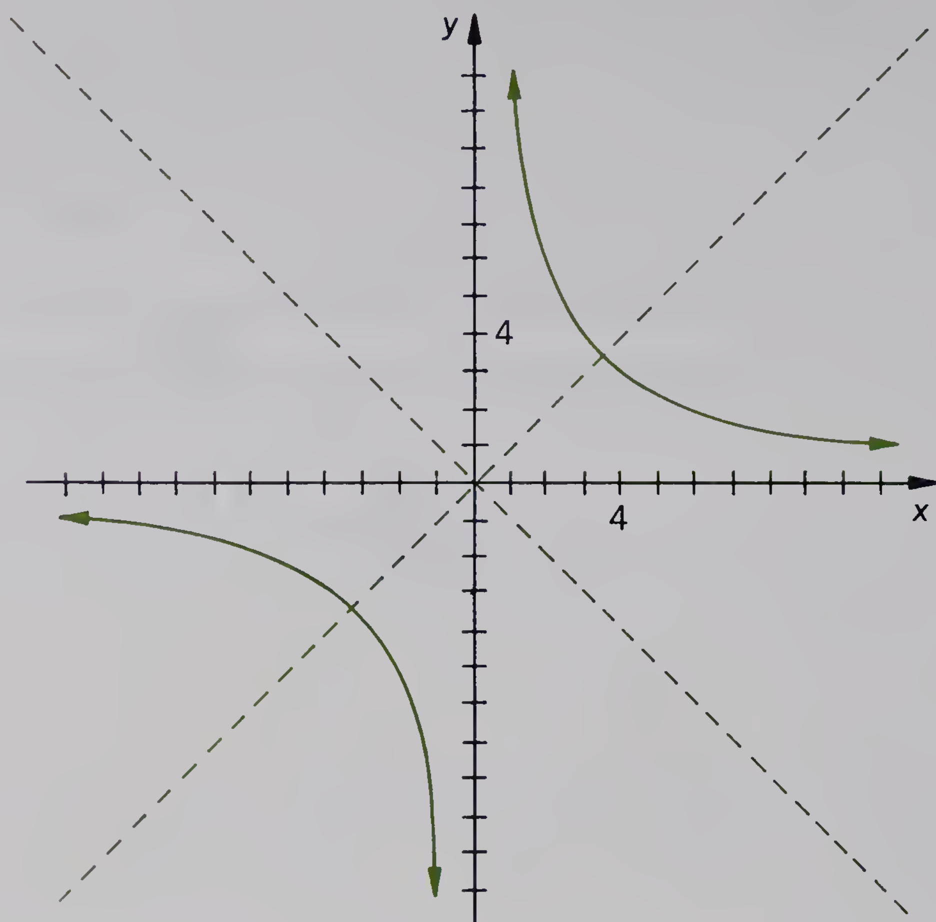
5. Explain why x cannot be 4. Explain why y cannot be 5.
The quantity under the radical sign cannot be negative.

6. Find ordered pairs and graph the ellipse of the equation $3x^2 + y^2 = 18$.
See front.

The Hyperbola

Physicists and other scientists are much concerned with the hyperbola. In many cases, movements of certain charged particles in the presence of other forces will move in a hyperbolic path. Knowing the mathematical properties of this hyperbolic path enables a scientist to explore the nature of the atom, for instance.

One of many equations whose graph is a hyperbola is $xy = 10$. Notice that the hyperbola has two parts and two axes of symmetry.



1. Find the following ordered pairs for $xy = 18$ and construct the graph.

x	1	-1	3	-3	9	$^-9$	10	$^-10$	5	3.6	6	$^-6$
y	18	$^-18$	6	$^-6$	2	-2	1.8	$^-1.8$	3.6	5	3	$^-3$

2. If the equation is $xy = 42$, what is the value of x when $y = 84$? $\frac{1}{2}$

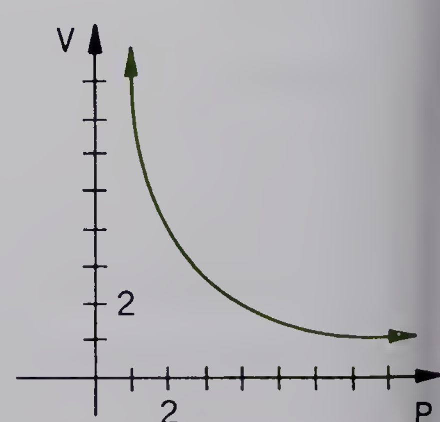
3. Describe the axes of symmetry of the hyperbola in Exercise 1.
two diagonal lines ($y = \pm x$) through the orgin

4. Find the ordered pairs and graph the equation $16x^2 - 4y^2 = 16$.

x	1	$^-1$	0	0	2	2	2.24	$^-2.24$
y	0	0	2	$^-2$	3.5	$^-3.5$	4	4

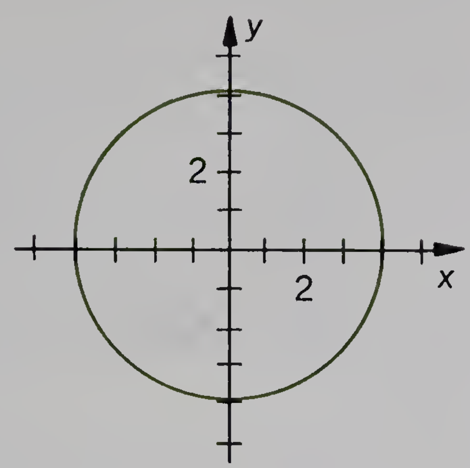
5. The graph in Exercise 4 should be a hyperbola. Describe its two axes of symmetry. *x and y axes*

6. Let us graph the relationship $PV = 8$. (P is pressure, V is volume.) This is one branch of a hyperbola. Why do we not use negative values for pressure and volume?
Pressure and volume cannot be negative.



The Circle

An equation whose graph is a circle is $x^2 + y^2 = 16$.



The properties and applications of a circle are very well known to us all and will therefore not be discussed at great length. We should note that the circle is the most completely symmetrical figure we can draw in a plane. Every line drawn through its center is an axis of symmetry. Since there are an infinite number of different lines that can be drawn through the center, every circle has an infinite number of axes of symmetry.

If you have 144 yards of fencing, what area could you enclose if the fence were in the shape of an equilateral triangle? a square? a rectangle that is twice as long as it is wide? a circle? Calculate the area of each region to the nearest square yard. *See front.*

1. Find the following ordered pairs of the equation $x^2 + y^2 = 36$ and graph them.

x	0	0	6	$^-6$	2	2	5.7	$^-5.7$
y	6	$^-6$	0	0	5.7	$^-5.7$	2	2

2. What is the radius of the circle in Exercise 1? *6 units*

3. As in Exercise 1 find ordered pairs and graph the circle of the equation $x^2 + y^2 = 25$. *See front.*

4. Explain why neither x nor y can be greater than 6 in Exercise 1. Explain why neither x nor y can be less than $^-5$ in Exercise 3.
The quantity under the radical cannot be negative.

5. Without graphing, give the radius of the circles of the following equations.

a. $x^2 + y^2 = 49$ *7*
b. $x^2 + y^2 = 16$ *4*
c. $x^2 + y^2 = 100$ *10*
d. $3x^2 + 3y^2 = 27$ *3*
e. $2x^2 + 2y^2 = 32$ *4*
f. $x^2 + y^2 = 10$ *$\sqrt{10}$*

We have learned in earlier chapters how to remove parentheses in a product by using the associative, commutative, and distributive properties.

EXAMPLE

$$4(3x - 7) = 4 \cdot 3x - 4 \cdot 7 = 12x - 28$$

We are able to do the same thing if both factors are binomials (factors containing two terms).

EXAMPLES

1. $$\begin{aligned}(2x + 5)(3x + 2) &= (2x + 5)3x + (2x + 5)2 \\ &= 3x(2x + 5) + 2(2x + 5) \\ &= 3x \cdot 2x + 3x \cdot 5 + 2 \cdot 2x + 2 \cdot 5 \\ &= 6x^2 + 15x + 4x + 10 \\ &= 6x^2 + 19x + 10\end{aligned}$$

2. $$\begin{aligned}(x - 4)(2x + 3) &= (x - 4)2x + (x - 4)3 \\ &= 2x(x - 4) + 3(x - 4) \\ &= 2x \cdot x - 2x \cdot 4 + 3 \cdot x - 3 \cdot 4 \\ &= 2x^2 - 8x + 3x - 12 \\ &= 2x^2 - 5x - 12\end{aligned}$$

We can observe that the product of two similar binomials generally gives four terms which combine to form a *trinomial,* an algebraic expression of three terms. You will notice in the two examples above that trinomials are quadratic in form. Looking at the product closely, we see that the first term is the product of the first terms of each factor.

$$(2x + 5)(3x + 2) = 6x^2 + 19x + 10$$

Also, we see that the third term is the product of the second terms of each factor.

$$(2x + 5)(3x + 2) = 6x^2 + 19x + 10$$

Finally, we observe that the middle term is the second term of the first factor times the first term of the second factor plus the first term of the first factor times the second term of the second factor.

$$(2x + 5)(3x + 2) = 6x^2 + 19x + 10$$

Another example is $(x - 4)(2x + 3) = 2x^2 - 5x - 12$

In general, $(ax + b)(cx + d) = acx^2 + (bc + ad)x + bd.$

Find the product of the following binomials by using this shortened method. Do as many as possible without using pencil and paper.

1. $(x + 2)(x + 3)$ x^2+5x+6

2. $(x - 1)(x - 7)$ x^2-8x+7

3. $(x - 4)(x - 1)$ x^2-5x+4

4. $(x + 4)(x + 5)$ $x^2+9x+20$

5. $(2x + 3)(x - 1)$ $2x^2+x-3$

6. $(2x + 1)(3x + 2)$ $6x^2+7x+2$

7. $(x - 5)(2x + 2)$ $2x^2-8x-10$

8. $(3x - 1)(x + 7)$ $3x^2+20x-7$

9. $(2x - 1)(2x - 1)$ $4x^2-4x+1$

10. $(5x + 7)(x + 1)$ $5x^2+12x+7$

11. $(3x - 3)(x + 7)$ $3x^2+18x-21$

12. $(x + 8)(x - 5)$ $x^2+3x-40$

13. $(3x + 4)(4x + 3)$ $12x^2+25x+12$

14. $(x + 4)(x + 4)$ $x^2+8x+16$

15. $(5x - 3)(2x + 1)$ $10x^2-x-3$

16. $(3x - 5)(2x - 3)$ $6x^2-19x+15$

17. $(4x - 2)(x + 3)$ $4x^2+10x-6$

18. $(7x - 4)(2x + 1)$ $14x^2-x-4$

19. $(3x + 1)(3x - 1)$ $9x^2-1$

20. $(5x - 3)(5x + 3)$ $25x^2-9$

Factoring

Suppose we start with a product of the form $(ax^2 + bx + c)$ and wish to find its binomial factors if possible. This is called *factoring.* Essentially we are undoing the multiplication of the previous section. Perhaps, then, we can use what we learned from the previous unit to factor some quadratics.

EXAMPLES

1. Factor $x^2 - 5x + 6$ into two binomial factors.

   a. You will recall from multiplying binomials that the first term x^2 is the product of the first terms of the two binomial factor. Since $x \cdot x = x^2$, the first step can be $(x \quad)(x \quad)$.

   b. Since the third term is the product of the second terms of the binomial factors, we have four choices for the second terms of the binomial. They are

   $$(6)(1),\ (^-6)(^-1),\ (3)(2),\ \text{and}\ (^-3)(^-2).$$

   c. By trial and error we find that only $^-3$ and $^-2$ as second terms of the binomials will add to $^-5$ and give the proper middle term of ^-5x in the product.

   d. Therefore, $x^2 - 5x + 6 = (x - 3)(x - 2)$.

2. Factor $x^2 - x - 6$.

a. The first term of each binomial factor must be x. Why? $x \cdot x = x^2$

b. After the x in the binomials, one sign will be $^+$ and the other will be $^-$ since the product of the second terms is $^-6$.

c. $^-6 = {^-6} \cdot 1, 6 \cdot {^-1}$, or $^-6 = {^-3} \cdot 2$, or $3 \cdot {^-2}$. By trial and error we find that only $^+2$ and $^-3$ add to $^-1$ and give the proper middle term, ^-x.

d. Therefore, $x^2 - x - 6 = (x + 2)(x - 3)$.

Following are several simple quadratic trinomials. Factor them into the product of two binomials by using the method illustrated in the previous examples.

1. $x^2 - 2x - 8$ $(x-4)(x+2)$

2. $a^2 - 9a + 8$ $(a-8)(a-1)$

3. $r^2 - 7r + 12$ $(r-3)(r-4)$

4. $y^2 + y - 20$ $(y+5)(y-4)$

5. $x^2 + 10x + 24$ $(x+6)(x+4)$

6. $y^2 - 12y + 36$ $(y-6)(y-6)$

7. $a^2 + 7a + 12$ $(a+4)(a+3)$

8. $a^2 + 3a - 10$ $(a+5)(a-2)$

9. $x^2 + x - 6$ $(x+3)(x-2)$

10. $y^2 + 5y - 24$ $(y+8)(y-3)$

11. $r^2 + 8r + 15$ $(r+3)(r+5)$

12. $y^2 + 5y - 50$ $(y+10)(y-5)$

13. $m^2 - 7m + 6$ $(m-1)(m-6)$

14. $x^2 - 18x - 40$ $(x-20)(x+2)$

15. $r^2 - 18r + 72$ $(r-6)(r-12)$

16. $a^2 - 4a - 32$ $(a-8)(a+4)$

17. $x^2 - 7x - 18$ $(x-9)(x+2)$

18. $b^2 + 10b + 16$ $(b+8)(b+2)$

19. $y^2 + 8y - 20$ $(y+10)(y-2)$

20. $a^2 + 9a - 10$ $(a+10)(a-1)$

Suppose you notice the middle term is missing. For example, $x^2 - 9$ could be considered $x^2 + 0x - 9$. The products of $3x$ and ^-3x add to give the proper middle term $0x$. Therefore

$$x^2 - 9 = (x + 3)(x - 3)$$

Binomials of the form $x^2 - y^2$ are called differences of squares. Factor.

21. $x^2 - 25$ $(x+5)(x-5)$

22. $x^2 - 144$ $(x+12)(x-12)$

23. $x^2 - 169$ $(x-13)(x+13)$

24. $x^2 - 625$ $(x+25)(x-25)$

25. $x^2 - 36$ $(x+6)(x-6)$

26. $x^2 - 256$ $(x+16)(x-16)$

27. $x^2 - 64$ $(x+8)(x-8)$

28. $x^2 - 324$ $(x+18)(x-18)$

29. $x^2 - 196$ $(x+14)(x-14)$

30. $x^2 - 2500$ $(x+50)(x-50)$

Note: It should be understood that we have touched lightly on the subject of factoring. To master this topic completely would take a great deal of time and practice. If you are interested in learning more about factoring, it is suggested that you study further in an algebra text.

Now let us use factoring to solve some quadratic equations.

EXAMPLE

What values of x satisfy the equation $x^2 - 4x - 32 = 0$?
Factoring, we get $x^2 - 4x - 32 = (x - 8)(x + 4)$
Therefore $(x - 8)(x + 4) = 0$

We know that the product of two factors equals zero if, and only if, either one or both equals zero. If $f_1 f_2 = 0$, then $f_1 = 0$ or $f_2 = 0$. Therefore

$$x - 8 = 0 \quad \text{or} \quad x + 4 = 0$$
$$x = 8 \quad \text{or} \quad x = {}^-4$$

The values of x which satisfy $x^2 - 4x - 32 = 0$ are $x = 8$ and $x = {}^-4$. The solution set is $\{8, {}^-4\}$.

These steps can help solve quadratics by factoring.

Arrange the quadratic equation in the form $ax^2 + bx + c = 0$.
Factor the quadratic if possible.
Set each factor equal to zero.
Solve each new equation for the variable. Check.

Solve the following quadratic equations by factoring and check your answer.

1. $x^2 - 3x + 2 = 0$ 1, 2
2. $a^2 - 3a - 10 = 0$ 5, $^-2$
3. $a^2 = 4a + 12$ 6, $^-2$
4. $r^2 - 7r = {}^-6$ 6, 1
5. $x^2 + 5x + 6 = 0$ $^-3$, $^-2$
6. $x^2 - 8x + 7 = 0$ 7, 1
7. $b^2 = 10b - 24$ 6, 4
8. $48 = 26x - x^2$ 24, 2
9. $a^2 + 5a - 24 = 0$ $^-8$, 3
10. $y^2 + 5y = 50$ 5, $^-10$
11. $x^2 + x = 20$ $^-5$, 4
12. $a^2 - 12a + 32 = 0$ 8, 4
13. $x^2 + 36 = 12x$ 6
14. $y^2 - 11y + 28 = 0$ 7, 4
15. $r^2 + 13r + 42 = 0$ $^-7$, $^-6$
16. $x^2 = 14x - 49$ 7
17. $m^2 - 7m + 12 = 0$ 4, 3
18. $y^2 + 18 = 9y$ 6, 3
19. $x^2 - 12x + 35 = 0$ 5, 7
20. $r^2 + 6r = 27$ $^-9$, 3

All quadratic equations can be solved but not always by factoring. Consult an algebra text for other methods.

Word problems may produce a quadratic equation that can be solved by factoring.

EXAMPLE

Find two numbers which differ by 4 and whose product is 96.

Let x = first number
Let $x + 4$ = second number

$$x(x + 4) = 96$$
$$x^2 + 4x = 96$$
$$x^2 + 4x - 96 = 0$$
$$(x + 12)(x - 8) = 0$$

If the product of two factors is zero, at least one of the factors is equal to zero.

Then $x + 12 = 0$ or $x - 8 = 0$
Therefore, $x = {}^{-}12$ or $x = 8$

If $x = {}^{-}12$, then $x + 4 = {}^{-}8$. Therefore ${}^{-}12$ and ${}^{-}8$ is one pair.
If $x = 8$, then $x + 4 = 12$. Therefore 8 and 12 is another pair.
Do they both check? *yes*

The following problems can be solved by the use of quadratic equations and factoring. Reject all answers which do not fit the conditions of the problem.

1. Find a pair of numbers such that their sum is 23 and their product is 60. *3 ; 20*

2. Find the dimensions of a rectangle if its area is 216 square inches and it is 6 inches longer than it is wide. *12 in. by 18 in.*

3. Find a number that is 42 less than its own square. *7*

4. The sum of two positive numbers is 10. If one of the numbers is 2 less than the square of the other, find both numbers. *3 ; 7*

5. The formula for the height that an object will reach in t seconds if thrown upward at a speed of 48 feet per second is $d = {}^{-}16t^2 + 48t$. Find the time the object will require to reach a height of 32 feet. HINT: Let $d = 32$ and then divide every term of the equation by 16 before trying to factor. *1 sec.*

6. There are two possible answers in Exercise 5. Why? *quadratic equation*

7. The width of a rectangle is 8 inches less than the length. If the area is 84 square inches, find the dimensions of the rectangle. *6 in. by 14 in.*

Part One

A. Solve the following equations.

1. $2x - 7 = x - 3$ *4*

2. $3x + 4 = 1 + 2x$ *⁻3*

3. $a - 17 = 3 - 4a$ *4*

4. $2r - 5 = 2(5 - 2r)$ *2.5*

5. $\frac{x}{5} = \frac{24}{15}$ *8*

6. $\frac{3}{4}(4x - 8) = 2x - 3$ *3*

B. Translate the following phrases into mathematical expressions.

1. 3 more than a certain number *x + 3*

2. Twice a number decreased by 5 *2x − 5*

3. The sum of 6 and twice a certain number *2x + 6*

4. A number diminished by 15 *x − 15*

5. 28 increased by three times a certain number *3x + 28*

C. Fill in the blanks with either the word *positive* or *negative* to make the statement true.

1. The product of two negative numbers is __?__. *positive*

2. The quotient of a positive number divided by a negative number is __?__. *negative*

3. The sum of ⁻11 and 13 is __?__. *positive*

4. The sum of ⁻11 and ⁻13 is __?__. *negative*

5. ⁻14 multiplied by itself is __?__. *positive*

6. The product of two numbers with opposite signs is __?__. *negative*

7. The reciprocal of ⁻7 is __?__. *negative*

8. If ⁻11 is subtracted from ⁻3, the difference is __?__. *positive*

9. ⁻12 ÷ ⁻48 is __?__. *positive*

10. 8(3 − 7) is __?__. *negative*

Part Two

A. Solve the following problems.

1. If Bill can do a job alone in 4 hours while it takes George 5 hours, how long will it take if they work together? *$2\frac{2}{9}$ hr.*

15 lb. of walnuts ; 45 lb. of pecans

2. Mr. Harris mixed walnuts worth 90¢ per pound with pecans worth \$1.50 per pound to make 60 pounds of a mixture worth \$1.35 per pound. How many pounds of each kind of nut did he use?

3. Mrs. Williams left home traveling south at 50 miles per hour. At the same time, Mrs. Harris left home traveling north at 45 miles per hour. They met in 3 hours. How far apart did they live? 285 mi.

4. Fred left home traveling at 20 miles per hour. One hour later his brother, Bill, traveling at 40 miles per hour, left in pursuit of Fred. How long will it take Bill to overtake Fred? 1 hr.

5. The concentration of 15 ounces of a salt solution is 60%. What will be the concentration if 5 ounces of water are added? 45 %

B. Solve the following systems of equations by elimination of a variable or by substitution.

1. $3x - y = 5$
   $2x + y = 10$ (3, 4)
2. $2a + 3b = 7$
   $a + b = 2$ ($^-1$, 3)
3. $4r - 2s = 6$
   $3r + 2s = {}^-2$ ($\frac{4}{7}$, $\frac{^-13}{7}$)
4. $x = 5y - 2$
   $3x + 2y = 28$ (8, 2)
5. $3r - s = 5$
   $s + 2r = 5$ (2, 1)
6. $2q + 3p = 5$
   $3q + p = 2$ ($\frac{11}{7}$, $\frac{1}{7}$)

C. Solve the following exercises by using two equations and two unknowns.

1. The sum of two numbers is 48 and their difference is 14. Find the numbers. 17 ; 31
2. Bill's age increased by Harry's age is 50 years. Bill is 4 years older than Harry. How old is each? Bill, 27 ; Harry, 23
3. Mary and Helen have \$58. Helen has \$10 more than Mary. How much does each girl have? Mary, \$24 ; Helen, \$34
4. Twice a certain number decreased by a second number is 44. The sum of the two numbers is 40. Find the numbers. 28 ; 12
5. Three times a certain number decreased by twice another number is 50. The sum of the first number and twice the second is 70. What are the two numbers? 30 ; 20
6. The sum of two numbers is 228 and their difference is 128. Find the numbers. 50 ; 178
7. Three times a certain number increased by twice another number is 43. The sum of the first number and three times the second is 33. What are the two numbers? 9 ; 8

STEPS FOR SOLVING APPLIED PROBLEMS

1. Understand the problem.
2. Note what the problem asks for.
3. Look for hidden questions.
4. Estimate a reasonable answer.
5. Set up and solve the conditional sentence(s).
6. Check your answer.

Part Three

A. Find the following products.

1. $(x-4)(x+7)$ $x^2+3x-28$
2. $(x+5)(x+2)$ $x^2+7x+10$
3. $(a-2)(a-6)$ $a^2-8a+12$
4. $(r-70)(r-7)$ $r^2-77r+490$
5. $(s-2)(s-7)$ $s^2-9s+14$
6. $(x+7)(x-3)$ $x^2+4x-21$
7. $(c-4)(c+9)$ $c^2+5c-36$
8. $(d-5)(d+5)$ d^2-25
9. $(y-7)(y+7)$ y^2-49
10. $(d+8)(d-7)$ d^2+d-56

B. Factor.

1. x^2-5x-6 $(x+1)(x-6)$
2. x^2-4x+4 $(x-2)(x-2)$
3. a^2-7a+6 $(a-1)(a-6)$
4. $r^2-2r-15$ $(r-5)(r+3)$
5. y^2+6y+9 $(y+3)(y+3)$
6. $x+x^2-2$ $(x+2)(x-1)$
7. $s^2+9s+20$ $(s+5)(s+4)$
8. b^2+5b-6 $(b+6)(b-1)$
9. $x^2-8x+16$ $(x-4)(x-4)$
10. $r^2-3r-10$ $(r-5)(r+2)$

C. Solve the following quadratic equations by factoring.

1. $x^2-4x-5=0$ $5, {}^-1$
2. $r^2+r-12=0$ ${}^-4, 3$
3. $y^2-8y+12=0$ $6, 2$
4. $a^2=a+2$ $2, {}^-1$
5. $x^2-5x+6=0$ $3, 2$
6. $x^2-7x+6=0$ $6, 1$
7. $b^2-7b+10=0$ $2, 5$
8. $r^2-11r+10=0$ $10, 1$
9. $a^2-10a+25=0$ 5
10. $x^2=2x+8$ $4, {}^-2$
11. The base of a triangle is 2 inches less than the height. If the area is 12 square inches, find the base and height. b, 4 in.; h, 6 in.
    HINT: The area of a triangle $= \frac{1}{2}bh$.
12. The quotient of the square of a certain number and the number decreased by 4 is 16. Find the number. 8

CONSUMER CREDIT

WORDS TO WATCH FOR

arithmetic series
average principal
balance
carrying charge
charge account
consumer credit
credit rating
depreciation
down payment
foreclosure
installment plan
modulus
mortgage
open book account
personal loan

"Buy Now–Pay Later" and "No money down on approved credit" are familiar slogans which appear in newspapers and periodicals, on radio and television, and on billboards along the highway. Offering the consumer immediate purchase of something for which he does not have ready cash is an evident part of the business operation.

When you purchase goods or services with a promise to pay later you are using *consumer credit.* If you think about it, you will recognize that many consumer transactions involve credit. This credit is an expression of confidence that a seller places in a buyer. The amount of credit extended is based on several important factors: (1) the individual's ability to pay, (2) his previous record in using credit, and (3) the amount of security he can provide. Security usually consists of other goods that have a cash value which can be used to meet an unpaid debt.

Consumer credit has a few basic identifiable forms: the *charge account* (30 day, 60 day, 90 day, and revolving), the *installment plan,* and the *personal loan.*

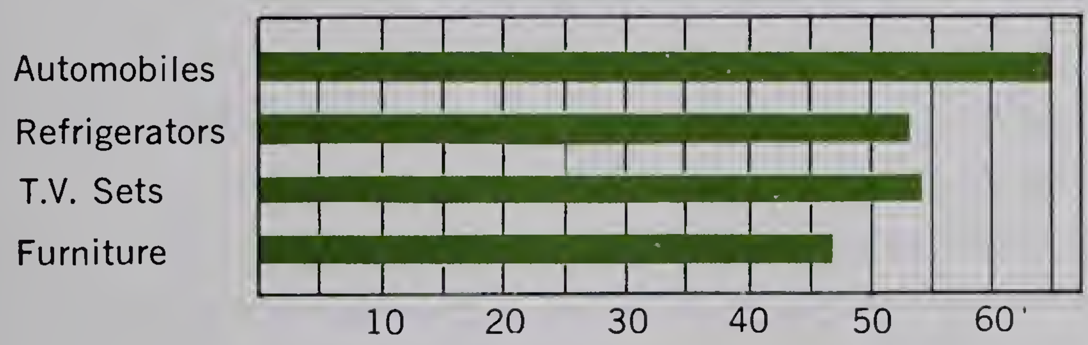
Everyone uses consumer credit, sometimes wisely, sometimes unwisely. To make the best use of credit privileges, the consumer needs to know how the credit operation works.

When credit is provided, the consumer should know how much he pays for credit and how he can figure out what additional expenses he is paying for when he uses it. This is especially important with installment credit, since it is the most prevalent type. In a recent report of debt distribution only $17.5 billion consisted of charge accounts and other non-installment debt out of a total personal debt of $263.5 billion. The remaining $246 billion was installment debt.

93.4%

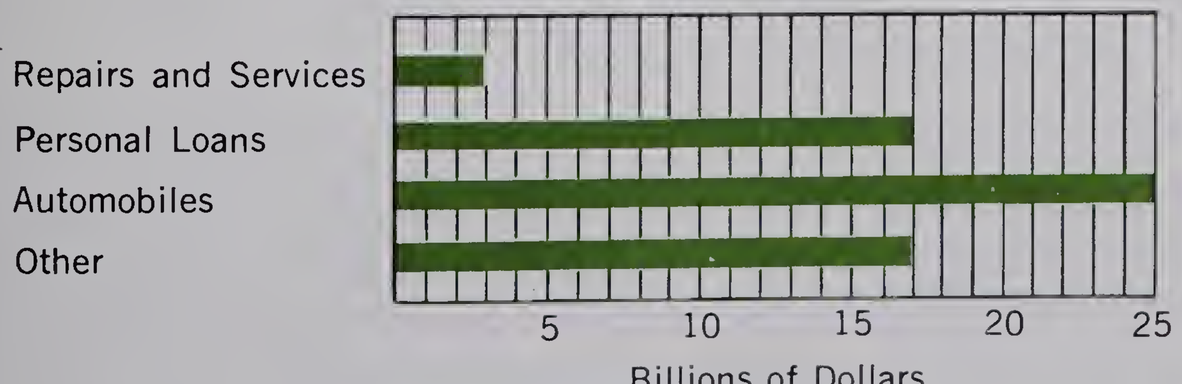
1. What per cent of the outstanding personal debt is installment debt?
2. At the time of the report there were 70 million employed workers in this country. How much was the installment debt per worker? $3514
3. The report stated that each month an additional $6 billion of new installment credit was incurred in this country. This was how much per worker? $ 85.71
4. The report added that each month $4.8 billion of installment debt was retired. At this rate, how much was the increase in installment debt per year? $ 14.4 billion
5. Some commodities are more commonly purchased on the installment plan than others. In the graph you can see what they are. What per cent of each commodity was purchased on the installment plan? See front.

INSTALLMENT PURCHASES OF COMMODITIES

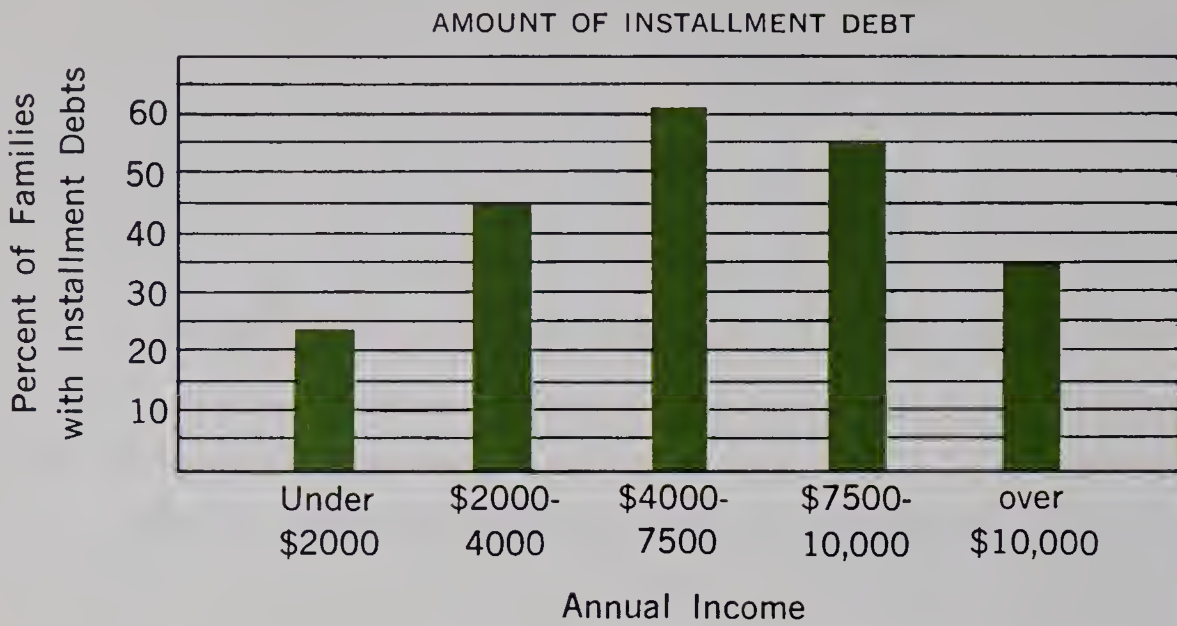


6. In the graph below, you can see how much installment credit was used in several ways. How much was used in each way? See front.

OUTSTANDING INSTALLMENT DEBT



7. The amount of installment credit used by a family is related to size of family income, as can be seen in the graph below. At what income level is the least amount owed? *under $2000*



8. At what level does the installment debt tend to be greatest? *$4000 – $7500*

FOR DISCUSSION AND RESEARCH

1. Give some examples in addition to those listed above of goods and services that are usually obtained on credit.
2. What goods and services are usually paid for in cash?
3. What articles besides those mentioned above are commonly purchased on the installment plan?
4. Many people carry credit cards in order to avoid carrying large amounts of cash, especially on trips out of town. What kinds of credit cards do you know about?
5. In arranging credit for a customer, a merchant will usually call the credit bureau to find out about the customer's credit rating. Is there a credit bureau in your community? If so, find out what information it collects about the individual to determine his rating.
6. A great convenience for the traveling motorist is the gasoline credit card. What expenses do you think the oil company incurs in providing credit through credit cards? Does the motorist who pays cash get his gasoline any cheaper?
7. In what ways can a grocery store save money by selling for cash only?
8. Should the saving in the "cash only" store be passed on to the customer? Compare the prices on some standard articles in two stores, one that sells only for cash, and the other that offers credit. Is there a saving in the store that does not offer credit?

THE CARRYING CHARGE

Since the merchant incurs added expenses for clerical work and possible failure of the customer to keep up his installment payments, there is usually a *carrying charge* added to the cash price of the merchandise sold on installments. The purchaser commonly makes a down payment, and the remainder of the price is paid in a certain number of weekly or monthly installments.

1. Give some examples of extra clerical work that might be incurred by the merchant selling on the installment plan. *See front.*

2. The price of a radio purchased on the installment plan is $50. There is a 10% reduction in price if it is purchased for cash. How much is the carrying charge on the installment plan? *$ 5*

3. An electric stove with a cash price of $278.00 sells on the installment plan for $10 down and $16 per month for 18 months. What is the total cost if the stove is bought on the installment plan? *$ 298*

4. The difference between the cash price and the total cost on the installment plan is the carrying charge. What was the carrying charge on the stove? *$ 20*

5. An advantage of buying on the installment plan is that the article may be used while the money is accumulated to pay for it. The Martins had to replace their refrigerator. The cash price on the model they wanted was $329.00. The installment price was $35 down and installments of $13.95 per month for two years. How much more than the cash price was the installment price? *$ 40.80*

6. Eddie Tolland bought a radio for $5.00 down and $3 per week for 12 weeks. The cash price was $36.95. How much did Eddie pay for the privilege of buying on a credit payment plan? *$ 4.05*

7. A large percentage of automobiles are now sold on some type of installment plan. An advertisement in a recent paper offered a new car $100 down and $88.48 per month for 36 months. What would be the total cost of the car? *$ 3285.28*

8. The cash price of the car in Exercise 7 is $2999. How much is the carrying charge? *$ 286.28*

9. Mrs. Henderson can buy a certain model refrigerator for a cash price of $250. On the installment plan it is $10 down and $15 a month for 18 months. What is the total cost if she buys it on the installment plan? How much carrying charge would Mrs. Henderson pay on the installment plan? *$ 280 ; $ 30*

10. Nat and Jim shopped for a used car. They found a car which they liked priced for \$550. They could buy the car for 10% down and \$43.50 per month for 12 months. How much is the carrying charge on the used car? $ 27

11. Martin wants to buy a radio priced at \$47.50. He can get it on the installment plan for \$8 down and \$2 a week for 28 weeks. How much is the carrying charge? $ 16.50

12. Jane averages \$6 a week earned by baby sitting. She can save \$3.50 a week and has saved \$16.50. She wants a new bicycle, priced at \$60 cash. The dealer will let her have it for \$15 down and \$2.50 a week for 22 weeks. How much is the carrying charge? $10

13. How many weeks would Jane have to wait before she has saved up enough money to pay cash for the bicycle? 13 weeks

14. Jim works eight hours each Saturday clerking at the supermarket. He is paid 85¢ an hour. How much does he earn each week? $6.80

15. Jim can save \$3.25 each week from his earnings. He has already saved \$9.75. He wants to buy a wrist watch that is priced at \$32.50. How many weeks will it take him to save the balance? 7

16. On the installment plan Jim can pay \$10.50 down and have his choice of terms:

\$2.50 for 10 weeks, or $ 3
\$9.00 a month for 3 months. $ 5

Find the carrying charge by each plan.

SPECIAL PROJECTS

1. What are the advantages of installment buying for the customer? for the merchant?
   What are the dangers and disadvantages of installment buying for the customer? for the merchant?
2. Set up a list of cautions for anyone who is considering purchasing on the installment plan.
3. If the installment payments are not met, the merchandise may be *repossessed* and sold again by the merchant. Do you think he is fully protected against loss by this arrangement? Explain your answer.
4. Select two or three advertisements that offer appliances, such as refrigerators, stoves, television, etc., that can be purchased under an installment plan. Calculate the carrying charge.
5. See if you can obtain a copy of a time-payment agreement that a person signs when making a major purchase. Report to the class.

A. Find the value for x:

1. $x + 9 = 27$ 18
2. $42x = 378$ 9
3. $x \div 9 = 25$ 225
4. $26 - x = 12$ 14
5. $176 \div x = 32$ 5.5
6. $19 + x = 37$ 18
7. $18 = x \div 7$ 126
8. $13x = 117$ 9

B. Find the value for n:

1. $\frac{7}{15} = \frac{n}{75}$ 35
2. $\frac{3}{25} = \frac{18}{n}$ 150
3. $\frac{6}{n} = \frac{96}{144}$ 9
4. $\frac{n}{5} = \frac{16}{20}$ 4
5. $\frac{9}{n} = \frac{72}{100}$ 12.5
6. $\frac{16}{25} = \frac{n}{100}$ 64

C. Copy and find the value for n.

1. 45 is 30% of n 150
2. 14% of 170 is n 23.8
3. 120% of 85 is n 102
4. 75% of 240 is n 180
5. n% of 120 is 24 20
6. 15 is n% of 750 2
7. 12.5% of 1200 is n 150
8. 60 is 37.5% of n 160
9. 200% of n is 320 160
10. 130% of 600 is n 780
11. 150% of n is 57 38
12. n is 2.5% of 720 18
13. n% of 64 is 96 150
14. n% of 8.40 is 4.80 57.1
15. 186 is 75% of n 248
16. 2.3% of 400 is n 9.2

D. Find the simple interest.

	Principal	*Rate*	*Time*
1.	$250	4%	90 days $2.50
2.	$150	5%	135 days $2.81
3.	$85.50	3%	200 days $1.43
4.	$350.75	5%	8 months $11.69
5.	$750	6%	10 months $37.50

If you need practice, turn to the exercises on page 486. If not, you may work in the Experts' Corner.

Modular Arithmetic

In the arithmetic we are familiar with, we say that $8 + 6 = 14$. This numerical relationship is true in most of the situations we encounter. On the other hand, with our everyday clock, six hours after 8 o'clock is not 14 o'clock, but 2 o'clock. That is, $8 + 6 = 2$. Similarly, 9 hours before 3 o'clock is 6 o'clock, or $3 - 9 = 6$.

Clearly, in our system of telling time, we use some special rules that are out of the ordinary, but which are typical of *modular arithmetic*. According to these rules, if two numbers differ by 12, or a multiple of 12, they are equal. In the first equation above $14 = 2$, because $14 = 2 + 12$. In the second equation $3 = 15$, because $3 + 12 = 15$. In like fashion all numbers can be expressed as integers 1 through 12. This relationship is referred to as modulo 12.

Since the equality here has a specialized meaning, we use the symbol $\equiv$ instead of the sign for equality and call the statement a *congruence* rather than an equation. Thus:

$$14 \equiv 2 \text{ modulo } 12 \qquad\qquad 15 \equiv 3 \text{ modulo } 12$$

The phrase *modulo 12* specifies the condition that the relationship is true only if we neglect multiples of 12.

1. You can find a number congruent to a given number, modulo 12, by adding or subtracting a multiple of 12. Find a number congruent to each of these and set up the congruence statement:

a. 15 *3*	**c.** 5 *5*	**e.** 56 *8*	**g.** 36 *0*
b. 27 *3*	**d.** 19 *7*	**f.** 4 *4*	**h.** 112 *4*

2. What happens if we encounter negative numbers? Does

$$^-25 \equiv 11 \text{ modulo } 12?$$

Think of a clock: What is 25 hours before noon?
Then add $^+36$ to $^-25$. Does it check? *yes*

3. If we multiply both sides of a congruence by the same number, will the new congruence be true?
Try it with this congruence and check the result:

$$\begin{array}{rcl} 18 & \equiv & 6 \text{ modulo } 12 \\ \underline{\times 5} & & \underline{\times 5} \\ \square & \equiv & \square \text{ modulo } 12 \end{array}$$

90 ≡ 6 modulo 12

Test the operation with several other congruences.

4. If we add two or more congruences, will the new congruence be true? Check this addition and see:

$$\begin{array}{rcl} 17 & \equiv & 5 \text{ modulo } 12 \\ +\underline{15} & \equiv & \underline{3} \text{ modulo } 12 \\ \square & \equiv & \square \text{ modulo } 12 \end{array}$$

32 ≡ 8 modulo 12

Test the operation with addition of several other congruences.

5. If we multiply two congruences, will the products constitute a true statement? Check this result:

$$\begin{array}{rcl} 14 & \equiv & 2 \text{ modulo } 12 \\ \underline{4} & \equiv & \underline{16} \text{ modulo } 12 \\ 56 & \equiv & 32 \text{ modulo } 12 \\ 8 & \equiv & 8 \text{ modulo } 12 \end{array}$$

Test the operation with several other products.

Some of these properties of the congruences become very interesting with modulus 9. We know that 10 ≡ 1 modulo 9, because if we remove 9 from 10 the remainder is 1. If we multiply the congruence by itself, we have: 100 ≡ 1 modulo 9.

1. Test this congruence. If you remove every multiple of 9 from 1000, do you have 1 as a remainder? *yes*

2. If we multiply these two congruences (of 10 and 100), we have 1000 ≡ 1 modulo 9. Check this to see if it is true.
   Show that any power of 10 is congruent to 1, modulo 9.
   Yes; one can repeatedly multiply 10 ≡ 1 modulo 9 by itself.

3. Explain how you could obtain this congruence from one of the above congruences: 50 ≡ 5, modulo 9.
   Multiply both sides of 10 ≡ 1 modulo 9 by 5.

4. Find a single-digit number congruent to each of these modulo 9:

a. 20 *2*	**c.** 40 *4*	**e.** 18 *0*	**g.** 800 *8*
b. 300 *3*	**d.** 2000 *2*	**f.** 60 *6*	**h.** 70 *7*

5. The number 562 may be written as: 500 + 60 + 2.
   Write the single-digit number congruent to each of these addends, modulo 9. *5, 6, 2*

6. Add the congruences and write the single-digit number congruent to 562, modulo 9. *562 ≡ 4 modulo 9*

7. Show that this statement is true: Any number is congruent to the sum of its digits, modulo 9. *Use the distributive property: 1000a ≡ a modulo 9*

8. Explain why this is true: Any number minus the sum of its digits is divisible by 9. *From exercise 7 it follows that 1000a − a = 999a, which is divisible by 9.*

9. An important application of these relationships is in checking an operation in addition. The sum of the digits in the result should equal the sum of the digits in the addends, if multiples of 9 are removed. For example:

$$\begin{aligned} 57 &\equiv 3 \text{ modulo } 9 \\ 283 &\equiv 4 \text{ modulo } 9 \\ \underline{46} &\equiv \underline{1} \text{ modulo } 9 \\ 386 &\equiv 8 \text{ modulo } 9 \end{aligned}$$

If we add two or more congruences, the sum is a congruence.

Is the congruence true for the sum? yes Why should it be?

10. This method of checking is commonly called "casting out 9's." Add and check by casting out 9's.

a.	**b.**	**c.**	**d.**
45	86	28	19
102	13	5	122
19	103	602	15
34 200	85 287	96 731	205 361

11. Since the product of two congruences is a true congruence, we can use the method for checking an operation in multiplication.

$$\begin{aligned} 159 &\equiv 6 \text{ modulo } 9 \\ \underline{48} &\equiv \underline{3} \text{ modulo } 9 \\ 7632 &\equiv 18 \equiv 0 \text{ modulo } 9 \end{aligned}$$

Multiply and check:

a.	**b.**	**c.**	**d.**
214	569	158	813
83 17,762	18 10,242	46 7268	55
			44,715

12. In subtracting congruences, you are likely to encounter negative numbers. Remember that you can add as well as subtract multiples of 9 to a term in a congruence modulo 9. Complete these congruences:

a. $^{-}5 \equiv$ _?_ 4 **b.** $^{-}3 \equiv$ _?_ 6 **c.** $^{-}7 \equiv$ _?_ 2 **d.** $^{-}8 \equiv$ _?_ 1

13. Use this example in explaining how you can check a subtraction operation by casting out 9's.

$$\begin{aligned} 257 &\equiv 5 \text{ modulo } 9 \\ \underline{166} &\equiv \underline{4} \text{ modulo } 9 \\ 91 &\equiv 1 \text{ modulo } 9 \end{aligned}$$

The difference of two congruences is a congruence.

14. Perform these subtractions and check by casting out 9's:

a.	**b.**	**c.**	**d.**
317	556	832	119
148 169	318 238	577 255	88 31

15. Casting out 9's is useful for finding errors, but it does not guarantee accuracy of computations. For example, if you wrote 459 in a product instead of 495, the error would not be revealed by casting out 9's. Explain why. The sum of the digits is the same.

To find the interest on a loan when we know the rate and the time the loan is to run before it is paid, we have been using the formula:

$$i = prt$$

in which the relationship *product* = *factor* × *factor* exists. Occasionally we need to find one of the other elements in the formula—the rate, the time, or the principal. Then the formula is more convenient to use if we *solve* it for the variable that is unknown.

Since we are familiar with the factor-factor-product relationship, which provides several equivalent forms of an equation, this new formula is not difficult. To determine the *rate* of interest, as we shall be doing in many of the problems in this chapter, the formula is in the most convenient form when it is solved for r.

EXAMPLE

Solve for r: $\quad i = prt$

Using the associative and commutative properties

$$i = (p \cdot t) \cdot r \qquad p = f_1 \cdot f_2$$

$$r = \frac{i}{pt} \qquad f_2 = p \div f_1$$

1. Mr. Anderson borrowed \$600 for a 90-day period and paid \$7.50 interest. What rate of interest did he pay? 5 %

   $$i = \$7.50 \quad p = \$600 \quad t = \frac{1}{4}$$

   Complete the solution.

2. Mr. Smith paid \$6.50 interest on a loan of \$300 that he paid up at the end of four months. What was the rate of interest? 6.5 %

3. Mr. Hanson borrowed \$150 for three months from a personal loan agency. The agency charged \$4.50 in interest plus a clerical fee of \$1.50. The total interest, therefore, amounted to \$6. What was the actual rate of interest? 16 %

4. Jim borrowed \$200 from a personal loan agency to purchase an outboard motor. In addition to interest at the rate of 8% for the six-month period of the loan, he was required to pay a service charge of \$3.50 for clerical expenses, checking his credit rating, and other details. The entire expense of making the loan is to be considered as part of the interest. What was the actual rate of interest? 11.5 %

5. Sometimes it is important to find how long it will take a given sum of money at a given rate of interest to earn a stated amount of interest. In that case we wish to use the formula solved for t:

$$i = (p \cdot r) \cdot t \qquad p = f_1 \cdot f_2$$

Write the formula solved for t. $t = \frac{i}{pr}$

6. How long will it take \$500 to earn \$7.50 interest at 6%? 3 mo.

7. Mr. Henderson borrowed \$750 at 5%. He repaid the loan when it was due with \$12.50 interest. What was the term of the loan? 4 mo.

8. How long will it take a loan of \$1250 to earn \$37.50 in interest at the rate of 4%? 9 mo.

9. When we need to find the principal that will yield a stated amount of interest at a given rate for a specified time, we need to solve the interest formula for p:

$$i = p \cdot (r \cdot t) \qquad p = f_1 \cdot f_2$$

Write the formula, solved for p. $p = \frac{i}{rt}$

10. What principal, loaned at 5%, will earn \$80 interest in eight months? \$2400

11. Mr. Edison paid \$30 interest on a loan that had run six months at 6%. How much had he borrowed? \$1000

Copy this chart and find the missing answers. Do not mark your book.

	Principal	*Rate*	*Time*	*Interest*
12.	\$1500	☐ 6%	90 days	\$22.50
13.	3000	5%	180 ☐ days	75.00
14.	300	☐ 6%	180 days	9.00
15.	800	☐ 4.5%	40 days	4.00
16.	750	8%	90 days	☐ \$15.00
17.	1500	6%	120 ☐ days	30.00
18.	500	☐ 7.5%	18 mo.	56.25
19.	360	☐ 8%	15 mo.	36.00
20.	525	☐ 13.3%	9 mo.	52.50
21.	750	☐ 7.5%	16 mo.	75.00
22.	875	☐ 8%	27 mo.	157.50

Arithmetic Series

A series of numbers that increases or decreases by the same difference from one number to the next is an *arithmetic series.* These are examples of arithmetic series:

1,2,3,4 18,15,12,9,6,3 2,4,6···20

Two interesting problems arise in dealing with arithmetic series. One is finding the sum of the series, and the other is finding the average value of a term in the series.

Suppose you are to find the sum of the natural numbers from 1 to 10; that is:

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = ?$$

1. Before adding, let us look for pairs of equal values. What is the sum of the first and last numbers (1 + 10)? *11*

2. What is the sum of the second and next to the last, that is 2 + 9? *11*

3. Following the same pattern, how many pairs can you find? *5* What is the sum of each pair? *11*

4. State how you can find the sum of the series without adding up the numbers in the series. *The sum of the series equals the sum of each pair times the number of pairs.*

5. What is the sum? *55*

6. If there are n numbers in the series, how many pairs will there be? $\frac{n}{2}$

7. If the initial number in the series is a and the last is l, what is the sum of each pair? $a + l$

$S = \frac{n}{2}(a + l)$

8. Using the variables n, a, and l, set up a formula for the sum of an arithmetic series. (If you do not see it right away, go back and look over your answers to Exercises 1 to 7. You may want to find the sum of another series, as for example, the odd numbers from 1 through 19. Try to figure it out before going on to the next exercise.)

9. If you were successful with Exercise 8, you came up with the formula: $S = \frac{n}{2}(a + l)$. Use the formula to find the sum of each of these series:

a. The numbers 1 through 16. *136*
b. The even numbers 22 through 48. *490*
c. The series 80,75,70···5. *680*
d. The series 10, 16, 22 ···70. *440*

10. What happens if we try to find the sum of an odd series of numbers, such as 1,2···9?

a. How many pairs are there? 4

b. What number is left over? 5

c. How does it compare in value with the sum of each of the pairs? $\frac{1}{2}$

11. What is the sum of the first nine numbers that are divisible by 3 (as 3,6,9···)? 135

12. Note that you do not need to count numbers in order to find the ninth term, in Exercise 11. If d is the common difference between the terms in the series and a is the initial term, the second term is $(a + d)$, the third term is $(a + 2d)$, and so on. What is the fourth term? the seventh? the ninth?
$a + 3d$ $a + 6d$ $a + 8d$

13. In Exercise 11, $a = 3$ and $d = 3$. What is the twelfth term? 36 the fifteenth? 45 the nth term? $a + (n-1)d$

14. Another way to interpret the formula for the sum of an arithmetic series is: The average value of a term in the series is $\frac{a + l}{2}$. Since there are n terms, we multiply the average value by n:

$$S = n\left(\frac{a + l}{2}\right) \qquad \text{Notice } l = [a + (n - 1)d]$$

What is the average value of the first ten numbers that are divisible by 5? 25

15. What is the average value of each term in each of these series?

a. The first 15 even numbers. 15

b. The first 18 odd numbers. 18

c. The first 20 numbers divisible by 6 (6,12,···) 60

16. Mr. Anderson borrowed \$250. He is to pay it back in ten payments of \$25 per month, plus interest on the unpaid balance. What is the average amount he owes during the period? Let l be the last payment. \$ 137.50

This is an arithmetic series: 250,225,200···and so on.

What is a? 250 What is l? 25 What is $\frac{a + l}{2}$? 137.50

17. Find the average amount owed in each of these transactions:

	Initial debt		*Monthly payment on principal*
a.	\$150	\$82.50	\$15
b.	\$275	\$150.00	\$25
c.	\$300	\$156.25	\$12.50
d.	\$60	\$32.50	\$5
e.	\$360	\$186.00	\$12

When buying on installment payments, the purchaser is concerned with two questions:

How much more does it cost on installments than for cash?

What rate of interest am I paying on the average debt?

The first question is readily answered. The answer to the second question is not so obvious. To use the formula $r = \frac{i}{pt}$ we need first to know, "How much is the *interest*?" Since the purchaser has the option of going to the bank to borrow what he needs to pay the cash price, anything above the cash price must be considered interest. In installment buying

$$\text{Interest} = (\text{Installment price}) - (\text{Cash price}).$$

Since the debt decreases with each installment payment, what principal shall we use for p in the formula? Actually we take the average amount owed. This is $\frac{\text{initial debt} + \text{final debt}}{2}$. For initial debt we take the sum one might borrow from the bank to pay the cash price of the article. If there is a down payment, the initial debt is the difference between the cash price and the down payment. Do you see why?

The final debt is the payment on principal in the last installment. It is found by dividing the initial debt by the number of installments.

There is little question about the variable t. This is the time between the purchase and the final payment.

Let us see in an example how we identify the numbers to substitute for the variables in the formula $r = \frac{i}{pt}$.

EXAMPLE

The cash price of a radio is \$60. On installments the terms are \$7.50 down payment and \$5 per week for 11 weeks. What rate of interest is the purchaser paying?

a. What is the interest?

Total price by installments	\$62.50
Cash price	60.00
Difference (this is Interest)	2.50

b. What is the average amount owed over the 11 weeks (p in the formula)?

We take as *initial debt* the cash price—\$60—less the down payment—\$7.50. This is \$52.50. This we label p_i

The *final debt* is $\frac{1}{11}$ of \$52.50, since it is paid off in 11 installments.

This is \$4.77. This we label p_f

Then the *average amount owed* is $\frac{\$52.50 + \$4.77}{2} = \$28.64$ which is p_a or p in the formula.

c. Time is 11 weeks, or $\frac{11}{52}$ of a year.

Substituting in the formula,

$$r = \frac{2.50}{28.64 \times \frac{11}{52}} = \frac{2.50}{6.06} = 0.413$$

To the nearest whole per cent, this is a rate of 41% in interest.

Let us follow the Barnes family as they calculated the interest rate while considering the purchase of a television set.

1. The TV set they were interested in was priced at \$265, with a 10% discount for cash. What was the *cash price*? *\$ 238.50*

2. The installment price was \$295 with a \$10 down payment and 19 monthly payments of \$15. How much was the *initial debt*, p_i cash price less down payment? *\$ 228.50*

3. How much was interest—total installment price less cash price? *\$56.50*

4. How much is each monthly payment on principal—initial debt divided by 19? (Where does the 19 come from?) *\$ 12.03*

5. How much is the *final debt*, p_f? *\$ 12.03*

6. Set up a statement to show how each monthly payment is divided between principal and interest.

Monthly payment on principal $\left(\frac{1}{19}\text{ of initial debt}\right)$	?	*\$ 12.03*
Interest (rest of \$15)	?	*2.97*
Total payment	\$15	

7. As you can see, the principal decreases regularly each month, constituting an arithmetic series. What is a? What is l? What is d? What is n? (Note l is the last payment on the principal.) *See front.*

8. To find the average principal p_a, Mr. Barnes made the calculation: $p_a = \frac{228.50 + 12.03}{2}$. Explain where the numbers came from and why this is the average principal. How much is the average principal? *$\frac{a+l}{2}$; \$ 120.27*

9. What is t in the formula? *$\frac{19}{12}$ years*

10. The amount of interest being paid is often overlooked by the purchaser. The interest is:

(Total installment price) – (Cash price)

You calculated this in Exercise 3.
Substitute the values in the formula and calculate the rate of interest Mr. Barnes is paying, to the nearest whole per cent. 30 %

11. Mr. Barnes could borrow the money to pay the cash price from the bank, which charges 6% interest on such a loan. What is the difference between the rate charged by the bank and the rate charged on the installment purchase? *The installment rate is five times as great.*

Installment Rate Of Interest

To summarize, the steps for finding the rate of interest on an installment purchase are as follows:

a. The conditional statement is based on the formula for finding rate of interest: $r = \frac{i}{pt}$. It is necessary to find values for p, t, and i before setting up the conditional statement.
b. To find i, subtract the cash price from the total amount paid using the installment plan.
c. To find the average principal take $\frac{\text{initial debt} + \text{final debt}}{2}$ $\quad \frac{a + l}{2}$
d. t is the time in years it takes to pay the installments.
e. Substitute in the formula; solve the conditional statement.

EXAMPLE

The cash price of a refrigerator is $200. On installments the terms are $10 down and $12 per month for 18 months. What rate of interest will the purchaser be paying?

The statement for each month is:

Payment on principal	$10.56
Interest payment	1.44
Total	$12.00

i = $26. (How was this determined?)

$p = \frac{\$190 + \$10.56}{2} = \$100.28 \qquad t = 1.5$ years

Substituting in the formula, $r = \frac{26}{100.28 \times \frac{3}{2}} = 26 \div 150.42$

Then $r = .173$, or 17%, to the nearest whole per cent.

Let us examine another example using the method illustrated on the previous page. To find the rate of interest on the installment plan for an article, cash price \$132 or \$15 down and \$11.25 a month for 12 months, we proceed by writing the statement for each month

Payment on principal	\$ 9.75
Interest payment	1.50
Total	\$11.25

$$i = \$18 \qquad p_i = \$117 \qquad p_f = \$9.75$$

Then
$$p_a = \frac{p_i + p_f}{2} = \frac{\$117 + \$9.75}{2} = \$63.38$$
$$r = \frac{i}{p_a} = \frac{18}{63.38} \approx .284 \text{ or } 28\%$$

Follow the method illustrated above in finding the rate of interest, to the nearest 1%, on each of these installment plans. Be sure to start by setting up the statement to show what each payment contributes to principal and to interest.

1. Jack Barnes bought a bicycle on the installment plan. The cash price was \$62.40. Jack paid \$10 down and made 12 monthly payments of \$4.90. 23%

2. A combination washing machine and drier was regularly priced at \$540 cash. On the installment plan it sold for \$140 down and 16 monthly payments of \$30. 28%

3. A vacuum cleaner priced at \$79.90 was offered for sale at \$10 down and six monthly payments of \$12.50 each. 25%

4. A car on the used lot was marked \$1000 cash. It was sold for \$333.33 down and 24 monthly payments of \$33.33. 19%

Here are some selected items from advertisements in the daily paper. Find the rate of interest on the installment plan in each.

5. A radio for \$180 cash. Terms, \$30 down and 11 monthly payments of \$15. 20%

6. A refrigerator, cash price, \$275. Terms, \$50 down, 12 monthly payments of \$20 each. 12%

7. A rug, cash price \$175. Terms, \$15 down, \$20 a month for nine months. 30%

8. A bicycle, cash price \$60. Terms, \$20 down, \$5 a month for 12 months. 92%

9. Golf clubs and bag, cash \$150. Terms, \$30 down, \$12.50 a month for 12 months. 46%

1. The cash price of a radio is \$42.30. The "easy payment plan" is \$4 down and \$1 a week for 43 weeks.

a. How much more does the radio cost on the installment plan than for cash? *\$ 4.70*

b. How much in addition to the down payment would be needed to buy the radio for cash? *\$ 38.30*

c. Your answer to **b** is the principal. It is being paid up in weekly payments for 43 weeks. How much is the interest? *\$ 4.70*

d. Prepare a statement showing how much of each payment is on the principal and how much is interest. What is t? $\frac{43}{52}$ *yr.*

e. What is p_i? p_f? p_a? *p, \$ .89 ; i, \$ .11*

f. What rate of interest is paid on the easy payment plan? *29 %*

\$ 38.30 ; \$ .89 ; \$ 19.60

2. Jim can purchase a bicycle for \$55 cash. The installment terms are \$7.50 down and \$1.25 a week for 44 weeks.

a. How much in addition to the \$7.50 down payment would Jim need to pay the cash price? This is the initial principal. *\$ 47.50*

b. How much more than this is the total of Jim's payments? This is the interest. *\$ 7.50*

p, \$ 1.08 ; i, \$ .17

c. Prepare a statement showing how much of each payment is on the principal and how much is interest. What is t? $\frac{44}{52}$ *yr.*

d. What is p_i? p_f? p_a? *\$ 47.50 ; \$ 1.08 ; \$ 24.29*

e. What rate of interest is Jim paying if he purchases the bicycle on installments? *36 %*

3. Jim has the \$7.50 he needs for the down payment. He can borrow \$47.50 from the bank so that he can pay cash for the bicycle. He will repay the loan in ten monthly payments of \$4.75 each plus $1\frac{1}{2}\%$ of the unpaid balance each month as interest.

a. How much is the interest at the end of the first month? *71¢*

b. What is the total payment for the first month? *\$ 5.46*

c. How much is the interest at the end of the second month? *64 ¢*

d. What is the total payment for the second month? *\$ 5.39*

e. List the total payments for each of the ten months. *See front.*

f. What is the total amount of interest Jim will pay to the bank? *\$ 3.92*

g. What is the total amount of interest Jim would pay if he buys the bicycle on installments? *\$ 7.50*

h. Which method is cheaper? How much cheaper? *bank ; \$ 3.58*

4. The cash price of a refrigerator is \$399.50. On the installment plan, terms are \$99.50 down payment and \$11.50 a month for 30 months.

a. After the down payment is made, what is the balance of the cash price? *\$ 300*

b. This is being paid in how many payments? 30
c. Prepare a statement showing how much of each payment is on the principal and how much is interest. What is t? p, \$10; i, \$1.50 $\frac{5}{2}$ yr.
d. What is p_i? p_f? p_a? \$300; \$10; \$155
e. What rate of interest is being charged on the installment plan? 12%

5. The marked price on a TV set is \$400. Terms are \$100 down and \$11.50 a month for 30 months. A customer paying cash is given a 10% discount from the marked price.
   a. What is the cash price of the TV set? \$360
   b. After the down payment is made, what is the balance the purchaser would need to pay the cash price? \$260
   c. The balance is paid up in how many payments on the installment plan? 30 What is t? $\frac{5}{2}$ yr.
   d. What is the total amount of interest paid? \$85
   e. Prepare a statement showing what part of each payment goes to paying the principal, and what part is interest. p, \$8.67; i, \$2.83
   f. What is p_i? p_f? p_a? \$260; \$8.67; \$134.34
   g. What rate of interest is charged on the installment plan? 25%

6. A used car dealer offers an old model car for \$260 cash or \$100 down and \$25 a month for eight months.
   a. What is the balance of the cash price after the down payment is made? \$160
   b. The balance is paid up in how many payments on the installment plan? 8 What is t? $\frac{8}{12}$ yr.
   c. Show in a statement what part of each payment is on the principal and what part is interest. p, \$20; i, \$5
   d. What is p_i? p_f? p_a? \$160; \$20; \$90
   e. What rate of interest is charged on the installment plan? 67%

7. Mike is thinking of purchasing the car referred to in Exercise 5. He has \$100 for the down payment. He can borrow \$160 from the bank, paying back \$20 each month plus 1% of the unpaid balance each month as interest.
   a. How much interest is due at the end of the first month? \$1.60
   b. What is the total payment to the bank at the end of the first month? \$21.60
   c. What is the total payment for the second month? \$21.40
   d. List the total payments for each of the eight months. See front.
   e. What is the total amount of interest Mike would pay to the bank?
   f. How much would Mike save by borrowing from the bank rather than buying on the installment plan? \$32.80
   \$7.20

Most automobiles are purchased on the installment plan. The common practice is to require a down payment equal to $\frac{1}{3}$ of the purchase price and payment of the balance in monthly installments which include interest. On a new car the payments may be spread over two years or more. On a used car they are usually completed in two years or less.

One reason for requiring a down payment is that the value of a new car decreases sharply during the first year. Decrease in value of property due to age is called *depreciation.* The standard depreciation of an automobile is 40% of its purchase price during the first year, 20% of its price in the second year, and 10% in each of the next two years.

The purchaser of a car on installments signs a contract stating that if he does not keep up the payments the car may be reclaimed by the holder of the contract and sold to meet the unpaid balance. If the selling price is greater than the unpaid balance, the one signing the contract receives the difference; if it is sold for less than the balance due, he is liable for the difference. The dealer usually sells the contract to a finance company for the total payments contracted for, less the interest.

1. The Grommons purchased a new car for \$2400. How much is its depreciation for each of the first four years? *1st year, \$960; 2nd year, \$480; 3rd year, \$240; 4th year, \$240*

2. What is its value at the end of the first year? *\$1440*

3. The Grommons paid $\frac{1}{3}$ of the purchase price in cash. How much is the balance that they owe? *\$1600*

4. The finance plan which they arrange with the dealer calls for 24 monthly payments of \$72.60 each. What is the total of the payments? *\$1742.40*

5. Each payment includes a payment on the principal amounting to $\frac{1}{24}$ of the original balance. How much is the payment on the principal? *\$66.67*

6. The remainder of each payment is interest. How much interest do they pay each month? *\$5.93*

7. Set up a statement showing what each payment includes, in this way:

Payment on principal	\$?	*66.67*
Interest	?	*5.93*
Total	\$72.60	

8. What rate of interest are the Grommons paying? *9%*

9. How much do the payments on the principal amount to during the first year? *\$800*

10. How much do they still owe on the car at the end of the year (not counting the interest payments)? $800

$640

11. How much less is this than the value of the car after depreciation?

12. Suppose that the Grommons cannot keep up the payments on the car after the first year, and it is reclaimed and sold for the depreciated value. How much should the Grommons receive? $640

13. How much did the Grommons pay for the use of the car during the year they had it? $1671.20 (down payment plus 12 X $72.60)

14. Mr. Henderson purchased a $3000 car with a down payment of $500. The balance was to be paid in 36 installments of $80 each. How much does he pay on the principal (or unpaid balance) each month?

$69.44

15. Set up a statement (as in Exercise 7) showing payments on principal and interest each month. p, $69.44; i, $10.56

16. What rate of interest is Mr. Henderson paying? 10%

17. What is the unpaid balance owed on the car at the end of the first year? $1666.72

18. What is the depreciated value of the car? $1800

19. Suppose Mr. Henderson cannot keep up his payments and the car is reclaimed and sold at its depreciated value at the end of the first year. How much would Mr. Henderson receive from the sale?

$133.28

20. Jim Brown bought a second-hand car for $800 at a sale advertising "No Money Down." He was given 18 months to pay with monthly payments of $56.44. Set up a statement showing how each payment is divided between principal and interest. p, $44.44; i, $12

21. What rate of interest was Jim paying? 34%

22. Jim had been working at the filling station at $15 a week. He was using his salary to keep up the payments. After six months, however, business slowed up and Jim lost his job. How much did he owe on the unpaid balance at that time? $533.36

23. The car was reclaimed and sold for $400. How much did Jim still owe? $133.36

24. Mr. Adams plans to purchase a new car costing $2700. He will make a cash payment of $900. He will pay the balance in 18 monthly payments of $106.85 each. Prepare a statement showing how each payment is divided between principal and interest. p, $100; i, $6.85

25. What rate of interest is Mr. Adams paying? 9%

At one time pawn brokers and "loan sharks" were the primary source of "consumer loans," which carried high rates of interest. Today most commercial banks have set up personal loan departments, where consumer credit loans are handled separately.

To provide personal credit at a reasonable rate of interest, credit unions have been organized among many groups such as teachers, farmers, and labor unions. Members of a credit union deposit their savings with the group and can borrow at a reasonable rate.

In most states the rate of interest to be charged is set by the law. Personal loans are usually $300 or less. The signatures of two persons of good credit standing, besides the borrower, who will guarantee payment of the loan are usually required. The loan is paid in monthly or weekly installments.

1. Mr. Henderson borrowed $100 at the personal loan department of the First National Bank. He is to make a payment of $10 each month plus $2\frac{1}{2}\%$ of the unpaid balance as interest for ten months. How much is the interest for the first month? *$ 2.50*

2. What is the total payment for the first month? *$ 12.50*

3. Calculate the interest for each of the other nine months and find the total payment for each month. *See front.*

4. What is the total amount paid by Mr. Henderson in the ten months? *$ 113.75*

5. A simple means of calculating the total interest is first to find the average amount owed over the ten months. How much is it? *$ 55*

6. The interest on this amount is $2\frac{1}{2}\%$ of it per month. How much is this for ten months? *$ 13.75*

7. If you add to this the principal of $100, you have the total amount to be paid. How much is this? *$ 113.75*

8. Is your answer the same as in Exercise 4? (It should be.) *yes*

9. Mr. Jones borrowed $150, to be repaid in 15 monthly payments of $10 each plus monthly payment of 3% of the unpaid balance as interest. Prepare a statement for each month to show how much the payment is, and find the total. *See front.*

10. Check your answer for Exercise 9 by the short method. *$ 186*

11. Mr. Jones borrowed $100 from a loan company, to be repaid in 20 weekly payments of $5.25. What is the average balance? *$ 52.50*

12. What is the total interest paid by Mr. Jones? *$ 5*

13. How much would this amount to in a year if continued? *$ 13*

14. What is the rate of interest paid by Mr. Jones? *25%*

15. Mr. Henderson borrowed $250, to be repaid in 10 monthly payments of $27.50 each.

a. Prepare a statement to show how each payment is divided between payment on principal and on interest. *p, $ 25 ; i, $ 2.50*

b. What is the average principal? *$ 137.50*

c. How much interest is paid over the ten months? *$ 25*

d. What rate of interest is being charged? *22%*

16. Most states have passed laws establishing the maximum interest rate that may be charged on personal loans by small loan companies. In one state the law permits a rate of 3% per month on the first $100 and $2\frac{1}{2}$% per month on anything over $100.

a. Suppose $100 is borrowed in that state with interest and principal payable in ten monthly payments, how much is the first payment? *$ 13*

b. Show in a statement how the payment is divided between principal and interest. *p, $ 10; i, $ 3*

c. What is the average amount owed during the ten months? *$ 55*

d. What is the total amount of interest paid? *$ 16.50*

e. What rate of interest is being paid? *36 %*

17. A simple method of approximating the rate of interest on an installment loan is to use the formula:

$$r = \frac{\text{annual interest}}{\frac{1}{2}\text{ of initial principal}}$$

Annual interest may readily be calculated by taking 12 times the monthly interest, or 52 times the weekly interest, as shown on the statement for the payment.

Use this method, and also the one you have been using, to find the rate of interest on the following loan. Then note which gives the higher rate, whether the difference is large enough to be significant, and which method you find simpler. You may use the method you prefer on the remaining problems. $250 is borrowed, to be repaid in 14 monthly payments of $21 each. What rate of interest is the borrower paying? Find the rate of interest on the following loans: *regular, 28 % ; short cut, 30%*

18. $40 to be repaid in 12 weekly payments of $3.50 each.
regular, 40 % ; short cut, 43 %

19. $60 to be repaid in three monthly payments of $22 each.
regular, 60%; short cut, 80 %

20. According to the plan used by a certain personal loan company, a borrower pays 6% of the loan as interest plus 2% of the loan as charges for investigation. These charges are deducted from the loan in advance. How much would be deducted if he made a loan of $200? *$16*

21. This loan is paid back in fifty weekly payments of $4. Show how much of the payment is payment of principal and how much is interest. (Deduct the charges in advance to find the principal of the loan.) *p, $3.68; $.32*

22. Find the rate of interest the borrower is paying in Exercise 21. *18%*

23. Amos Jensen borrowed $150 for one year on the following terms: 10% of the loan was deducted in advance; the loan was to be repaid in 12 monthly installments of $12.50 each. What rate of interest was Amos Jensen actually paying? *regular, 21%; short cut, 24%*

24. Arnold Anderson needed $300 to make a down payment on a car. He borrowed $300 on the following terms: 5% of the loan deducted in advance as interest, the loan to be repaid in 12 monthly installments of $25.00. What rate of interest was he paying? *regular, 10%; short cut, 11%*

25. The Liberty Personal Finance Company will lend sums up to $300. It charges $2\frac{1}{2}$% a month on the first $100 of balance, 2% on the second, and 1% on the third. How much is the interest per month on a loan of $100? $200? $300? *$2.50, $4.50, $5.50*

26. Here is an advertisement by a loan company.

HOME OWNERS—LOANS!

You Borrow	*Pa*	*You Pay*	*i*	*Term*	*r*
$ 500.00	*$256.91*	$15.00 per mo.	*$85.02*	39 months	*10%*
$1,000,00	*510.42*	25.00 per mo.	*200.16*	48 months	*10%*
$1,500.00	*765.63*	37.50 per mo.	*300.00*	48 months	*10%*
$2,000.00	*1018.52*	45.00 per mo.	*429.84*	54 months	*9%*
$2,500.00	*1273.15*	55.50 per mo.	*496.80*	54 months	*9%*
$3,000.00	*1525.00*	60.00 per mo.	*600.00*	60 months	*8%*

These payments include principal and interest.

Select three of the amounts and calculate:

a. Average principal of the loan over the term of the loan

b. Amount of interest paid

c. Rate of interest

27. Obtain some local information regarding loans, schedules for repayment etc. and prepare a report of your findings for the class. Personal loan companies, banks, and advertisements are sources to check.

Usually the purchaser of a home pays only part of the cost in cash. He borrows the remainder from a bank or savings and loan association. Savings and loan associations invest a large portion of their funds in building loans, which, because the property is security for repayment, is considered a sound investment. A loan with real estate as security is called a *mortgage*. If the loan is not repaid, the property may be sold, and the loan paid from the receipt of the sale. This procedure is called *foreclosing* a mortgage.

1. The Donaldson family wishes to buy a house priced at \$35,000. They have \$15,000 in cash and can borrow \$20,000 from a savings and loan association. Two plans for borrowing are available. One is to borrow the \$20,000 for five years at $6\frac{1}{2}\%$ interest per year. Interest is payable each six months, and at the end of five years the principal is due and payable. This is called a *straight mortgage*. How much interest is due each six months? \$650

2. How much must the Donaldson family save each month in order to pay back the \$20,000 at the end of five years? \$333.33

3. How much must they save monthly to cover interest plus principal? \$441.67

4. How much will they have paid by the time the mortgage is paid off? \$26,500

5. A second plan is the *installment* mortgage. By this plan both principal and interest are paid in regular monthly installments. Because the outstanding principal is decreasing regularly and the risk becomes less, the lender is willing to have the loan run for a longer time. A term of 15, 20 or even 25 years is not unusual, and the interest rate is slightly lower than with a straight mortgage. The Donaldsons found that they could borrow \$20,000 for 20 years at 6% interest on an installment mortgage. The monthly payments are \$150. How much of the first month's payment is interest? \$100

6. How much of the first month's payment is principal? \$50

7. What is the balance of the debt after the first month's payment? \$19,950

8. How much is the interest for the second month? \$99.75

9. Mr. Donaldson made the loan on June 1 and his first payment was made on July 1. What is the interest for the month of July? \$99.75

10. How much will be paid on the principal? \$50.25

11. Complete the chart to show how the payment of August 1 is divided between interest and payment on principal and what the balance is.

Date	Payment	Interest	Paid on Principal	Balance
6/1				$20,000.00
7/1	$150.00	$100.00	$50.00	19,950.00
8/1	$150.00	$99.75	$50.25	$19,899.75

12. In 1934 the Federal Housing Administration was established to enable people to borrow money at reasonable interest rates in order to buy or build a new home. At the time Mr. Donaldson was making inquiries the monthly payment on a 20-year loan of $20,000 at the bank was $140. Assuming that $\frac{1}{240}$ of the principal is being paid each month, how much of the $140 is being paid on the principal? $83.33

13. Explain where the fraction $\frac{1}{240}$ in Exercise 11 came from. 20 yr. = 240 mo.

14. Prepare a statement to show how each payment of $140 is divided between principal and interest:

Payment on principal $\left(\frac{1}{240} \times \$20,000\right)$	...	? $83.33
Interest	...	? 56.67
	Total	$140

15. Calculate the rate of interest being charged on the FHA loan. 6.8%

16. The FHA practice of making long-term installment mortgage loans was an important factor in making this 20-year mortgage popular. The agency also has helped to establish standards of building and of financial responsibility in the borrower. Before approving a loan, the FHA investigates the construction of the house to see that it will outlive the mortgage. It also investigates the buyer's financial standing and his ability to make the payments on the home. In addition, during the first year, a charge is added to the payments at the rate of $\frac{1}{2}\%$ of the average outstanding balance during the first year. What is the average outstanding balance during the first year if a debt of $20,000 is being repaid in 20 years by monthly payments of $140 each? $19,691.78

17. What is $\frac{1}{2}\%$ of the average outstanding balance during the first year? $98.46

18. This charge is divided into 12 monthly payments. How much is each payment? $8.21

19. What is the total monthly payment during the first year? $148.21

Part One

A. **1.** Find the sum of each of these arithmetic series:

a. 1,3,5···(to 17 terms) 289
b. 18,15,12···3 63
c. 2,7,12···(to 12 terms) 354
d. 105,108,111···(to 10 terms) 1185

2. Find the average value of a term in each series in Exercise 1.
a. 17 b. 10.5 c. 29.5 d. 118.5

B. Find the average principal owed during the term of each of the following transactions:

1. \$200 cash; terms, \$10 down, \$20 a month for 10 months. \$104.50

2. \$75 cash; terms, \$15 down, \$5 a week for 15 weeks. \$32.00

3. Regular price, \$400. 10% discount for cash. \$161.46
Terms, \$50 down, \$15 a month for 24 months.

4. Regular price, \$350. 12% discount for cash. \$164.27
Terms, nothing down, \$25 a month for 15 months.

C. Express as per cent to the nearest tenth of 1%.

1. $\frac{5}{9}$ 55.6% **3.** $\frac{7}{8}$ 87.5% **5.** $\frac{5}{7}$ 71.4%

2. $\frac{3}{16}$ 18.8% **4.** $\frac{13}{25}$ 52.0% **6.** $\frac{9}{20}$ 45.0%

D. Copy and find the value for n that makes each statement true.

70 **1.** 35 is n% of 50
80 **2.** 56 is n% of 70
120 **3.** 96 is 80% of n
60 **4.** 48 is n% of 80
80 **5.** n% of 85 is 68
8.4 **6.** 15% of 56 is n
28 **7.** n is 35% of 80
125 **8.** 45 is n% more than 20
900 **9.** 135 is 15% of n
240 **10.** 120 is 50% less than n
104 **11.** 80 increased by 30% is n
200 **12.** n decreased by 14% is 172

E. Find the interest.

\$6.19 **1.** \$660 for 75 days at $4\frac{1}{2}$%
\$14.30 **2.** \$880 for 90 days at $6\frac{1}{2}$%
\$21.75 **3.** \$725 for 180 days at 6%
\$40.33 **4.** \$4000 for 66 days at $5\frac{1}{2}$%
\$15.00 **5.** \$1500 for 3 months at 4%
\$13.75 **6.** \$500 for 180 days at $5\frac{1}{2}$%
\$3.25 **7.** \$650 for 60 days at 3%
\$36.67 **8.** \$2500 for 88 days at 6%
\$94.50 **9.** \$3600 for 7 months at $4\frac{1}{2}$%
\$56.25 **10.** \$750 for 18 months at 5%

Part Two

1. What are the advantages of installment buying for the customer? for the merchant? *See front.*
2. What are the dangers and disadvantages of installment buying for the customer? for the merchant? *See front.*
3. Set up a list of cautions for anyone who is considering purchasing on the installment plan. *See front.*
4. If the installment payments are not met, the merchandise may be *repossessed* and sold again by the merchant. Do you think he is fully protected against loss by this arrangement? Explain your answer. *No, the merchandise may be damaged.*
5. The cash price of a TV set is \$325. On the "easy payment plan" the terms are \$45 down and \$6 a week for 52 weeks.
   - **a.** How much is the carrying charge? *\$32*
   - **b.** What is the balance of the cash price after the down payment is made? *\$280*
   - **c.** Show in a statement how much of each weekly payment may be considered as payment on the principal and how much is interest. *principal, \$5.38; interest, \$.62*
   - **d.** What is the average balance owed? *\$142.69*
   - **e.** Using the interest formula, how much is i? How much is p? What is t? *i, \$32; p, \$142.69; t, 1yr.*
   - **f.** What rate of interest is being charged? *22%*
6. A bank charges a higher rate of interest on a personal loan than on a commercial loan to a business firm because it is more expensive to handle a personal loan. Give some reasons for the greater expense. *clerical costs; investigation of credit rating; bad debts*
7. Find the value of n in each of the following:

	Principal	*Rate*	*Time*	*Interest*
a.	\$500	5%	*6* n mo.	\$12.50
b.	\$600	*6* n%	9 mo.	\$27.00
c.	\$750	4%	18 mo.	\$ n *45*
d.	\$ n *120*	6%	15 mo.	\$ 9.00
e.	\$800	*8* n%	18 mo.	\$96.00

Part Three

1. Mrs. Jones can buy a refrigerator for a cash price of \$250. On the installment plan it is \$10 down and \$15 a month for 18 months. What is the total cost on the installment plan? *\$280*

2. How much greater is the total cost on the installment plan than the cash price? This difference is the carrying charge. $30

3. Henry Jones wants to buy a radio priced at $37.50. He can get it on the installment plan for $8 down and $1.25 a week for 28 weeks. How much is the carrying charge? $5.50

4. Lester Anderson makes $6 a week from his paper route. He can save $2.50 a week and has saved up $12. He wants a new bicycle, priced at $56 cash. The dealer will let him have it for $12 down and $2.50 a week for 20 weeks. How much is the carrying charge? $6

5. How many weeks would Lester Anderson have to wait before he has saved up enough money to pay cash for the bicycle? 18 wk.

6. Under what conditions would you advise Lester to buy the bicycle on the installment plan? *if he needs it for his paper route*

7. Helen works eight hours each Saturday clerking at the five-and-ten-cent store. She is paid $1.25 an hour. How much does she earn each week? $10

8. Helen can save $1.25 each week from her earnings. She has already saved up $7.50. She wants to buy a wrist watch that is priced at $29.95. How many weeks will it take her to save up the balance? 18 wk.

9. On the installment plan Helen can pay $7.50 down and have her choice of terms:

   $1.25 for 20 weeks or 57%
   $4.50 a month for six months 70%

   What rate of interest would she be paying by each plan?

10. Mr. Adams borrowed $500 to be repaid in 61 monthly payments of $10.50 each. What rate of interest is he paying? 11%

11. In the series 2,4,6 · · · and so on,
    a. What is a? 2
    b. What is d? 2
    c. What is the eighth term? 16 the tenth term? 20
    d. What is the sum of the first 20 terms? 420

12. How many terms are in the series 48,44,40 · · · 4? (See if you can calculate the number without counting them.) 12
    a. What is the fifth term in the series? 32
    b. What is the average value of a term in the series? 26
    c. What is the sum of the series? 312

THE AUTOMOBILE

WORDS TO WATCH FOR

depreciation	*liability*	*policy*
fixed costs	*license*	*premium*
horsepower	*personal liability*	*variable costs*

During the last 40 years, the automobile has brought about a tremendous change in our way of life. Many people consider a car a necessary element in their lives, not only for business but also for pleasure. The manufacture of the car has provided jobs for millions of Americans. One out of seven wage earners in this country works in the automobile industry or in an industry allied with it.

The automobile is more expensive to own, however, than many people realize. The cost of owning and operating a car comprises about 12% of the average family budget. But besides this obvious expense, we pay for our automobiles through taxes to maintain streets, parking lots, and highways. The cost of these services is a principal item in local, state, and national budgets.

We also pay for cars through the problems they cause in our crowded cities. Serious hazards of traffic congestion, downtown parking for commuters and shoppers, greater air pollution, and the increasing accident toll—all results of an increased use of the automobile—are part of the price we pay for the automobile and its place in our way of life.

1. In a recent year $15 billion worth of automobiles and trucks were produced. The value of all goods produced in that same year was $400 billion. The value of the automobiles and trucks was what per cent of the total value of all goods produced? 3.8%

2. In a recent year 70 million automobiles were registered in this country. The average value was $900 per automobile. What was the total value of registered cars? *$ 63 billion*

3. In the same year it was reported that each car used an average of 900 gallons of gasoline per year. What was the total amount of gasoline used by automobiles that year? *63 billion gallons*

4. If the average price of gasoline was 35¢ per gallon, what was the cost of the gasoline consumed by automobiles during the year? *$ 22.05 billion*

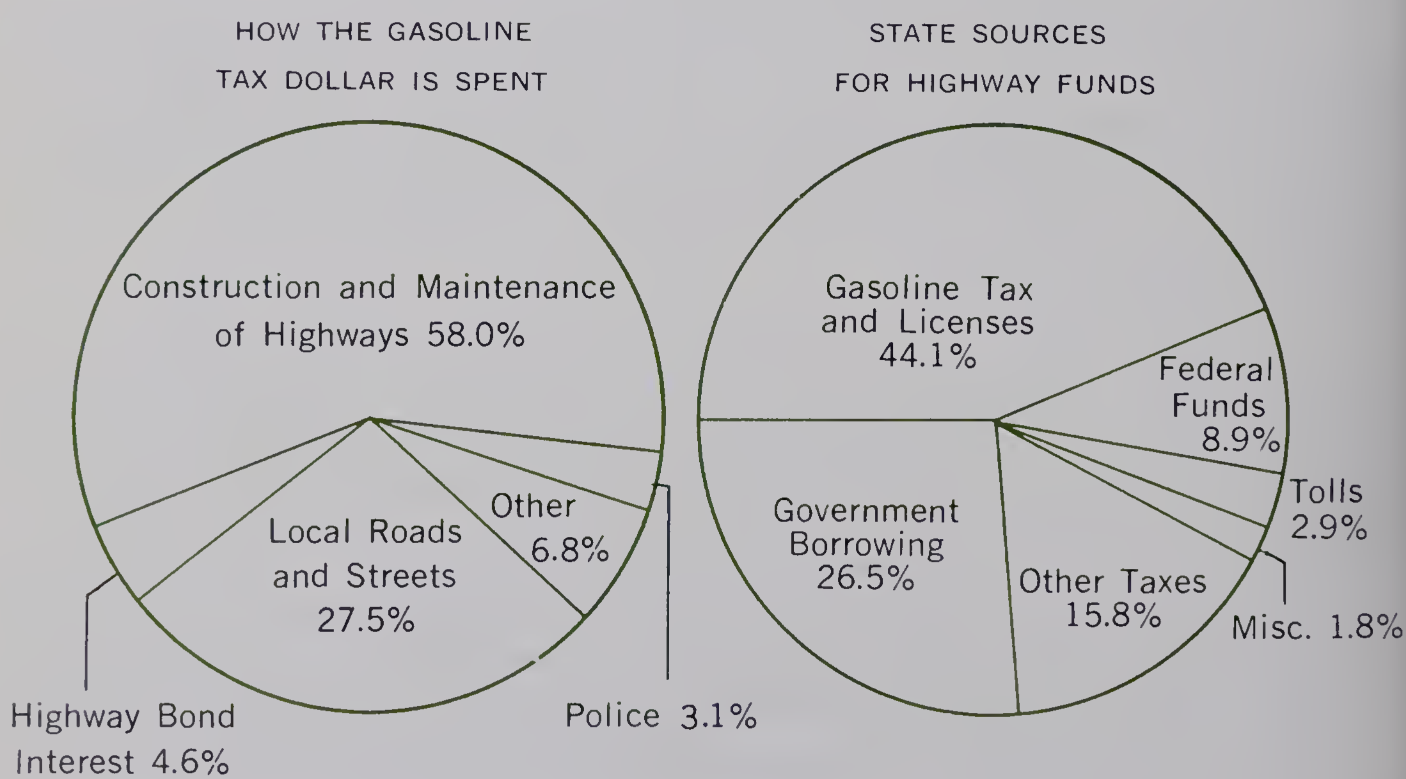
5. These cars were owned by 48 million families. What was the average expenditure for gasoline per family to the nearest dollar? *$ 459*

6. The population of Los Angeles, according to a recent census, was about 2,490,000. The ratio of cars to persons in the city was 1 to $1\frac{1}{2}$. How many cars were in Los Angeles? *1,660,000*

7. It was recently announced that there were 72,100,000 employed persons in the United States. Of these, $\frac{1}{7}$ were employed in business or industry related to automobiles. How many were employed in this way? *10,300,000*

8. To build a mile of graded concrete highway in the open country costs about $80,000. In a recent period of ten years about 130,000 miles of this type of road were built. About how much did this construction cost? *$10,400,000,000*

9. In Figure 1 you can see how gasoline taxes were spent in a recent year. The amount of such taxes was $7 billion. How much was spent in each way? *See front.*



10. Figure 1 includes the expenditures for federal highways. In Figure 2 you can see how \$6 billion was raised by the states for the building and maintenance of state highways. How much was raised in each way? *See front.*

11. In 1965, 8.2 million automobiles were produced in the United States. The 1960 production was 81% of the 1965 production. How many automobiles were produced in 1960? *6.642 million*

12. The automobile production in 1950 was 6.7 million. This was 85% of the 1955 production. What was the 1955 production? *7.9 million*

13. Recently it was predicted that in 1972 the population of the United States would be 23 million more than the 1965 population, which was 195 million. The increase predicted is what per cent of the 1965 population? *11.8%*

14. At the same time it was predicted that the average family income in 1972 would be 40% greater than 1965, when they averaged \$6250 per year. What income was predicted for 1972? *\$8750*

15. It was also predicted that the number of automobiles on the road would increase to 96 million in 1972, a 50% increase. How many cars were on the road at the time of the prediction? *64 million*

QUESTIONS FOR DISCUSSION

1. Suppose there were no cars in your community. What changes would there be in business? in recreation? in other ways?

2. What are some of the ways in which things have changed in the past 40 years because of the automobile? Consult an encyclopedia. Discuss this at home.

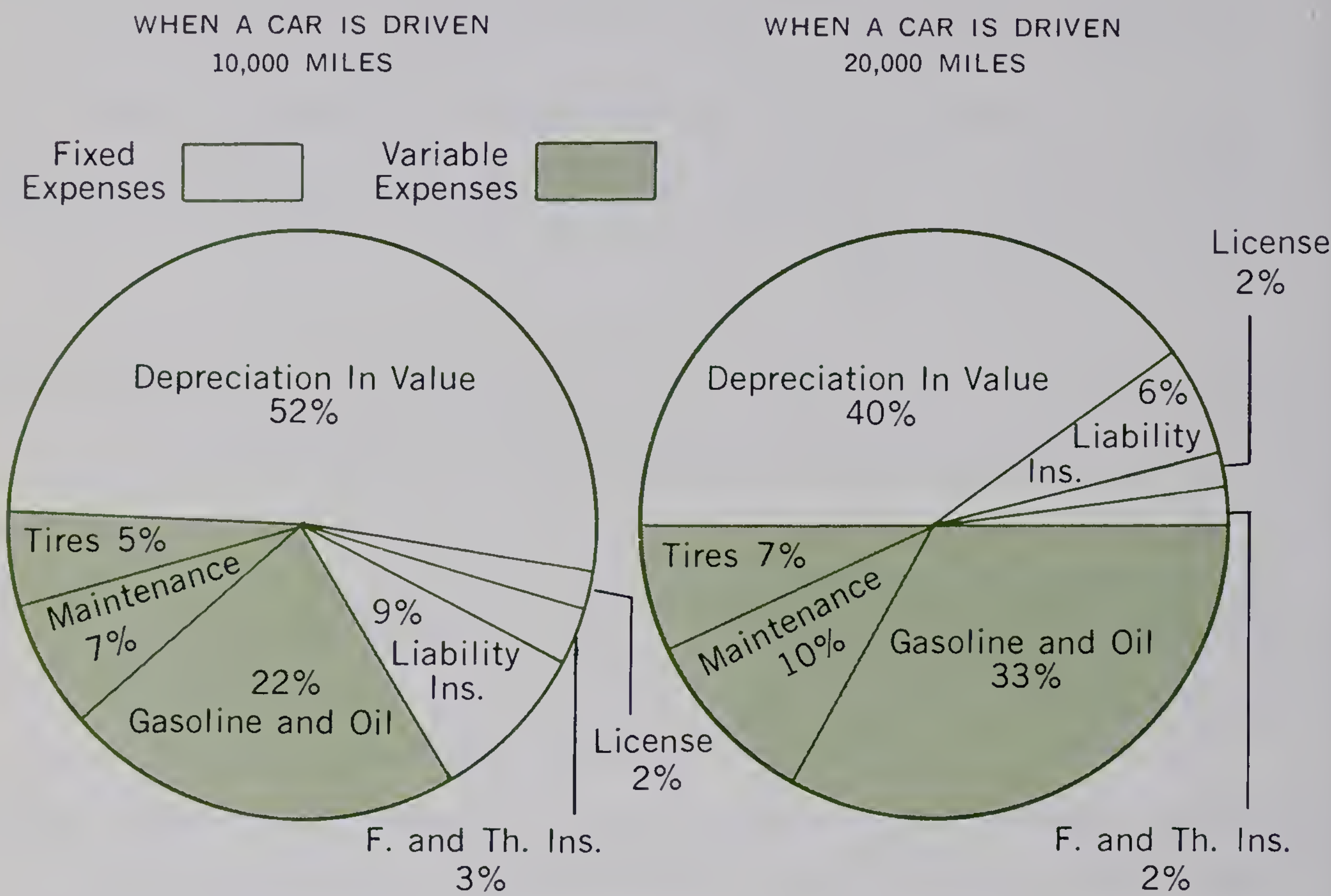
3. Other inventions that have also brought about great changes are the telephone, radio, television, and the airplane. Have any of these brought about greater changes than the automobile? Explain your answer.

4. There are many ways in which the automobile is an asset to the community. It is also a liability because it (a) is expensive, (b) is the source of traffic injuries and fatalities. Do you think that, on the whole, the automobile is more of an asset or a liability to your community?

5. Make a survey of your class to find out how many parents or close relatives earn their living directly or indirectly from the automobile industry.

Car expenses may be considered in two general classes: fixed expenses and variable expenses. Fixed costs, such as a car license, are the same no matter how much a car is driven. Variable expenses, such as the cost of gasoline, depend on how far a car is driven.

AVERAGE COSTS OF OWNING AND OPERATING A CAR



1. Which items increase when a car is driven 20,000 miles, as compared to 10,000 miles? *gasoline and oil, maintenance, tires*
2. Which items make up a smaller part of the expenses when a car is driven 20,000 miles? *license, insurance, depreciation*
3. The average cost for driving a car 10,000 miles is $1200. How much is this per mile? *$ .12*
4. What is the cost of gasoline and oil when a car is driven 10,000 miles? *$ 264*
5. How much did the gasoline and oil cost per mile? *2.64¢*
6. How much was allowed for depreciation in value of a car? *$ 624*
7. When a car is driven 10,000 miles what is the cost per mile for maintenance? *0.84¢* What is the cost per mile for liability insurance? *$1.08*
8. The average cost of driving a car 20,000 miles is $1600. What is the cost per mile for each item? *See front.*

A. Add or subtract as indicated:

1. $14 + (^{-}9)$ 5
2. $26 - (^{+}19)$ 7
3. $34 - (^{-}13)$ 47
4. $18 + (^{+}7)$ 25
5. $19 + (^{-}11)$ 8
6. $27 + (^{-}19)$ 8
7. $46 - (^{+}16)$ 30
8. $73 + (^{-}65)$ 8
9. $17 - (^{+}16)$ 1
10. $32 - (^{-}17)$ 49
11. $41 + (^{+}3)$ 44
12. $15 - (^{+}6)$ 9

B. *Solve*:

1. $4n = 36$ 9
2. $5x = 47$ 9.4
3. $3y + 3 = 33$ 10
4. $n - 11 = 11$ 22
5. $41x = 164$ 4
6. $2x + 3x + x = 48$ 8
7. $48 - 12y = 4y$ 3
8. $3n - n = 26 - 12$ 7
9. $3z = 22$ $7\frac{1}{3}$
10. $13y = 52 - 13y$ 2
11. $24 + 13 = 5n + n$ $6\frac{1}{6}$
12. $\frac{n}{2} = \frac{5}{2}$ 5
13. $6y = 45$ 7.5
14. $\frac{2n}{6} + 5 = 7$ 6
15. $\frac{n}{3} = 15$ 45

C. Find the value of the variable in the following proportions:

1. $\frac{n}{2} = \frac{4}{8}$ 1
2. $\frac{3}{45} = \frac{r}{15}$ 1
3. $\frac{x}{3} = \frac{15}{6}$ 7.5
4. $\frac{1}{y} = \frac{17}{85}$ 5
5. $\frac{18}{32} = \frac{x}{24}$ 13.5
6. $\frac{14}{n} = \frac{56}{100}$ 25
7. $\frac{2}{3} = \frac{30}{y}$ 45
8. $\frac{z}{21} = \frac{21}{77}$ $5\frac{8}{11}$
9. $\frac{21}{31} = \frac{7}{x}$ $10\frac{1}{3}$
10. $\frac{24}{60} = \frac{y}{15}$ 6
11. $\frac{5}{6} = \frac{n}{6}$ 5
12. $\frac{99}{108} = \frac{22}{z}$ 24

D. Find the values for n:

1. 18 is $n\%$ of 45 40
2. 17.5% of 56 is n 9.8
3. 15 is 30% of n 50
4. 185% of 60 is n 111
5. 25 is $n\%$ of 6.25 400
6. 85 is $n\%$ of 200 42.5
7. 0.3% of 250 is n 0.75
8. 56 is 40% of n 140
9. n is 0.2% of 40 .08

E. Find the interest:

1. \$280 for 72 days at $7\frac{1}{2}\%$ \$ 4.20
2. \$500 for 90 days at 8% \$ 10

If you had difficulty with any section, use the Practice Exercises on page 493. If not, you may work in the Experts' Corner.

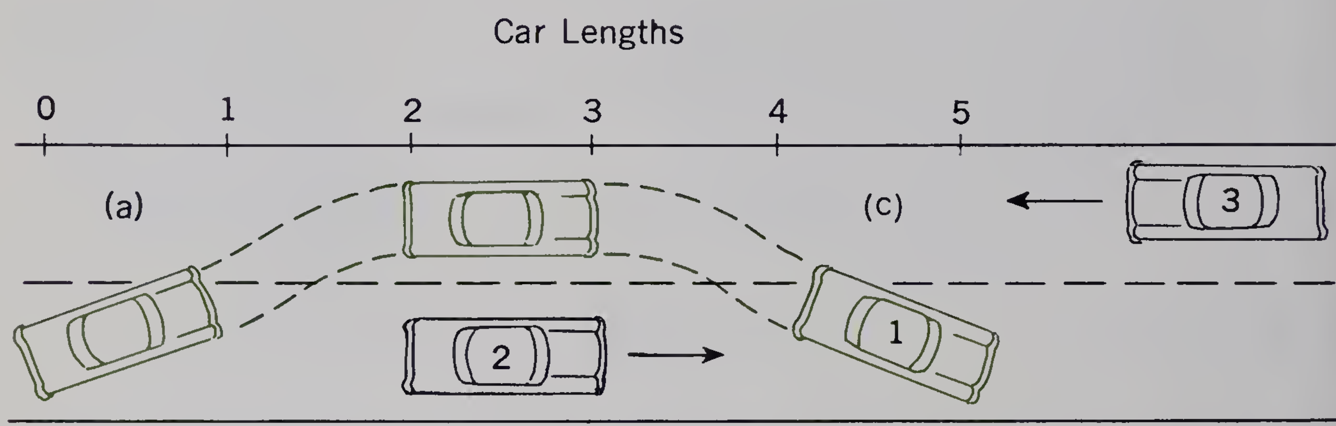
Is There Room To Pass?

Investigations reveal that a driver's estimate of the distance required to overtake another car, pass it, and get back on his own side of the road is generally too low. A safe rule to follow is to allow ten seconds or more from the time you pull out of your lane until you can return to it. If a car is approaching, there may be a problem.

1. Suppose your car is traveling 40 m.p.h., how many feet is it traveling per second? 58.7 ft.

2. A car coming toward you is traveling at 50 m.p.h. How far will he travel toward you in ten seconds? 733.3 ft.

3. While traveling 50 m.p.h., you turn into the left-hand lane to pass another car traveling at 45 m.p.h. in the same direction. How long will it take to get two car lengths ahead? (*1 car length = 18 feet*) 10 sec



Car 1, traveling 50 m.p.h., turns out at (a) to pass car 2, traveling 45 m.p.h. Since car 1 is traveling 5 m.p.h. faster than car 2, the time it takes to pass is the same as if car 1 were traveling at 5 m.p.h. and car 2 were standing still. From where it turns out (a) to where it is back in its own lane (c), car 1 has traveled four car lengths, or about 72 feet, farther than car 2. At 5 m.p.h. (or 7.3 ft. per sec.) this takes 10 seconds. During this time car 3, approaching at 50 m.p.h., has traveled 730 feet.

4. How far will your car travel in the number of seconds it took in Exercise 3? 730 ft.

5. A car was coming toward you at 55 m.p.h. when you turned out to pass (in Exercise 3). When you passed and turned back into your own lane, he was still 200 feet away. Allowing the time it took for passing, how far away was he when you started to pass? 1740 ft.

Most automobiles are purchased on the installment plan. Automobile purchases account for more installment debt than any other item of family expenditure.

The purchaser may finance his purchase either from the auto sales agency or through the bank. In either way the interest adds to the cost of owning the car.

Usually the bank requires the borrower to pay off the loan in monthly payments, which include interest. A loan on a new car may run for 24 months. On a used car the loan is usually for 24 months or less.

Here is a table of rates published by a bank:

New Cars			*Used Cars*		
Amount	*Monthly Payment*		*Amount*	*Monthly Payment*	
Borrowed	*18 months*	*24 months*	*Borrowed*	*12 months*	*18 months*
$ 600	$35.62	$27.25	$ 600	$53.00	$36.33
$ 800	47.49	36.30	$ 800	70.67	48.44
$1000	59.36	45.41	$1000	88.33	60.55
$1200	71.19	54.50	$1200	106.00	72.66
$1500	88.95	68.13	$1500	132.50	90.83
$2000	118.22	90.84	$2000	176.67	121.11

1. Mr. Smith bought a new car for $2700. He paid $2000 cash, and borrowed the rest from the bank for one year at 6% interest. How much was the interest? $42

2. What was the total cost of the car, including interest? $2742

3. The Andersons bought a car priced at $3200. They had $2000 in cash. The rest they borrowed from the bank for 18 months, paying interest at the rate of $6\frac{1}{2}\%$ a year. What was the total cost of the car including interest? $3317

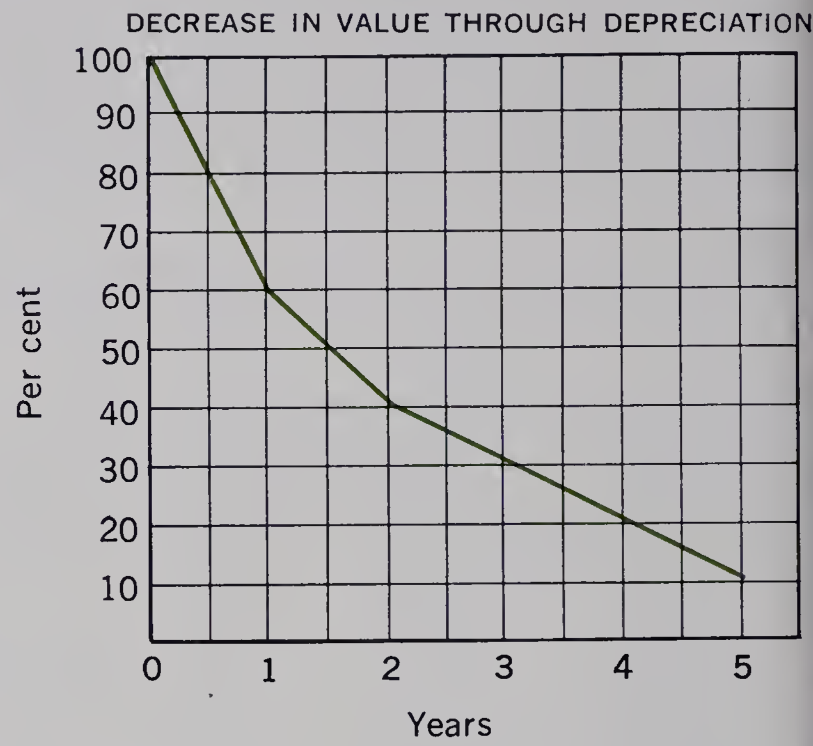
4. The Adams family borrowed $800 to purchase a new car. They paid the loan in 18 monthly payments according to the rates in the table above. What was the total amount of the 18 payments? $854.82

5. The amount the Adams family had to pay in addition to the $800 was interest. How much was the interest? $54.82

6. Compare the interest paid on a $1500 new car loan with the interest paid on a $1500 used car loan. Consider both using 18 monthly payments. new, $101.10; used, $134.94

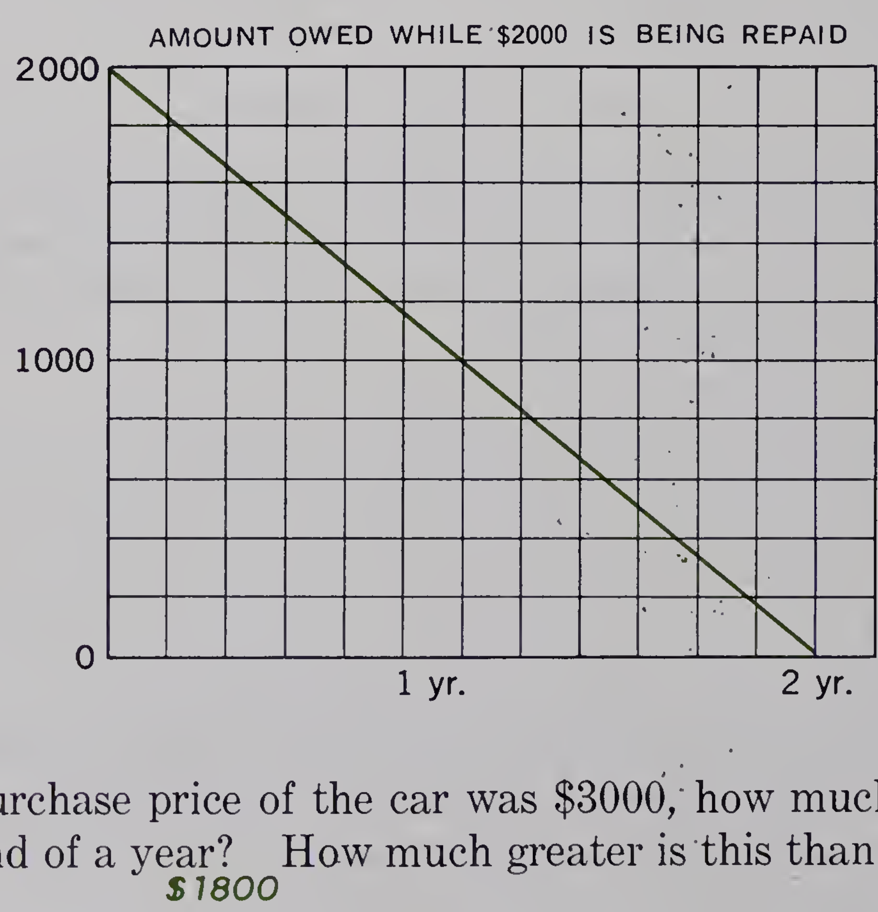
7. The Jensens purchased a new car for $3000. They had $1500 in cash and borrowed the remainder, paying back the loan in 24 monthly payments. What was the total cost including interest? Compare the cost with the 18 monthly payment plan. $3135.12
18 monthly payment plan costs $ 3101.10

8. The decrease in value of a car as it gets older is called *depreciation*. In the graph below you can see the typical per cent of its cost by which a car depreciates each year. What is the per cent of depreciation the first year? the second year? the third year? 40%; 20%; 10%



9. Using the rate of depreciation shown in the graph, calculate the depreciation for the first four years on a car purchased new for $2400.
See front.

10. Because a car depreciates so rapidly the first two years, most banks will not lend more than two-thirds the price of the car. The figure below shows how much is owed on a $2000 24-payment loan at the end of two-month periods. How much is owed at the end of a year?
$ 1000



11. If the purchase price of the car was $3000, how much was it worth at the end of a year? How much greater is this than what is owed?
$1800 $800

Even a careful driver may have a traffic accident, have his car stolen, or have it catch on fire. Insurance provides protection against heavy losses as a result of such accidents, both for his own car, and for liability for damage he may have caused to others. There are three major types of car insurance.

Collision insurance. This type of insurance protects the owner of the car against the cost of repairing damage to his car resulting from collision with another car or with a fixed object, such as a tree. Usually the car owner accepts responsibility to pay the first \$50 or \$100 of the cost of repairs, in order to secure lower cost for the insurance. Collision insurance is expensive, and is commonly carried only on expensive cars or new cars. The *premium* is the money paid to the insurance company for the *policy,* which is the written contract to insure a car.

Comprehensive insurance. This type of insurance protects against loss by fire or theft, plus a variety of other causes, such as falling objects or vandalism. The insuring company agrees to restore the car to its current value or to make a cash settlement. The cost of comprehensive insurance varies with the locality. In an area where the traffic volume and the accident rate are high, the insurance costs are high. It costs more to insure an expensive car than a low-priced car, and less to insure a car as it becomes older.

Liability insurance. If a driver is held responsible for an accident, he may be liable for injury or death of persons in the accident, and damage to property resulting from the accident. Most drivers consider that insurance against loss of money from such claims is the most important kind of insurance we have, and will not drive a car without such insurance. In many states a car cannot be licensed unless it is insured for liability.

Public liability insurance protects the car owner against liability for personal injury up to a specified amount. Most policies use the expression: \$10,000-\$20,000, to mean that up to \$10,000 will be paid for injury to one person, and up to \$20,000 for more than one in the same accident. For a small additional premium these amounts can be doubled.

Property damage insurance protects the owner against claims for damage to property, such as an automobile, a building, or other property, which he is responsible for damaging.

The premium for liability insurance varies with the locality, the age of the driver or drivers of the car, and the purposes for which the car is used. For a family car driven by adults, a certain company has annual premiums:

Public liability, \$10,000-\$20,000, \$59 Property damage, \$5000, \$20

For three well-known cars, car X in the low-priced field, car Y in the medium-price range, and car Z in the luxury class, annual premiums for comprehensive insurance in one locality are as follows:

Type of Car	*New*	*1 to 3 Years Old*	*4 or More Years Old*
Car X	$25	$21	$15
Car Y	31	26	19
Car Z	50	43	30

1. Mr. Henderson bought a car, make X, for $2250. He took out comprehensive insurance, $10,000-$20,000 public liability, and $5000 property damage insurance. How much was the annual premium? $104

2. Mr. Smith has a car of make Z that is four years old. How much is the premium for comprehensive insurance on the car? $30

3. What will it cost Mr. Smith to take out $10,000-$20,000 public liability and $5000 property damage insurance? $79

4. Mr. Smith has been reading in the paper about damages amounting to much more than $20,000 in cases arising from automobile accidents. He finds, on inquiring, that he can take out a policy for $20,000-$40,000 public liability insurance for an additional 15% of the annual premium. How much will the annual premium be for this insurance? $67.85

5. Mr. Johnson purchased a new car of make Y for $3800. He took out comprehensive insurance, $10,000-$20,000 public liability, and $5000 property damage insurance. What was his total annual premium? $110

6. The following year Mr. Johnson's 16-year-old son, Henry, was qualified for a driver's license, and was allowed to drive the car. The insurance premiums for public liability and property damage increased by 80%. How much was the new premium? $142.20

QUESTIONS FOR RESEARCH AND DISCUSSION

1. Find out from a local agent the rates for the types of insurance dealt with above, for various makes and ages of cars. The premiums quoted are those for a metropolitan area with heavy traffic. Are those in your community higher or lower? Can you explain why?
2. Are liability premium rates higher in your community for teen-age drivers? Does it make a difference if they have taken driver education courses? At what age do adult driver rates apply?

Variable Expenses

The cost of owning a car is usually underestimated. Costs of tires, lubrication, and gasoline are only a part of the expense. Even if a car is not being used, expenses continue to accumulate. Jim Brown had purchased a used car and found that it was costing him more to own and operate than he had expected. He decided to maintain a record of his expenditures on the car. Variable and fixed costs were kept separately, so he could obtain a clear idea of what it cost to own and operate his car.

1. To find out how much the gasoline cost per mile, he read the odometer as 14,675.8 miles when he filled the gasoline tank. When he next filled the tank, the odometer reading was 14,881.6 miles. How far had the car been driven? *205.8 mi.*
2. It required 14.7 gallons of gasoline to fill the tank. What did the gasoline cost at 33.9¢ per gallon? *$ 4.98*
3. To find the cost of gasoline per mile, Jim divided the cost by the number of miles traveled. To the nearest tenth of a cent, what was the cost of gasoline per mile? *2.4¢*
4. To get a more accurate figure on the cost of gasoline per mile, Jim kept a record of the amount used by the car for the next 1080 miles. He found it took 80 gallons. How much did the gasoline cost at 33.9¢ per gallon? *$ 27.12*
5. To the nearest tenth of a cent, what was the cost of gasoline per mile? *2.5¢*
6. Each 2000 miles Jim had the car lubricated and the oil changed. Five quarts of oil were required, at 45¢ per quart. What was the cost of the oil? *$ 2.25*
7. The lubrication job cost $1.75. What was the total cost of oil and lubrication each 2000 miles? *$ 4*
8. What was the cost of oil and lubrication per mile, to the nearest tenth of a cent? *0.2¢*
9. Another item of expense was maintenance of the car. Jim did most of the work on the car, but he found that parts he must purchase and repairs he could not make amounted to an average of $5.25 a month. He drove the car about 750 miles per month. What did car maintenance cost per mile? *0.7¢*

10. During the year Jim paid $60 for two new tires. He figured that this was about average wear. What was the average expense per mile? *0.7 ¢*

11. Copy and complete this table of variable costs per mile, using the answers to Exercises 5 through 10 above:

Gasoline	☐	*2.5¢*
Oil and lubrication	☐	*0.2*
Car maintenance	☐	*0.7*
Tires	☐	*0.7*
Total cost (variable) per mile	☐	*4.1¢*

Fixed Expenses

1. The state law required Jim to have liability insurance before he could get a license for his car. He insured it for $5000 property damage and for $20,000-$40,000 personal liability. What was the cost of the insurance per year? *$87.85*

2. The car was three years old when Jim purchased it. He figured that its depreciation per year was 10% of the price he paid for it, which was $800. Is this reasonable, according to the graph on page 430? How much was the depreciation as Jim calculated it? *yes ; $80*

3. Jim could have deposited the $800 in his savings account and earned 5% interest on it per year. This should be included in the expense of owning the car. How much is this per year? The license for the car costs $32 per year. Complete this table of fixed costs on Jim's car. *interest, $40*

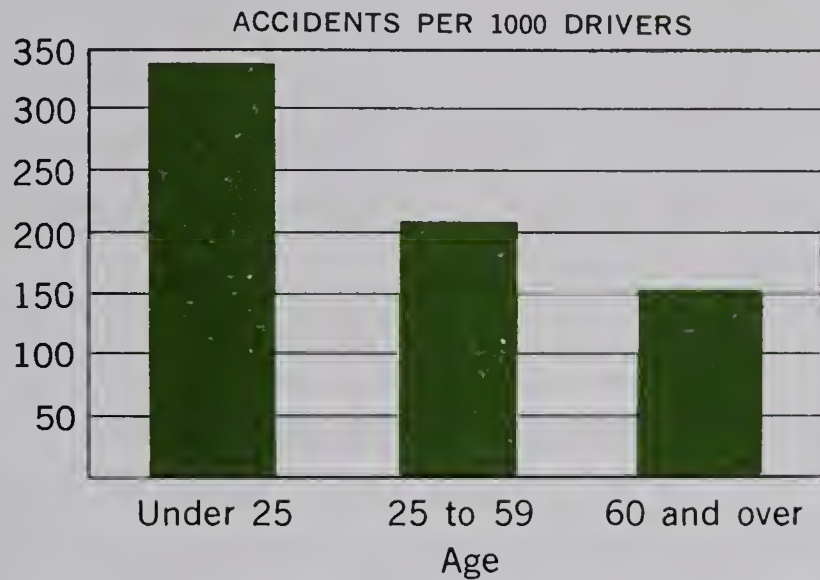
Insurance	☐	*$87.85*
Depreciation	☐	*80.00*
Interest	☐	*40.00*
License	☐	*32.00*
Total fixed costs	☐	*$239.85*

4. What does the car cost per day to own, even though it is not being used? *66 ¢*

5. Jim and some friends were planning to take a fishing trip, sharing the expenses of his car. Jim said that they should consider only variable expenses. Do you agree with this? Give the reason for your answer. *Yes, the trip will not increase the fined expenses.*

6. What would be the expected cost of a 300-mile trip? *$12.30*

7. Calculate the difference in cost per mile (if a variable expense) or per year (if a fixed expense) for each of these changes in the situation with Jim's car.
   a. The car Jim purchased was a compact model that traveled 35 miles on a gallon of gasoline. *a decrease of 1.5¢*
   b. Jim purchased a more modern car that needed oil change and lubrication only at 6000-mile intervals. *a decrease of 0.13¢*
   c. Since Jim was 18 years old and had taken a course in driver education there was no increase in cost of insurance over that for adults. Suppose he was 16 years old, with no driver education, what would be the added cost? *an increase of $70.28*



8. How many accidents occur per 1000 drivers age 25–59? *210*

9. What is the difference in the number of accidents per 1000 drivers between the 25–59 group and the group under age 25? *130 more*

10. A 1967 study found that the driver who turns in his car every 3 years has $982 per year fixed expenses. Give some reasons why Jim's expenses were less than this. *See front.*

RESEARCH REPORTS

1. Both fixed and variable costs vary greatly in different parts of the country on account of climate, costs of insurance and gasoline, and road conditions. Find out what the various costs are for your family car and have a report ready for class.

2. Prepare a circle graph showing the fixed and variable costs for your family car, and present it with your report.

3. Be ready to discuss this question: Suppose a high school student is planning to go to college, and will have to earn part of the cost of his education. Should he plan to own a car before he graduates from college?

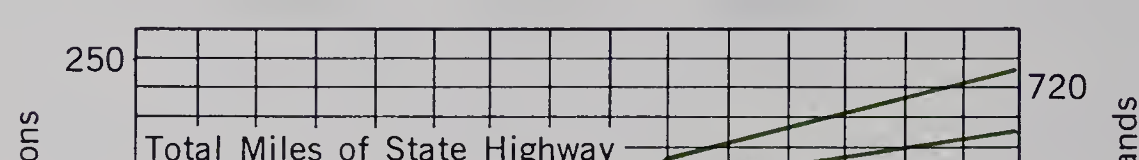
Since World War II, travel by autos, trucks, and buses has increased greatly. This has put a heavy burden on the highways. For purposes of measuring the amount of travel on the highway, the unit commonly used is the vehicle-mile. (This is one car driven one mile.)

1. If a car is driven 1000 miles in a year, this is 1000 vehicle-miles. How many vehicle-miles are traveled if a car travels 5000 miles in a year? *5000*

2. In a recent year it was reported that there were about a million registered passenger cars in the state of California. Each of the cars traveled an average of 9300 miles that year. How many vehicle-miles were driven in California? *9.3 billion*

3. Below, you can see how the highway load increased in 15 years following World War II. On which side of the graph do you read the number of vehicle-miles? On which side do you read the number of miles of highway? *left; right*

STATE HIGHWAY INFORMATION



4. Above, you see that the line showing vehicle-miles has risen sharply while the total miles of state highway has increased only gradually. How much was the increase in vehicle-miles from 1945–1960? (Read the graph to the nearest billion vehicle-miles.) *approximately 170 billion*

5. How much was the increase in miles of state highway from 1945–1955? (Read the graph to the nearest 10,000 miles.) 110,000 mi.

6. The increase in vehicle-miles was what per cent of the 1945 figure? 227%

7. The increase in miles of state highway was what per cent of the 1945 figure? 20%

8. While the vehicle-miles of travel increased 227 _?_ per cent in the ten years following World War II, the number of miles of state highway increased only _?_ per cent. 20

9. A highway must be paved in order to stand up to modern traffic conditions. Figure 1 shows the number of miles of unpaved highways in each year. How many miles of unpaved state highway were there in 1945? 100,000 mi.

10. What was the total number of miles of state highway in 1945? 560,000 mi.

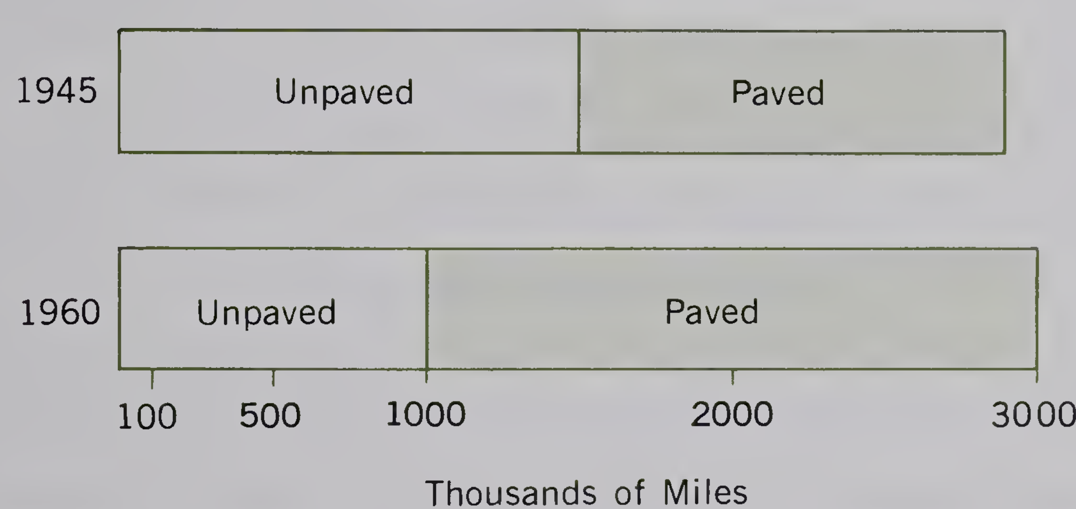
11. How many miles of state highway were paved in 1945? 460,000 mi.

12. How many miles of state highway were paved in 1960? 630,000 mi.

13. The increase in miles of paved highway was what per cent of the number of paved miles in 1945? 37%

14. Is this per cent greater than, or less than, the per cent of increase in vehicle-miles of traffic? less

MILES OF STATE AND LOCAL HIGHWAYS



15. Besides the state highways there is a network of secondary roads in each state that are maintained usually by the counties. The total miles of highway, state and local, are shown for 1945 and 1960 above. What per cent of the highways were paved in 1945? (Read the graph to the nearest 100,000 miles.) 48%

16. What per cent of the highways were paved in 1960? 67%

17. The distance around the earth at the equator is about 25,000 miles. How many times would the paved roads in 1960 go around the earth at the equator? 80

18. The distance to the moon is about 240,000 miles. How many times this distance is the total mileage of roads in 1960? 12.5

19. To take care of modern traffic requirements, many states with federal aid are building freeways, or expressways, as they are sometimes called. These "superhighways" have features to promote rapid travel with safety: overpasses and underpasses to eliminate cross traffic; center strips to separate traffic in two directions; and few stop signals. Recently it was reported that the planned and constructed freeways in this country were as follows:

	Miles	*Cost*
Total planned	7500	$10,098,049,000
Completed	2282	3,000,296,000

What is the cost per mile (to the nearest thousand dollars) of the planned freeways? $ 1,346,000

20. One superhighway recently completed is between Chicago and New York. Its mileage and cost in each state it crosses is:

(ex. 21)	(ex. 22)	*Miles*	*Cost*
Indiana 15.6%	25.0%	140	$ 758,000,000
Ohio 26.9%	10.8%	241	326,000,000
Pennsylvania 44.4%	53.7%	397	1,627,164,000
New Jersey 13.1%	10.5%	117	318,952,000
		895 mi.;	$3,030,116,000

Find the total mileage and total cost of this superhighway.

21. What per cent of the highway is in each state? See above.

22. What per cent of the cost of the highway was spent in each state? See above.

QUESTIONS FOR DISCUSSION

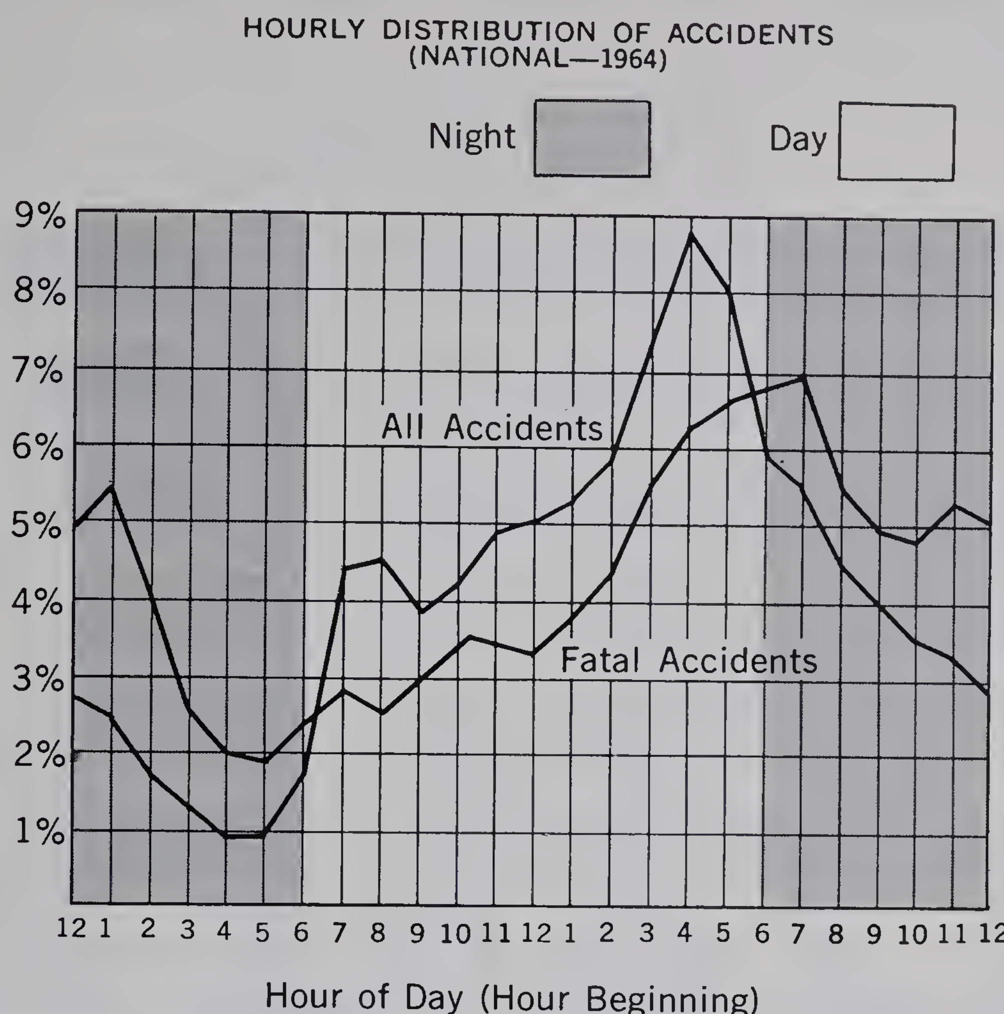
1. Are any of the highways in your neighborhood becoming overcrowded? If so, what plans are being made to relieve the traffic?

2. Find out about the superhighways that are being built in different parts of the country; report on the interstate system.
How is their design different from that of the ordinary highway?
What is the speed limit?
What tolls are charged for driving on them?
Make some recommendations for safe driving on superhighways.

3. The Browns are trying to decide whether it is cheaper to drive their car to Omaha or buy a railway excursion ticket.

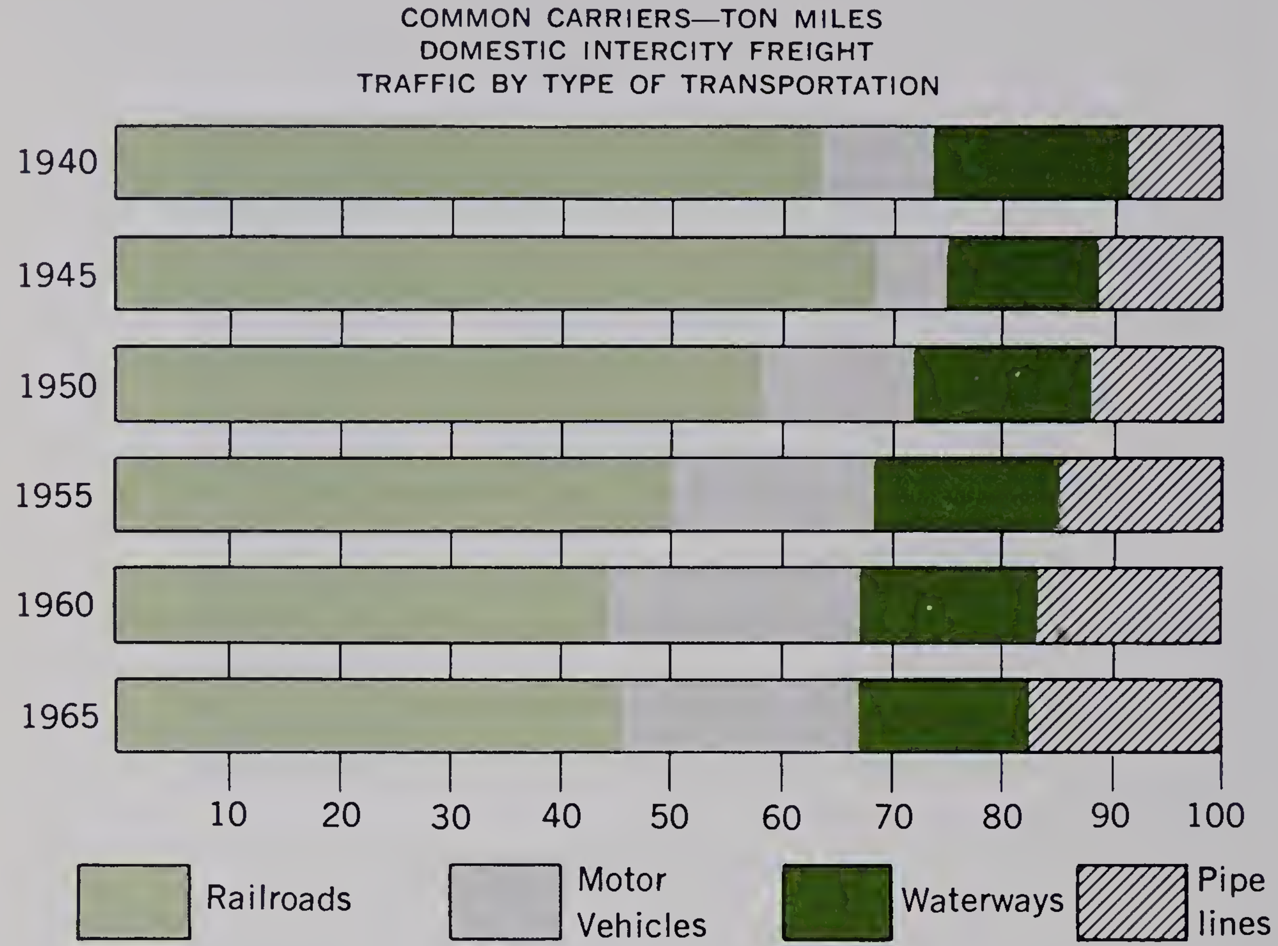
4. The manager of a company that rents cars to the public is keeping track of expenses for the purpose of fixing rates.

The graph shows the average hourly distribution of accidents for a recent year. The graph reveals that less than 1% of all accidents occur between 4:00 and 5:00 A.M.



1. Between what two hours of the day do most accidents occur? 2 - 4 P.M.
2. What per cent of the accidents occur between these two hours? Approximately 15%
3. Between what two hours of the day do the most fatal accidents occur? 2 - 4 P.M.
4. What per cent of all accidents occur between 10:00 and 11:00 P.M.? .5%
5. At what two times during the day do 5% of the accidents occur? 12 P.M. and 7:30 P.M.
6. What per cent of the fatal accidents occur between 9:00 and 10:00 P.M.? .25%
7. At what times during the daytime hours does 4% of the fatal accidents occur? between 1 P.M. and 2 P.M.
8. Explain why the greatest number of all accidents and the greatest number of fatal accidents occur at different times. See front.
9. Explain why the per cent of all accidents decreases between midnight and 6:00 A.M. while the per cent of all fatal accidents increases during these hours. See front.

Trucks and buses play an important role in our society as means for transporting freight and passengers. In this capacity they become *common carriers.* In this graph you can see how the freight traffic was distributed among the most important carriers in this country over a 25-year period.



1. What per cent of the freight was carried by motor vehicles (trucks) in each of the years shown? *See front.*

2. The per cent of freight carried by some of the agencies increased over the 25-year period, while the per cent carried by others was decreasing. Which of the carriers transported a lesser per cent in 1965 than in 1940? *railroads and waterways*

3. Which of the carriers transported a greater per cent in 1965 than in 1940? *motor vehicles and pipe lines*

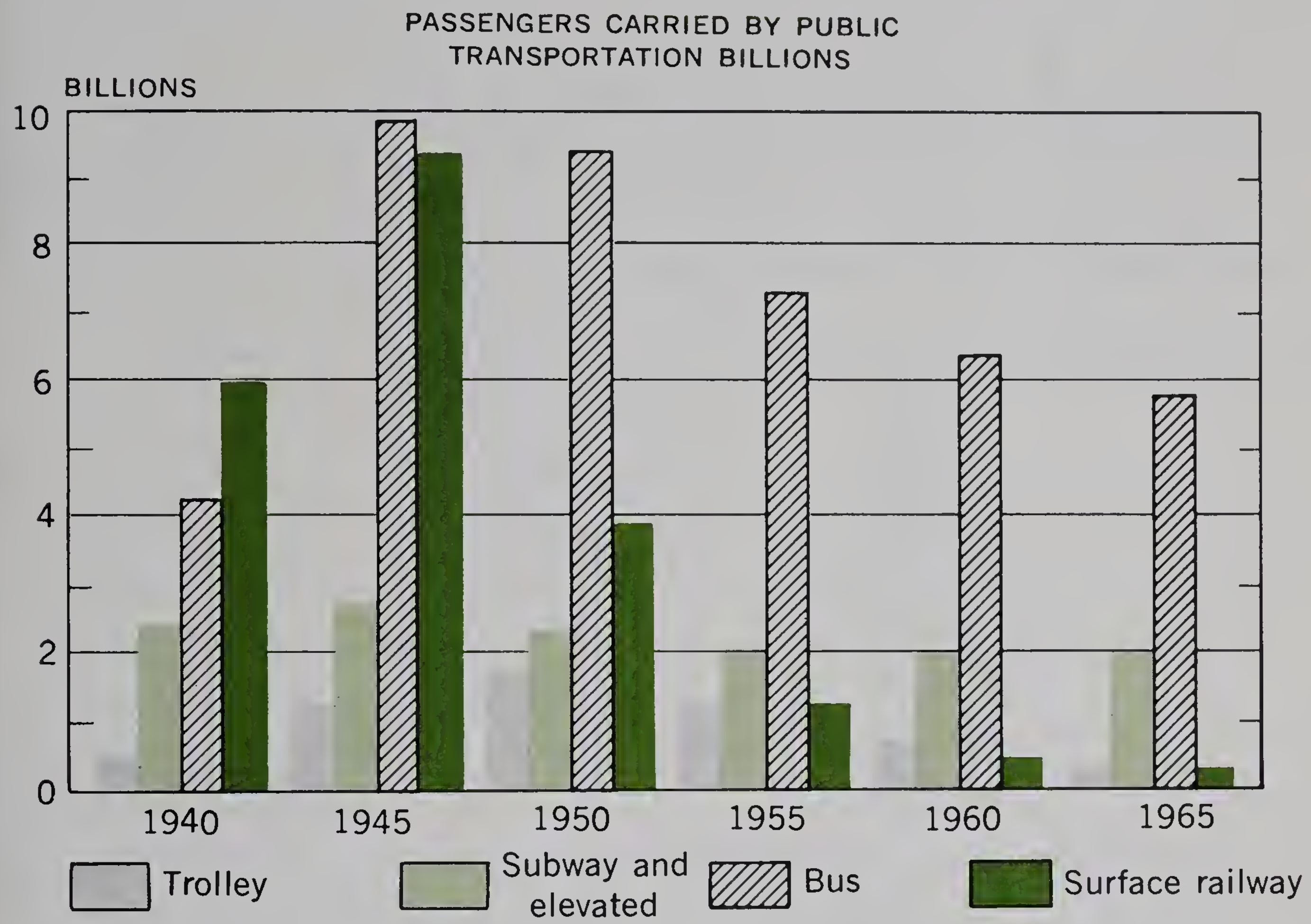
4. How do you account for the decrease in per cent of freight carried by trucks between 1940 and 1945? *gasoline rationing due to World War II*

5. A ton-mile of freight is one ton of freight carried one mile. If 5 tons of freight are carried eight miles, 40 ton-miles of freight have been transported. How many ton-miles of freight have been transported if 60 tons are carried 150 miles? *9000 ton-miles*

6. A typical freight car has a capacity of 50 tons. If loaded to capacity and transported 2000 miles, how many ton-miles does it represent? *100,000 ton-miles*

7. A barge being towed on the Mississippi River from New Orleans to St. Louis, a distance by river of about 800 miles, contains 150,000 gallons of gasoline. A gallon of gasoline weighs about 7 pounds. How many ton-miles of freight are being transported? *420,000 ton-miles*

8. The number of passengers transported by each of the major common carriers over a period of 25 years may be seen in this graph. To the nearest half billion, how many were carried by each agency in 1940? in 1945? in 1950? in 1965? *See front.*



9. What was the total number of passengers, to the nearest half billion, carried by the agencies represented on the graph during the years mentioned in Exercise 8? *See front.*

10. Does your answer to Exercise 9 indicate that people are traveling less since World War II than before? Explain your answer. *See front.*

11. In 1964 the airlines carried about 60 million passengers. Why are airlines not included among the agencies shown in the graph? *See front.*

12. What per cent of the passengers were carried by motorbus in 1940? in 1950? in 1965? *33% ; 53% ; 67%*

13. What per cent of the passengers were carried by surface railway in the years listed in Exercise 12? *1940, 44%; 1950, 22%; 1965, 6%*

14. It is commonly said that the motorbus is taking passengers away from the railroads. Do your answers to Exercises 12 and 13 support this point of view? *yes*

You have learned to read and construct several kinds of graphs: line, bar, divided bar, pictograph, and circle. Which graph might you use for each of the following purposes:

circle *a.* To show how a city spends its funds?

bar *b.* To compare the lengths of the seven longest bridges in the world?

line *c.* To show the population of the United States each ten years since 1900?

Prepare two posters using the types of graphs that would be most effective for presenting facts relating to two of the causes of highway accidents.

Students should make their own graphs.

1. Do the road conditions have an effect on highway accidents? Reports show that 78.1% of fatal accidents occur on dry roads, 17.8% on wet roads, 1.6% on snowy roads, and 2.5% on icy roads. Does this show that dry roads are more dangerous? yes ; not necessarily

2. On what day of the week do most accidents occur? Sunday

Day	*Per Cent of Accidents*
Monday	12.2
Tuesday	11.2
Wednesday	11.6
Thursday	11.9
Friday	14.1
Saturday	19.4
Sunday	19.6

3. What are the most common causes of accidents? collisions with pedestrians and automobiles

Cause of Accident	*Per Cent of Accidents*
Collision with pedestrian	53.6
Collision with automobile	34.1
Collision with horse-drawn vehicle	.5
Collision with train	1.6
Collision with streetcar	.7
Collision with bicycle	.6
Collision with fixed object	3.8
Non-collision	4.5
Miscellaneous	.6

4. At what time of day do most accidents occur? 6 P.M. to 12 midnight

Time	*Per Cent of Accidents*
12 p.m. midnight to 6 a.m.	13.6
6 a.m. to 12 noon	15.4
12 noon to 6 p.m.	30.3
6 p.m. to 12 midnight	40.7

5. Is an accident usually due to a defective car? no

Condition of Car	*Per Cent of Accidents*
Good condition	94.9
Defective brakes	1.7
Defective steering	.4
Glaring headlights	.3
Poor lights	.5
Tire failure	.7
Other defects	1.5

6. Do weather conditions increase highway accidents? yes

The per cent of accidents occurring in clear weather is 82.5%, in fog 2.0%, in rain 13.6%, in snow 1.9%.

7. What are pedestrians doing when they are involved in accidents?
any of the actions listed below

Action of Pedestrians	*Per Cent Killed*	*Per Cent Injured*
Crossing between intersections	27.9	28.8
Walking on highway	20.2	4.5
Crossing intersection with no signal	14.9	14.8
Coming from behind parked car	7.9	12.7
Playing in street	6.6	13.9
Working on highway	2.1	2.0
Getting on and off vehicle	2.2	2.5
Crossing intersection with signal	1.7	3.9
Miscellaneous	16.5	16.9

8. What are drivers doing when they get into accidents?
any of the actions listed below

Actions of Drivers	*Per Cent of Fatal Accidents*	*Per Cent of Non-Fatal Accidents*
Exceeding speed limit	39.7	28.7
Reckless driving	19.3	16.7
Driving on wrong side of road	13.7	14.3
Did not have right of way	12.1	22.1
Cutting in	2.9	3.9
Passing on curve or hill	2.8	1.5
Improper signalling	1.2	5.7
Passing on wrong side	.9	1.3
Miscellaneous	7.4	5.8

9. Write a paragraph which answers these questions about the graphs you have prepared. Answers will vary.

a. How can each of the important causes result in an accident?

b. How can these accidents be prevented?

1. When the Jensens started on their vacation trip, the odometer mileage was 16,445.0 miles and the gas tank was full. When they stopped for gasoline, the reading was 16,595.0.
   a. How far had they traveled? 150mi.
   b. It took 8 gallons of gasoline to fill the tank. How many gallons should they use on a trip of 210 miles? (Set up and solve a proportion.) 11.2 gal.
   c. They had left home at 9:00 A.M. and stopped for gasoline at noon. How many miles per hour had they averaged? 50 m.p.h.

2. Mr. Jones purchased a set of tires for his car. He paid \$18 in cash, which was 15% of the cost of the tires.
   a. Set up a proportion to find the entire cost of the tires. Solve the proportion. $\frac{18}{x} = \frac{3}{20}$; \$120
   b. How much does he still owe on the tires? \$ 102

3. The Purdys were driving from Nashville to Charleston. When they had traveled 120 miles Mr. Purdy said they had covered $\frac{2}{5}$ of the distance.
   a. Set up a proportion to find the total distance. $\frac{120}{x} = \frac{2}{5}$
   b. What is the total distance? 300 mi.

4. When they bought their new car, the Pierces paid 30% of the price in cash. The payment was \$900.
   a. Set up and solve the proportion to find the total cost. $\frac{30}{100} = \frac{900}{x}$
   b. How much was the amount still to be paid? \$ 2100

5. The depreciation on Mr. Edward's car last year was \$500. This was 20% of its value when new.
   a. Set up the proportion to find its value when new. $\frac{20}{100} = \frac{500}{x}$
   b. What was its value when new? \$ 2500
   c. Explain how you could express 20% as a fraction, and solve the problem without using a proportion. $\frac{1}{5}x = 500$

6. About 9% of the cars manufactured in 1920 were closed cars. About 180,000 cars manufactured in that year were closed cars.
   a. 180,000 is what per cent of the total number of cars manufactured? 9%
   b. Set up and solve the proportion to find the total number of cars manufactured. $\frac{9}{100} = \frac{180,000}{x}$
   c. How many of the cars manufactured in that year were not closed cars? 1,820,000

7. When the Harmons had driven 177 miles, the gauge showed that they had used $\frac{3}{5}$ of a tank of gasoline.

a. Set up and solve a proportion to find how far they could travel on a full tank. $\frac{177}{x} = \frac{3}{5}$

b. How many miles farther can they travel at that rate before the tank is completely empty? 118 mi.

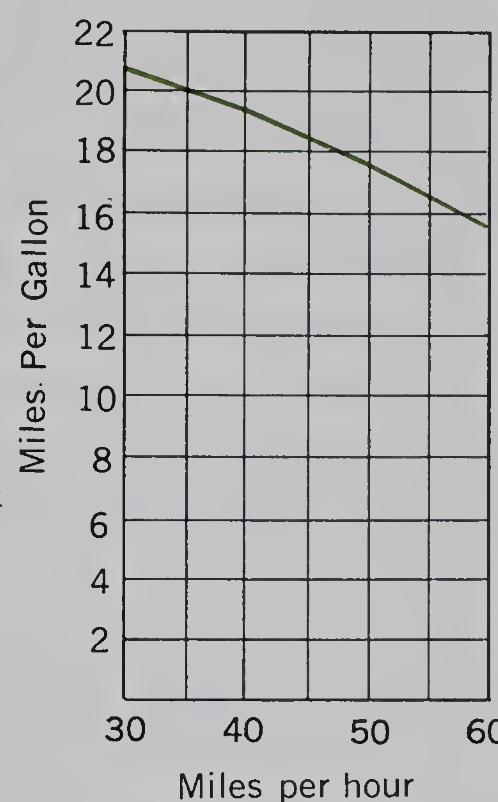
8. It is 280 miles from Albany to Buffalo. This is $\frac{7}{20}$ of the distance from Albany to Chicago by way of Buffalo.

a. Set up and solve a proportion to find the distance from Albany to Chicago. $\frac{280}{x} = \frac{7}{20}$

b. How far is it from Buffalo to Chicago? 800 mi.

9. The graph shows how the number of miles per gallon decreases as the speed increases in a certain make of car.

HOW THE CONSUMPTION OF GASOLINE INCREASES WITH SPEED



a. How many miles does the car travel on a gallon at 30 miles per hour? 20.5 mi.

b. How many miles will the car travel on a tank of 18 gallons at 30 miles an hour? 369 mi.

c. How many miles does the car travel on a gallon at 60 miles an hour? 15.4

d. How many miles will the car travel on a tank of 18 gallons of gasoline at 60 miles per hour? 277.4 mi.

e. The number of miles the car travels per gallon of gas at 60 miles per hour is what per cent of the number at 30 miles per hour? 75.1%

f. How many miles more per gallon did the car travel at 30 miles per hour than at 40 miles per hour? 1.5 mi.

g. How many miles less did the car travel per gallon at 50 miles per hour than at 40 miles per hour? 1.8 mi.

h. By what per cent is the number of miles per gallon decreased by driving 50 miles per hour instead of 40 miles per hour? 9.1%

i. The report stated that increasing the speed from 40 to 60 miles per hour increased by $11\frac{1}{2}$ gallons the amount of gasoline used for each 1000 miles traveled. Can you figure this out from the graph? yes

j. If your answer to **i** was yes, how many gallons are saved in 1000 miles of driving by traveling at 30 miles per hour instead of 60 miles per hour? 16.1 gal.

10. In budgeting for their vacation trip, the Browns are allowing 40% for car expenses. They figure the expenses of the car should be about \$10 a day.

a. Set up and solve the proportion to determine their estimate of total expenses. $\frac{10}{x} = \frac{40}{100}$

b. What is their estimate of expenses other than car expenses? \$ 25

11. Mr. Nelson is buying a new car. He is paying 35% of the purchase price in cash, which amounts to \$2100.

a. Set up and solve the proportion to determine the total cost of the car. $\frac{2100}{x} = \frac{35}{100}$

b. How much does he still owe on the car? \$ 3900

12. Ben worked part-time in a filling station. All employed in the station were given a 10% discount on gasoline. Last month Ben bought 40 gallons of gasoline. The regular price of the gasoline was 36.9¢ per gallon. How much did Ben save? \$ 1.48

13. Ben's convertible needed two new tires. Ben can buy them at the filling station for \$22 each. The regular price is \$28 each. What per cent discount does Ben receive? 21.4 %

14. The filling station sold premium oil in bulk at 40% off. To receive the discount, the customer had to buy 5 gallons or more. The regular price of the oil was 50¢ a quart. How much was saved in buying 5 gallons? \$ 4

15. The variable costs of driving a car in the lower price range total about 4¢ a mile. Of this, 2.6¢ is cost of gasoline. What per cent of the total variable costs is cost of gasoline? 65%

16. The Sanders family car averages 16 miles on a gallon of gasoline. The mileage of the economy car that Mr. Sanders drives to work is 175% of the family car's mileage. How many miles per gallon does Mr. Sanders get from his economy car? 28 mi.

17. Ben Meade found that the tire pressure gauge used at the filling station was not accurate. When a tire actually had 22 pounds of pressure, the gauge read 20 pounds. A tire with 33 pounds of pressure tested 30 pounds with the gauge. What correction would be applied to the reading to get the correct pressure with the gauge?

Add 10 % to the reading.

18. Mike and his father drove from Seattle to San Francisco, a distance of 864 miles, using 54 gallons of gasoline. What mileage were they getting from their car? 16 mi. per gal.

19. The gasoline cost 35.9¢ per gallon. What was the cost of gasoline per mile, to the nearest tenth of a cent? 2.2¢

THE FORMULA FOR DISTANCE, RATE, AND TIME

Many problems about automobiles are concerned with (a) How far did you go (distance)? (b) How long did it take (time)? or (c) How fast did you travel? Knowing two of these things, you can find the third. It is easy if you use the formula $d = rt$.

EXAMPLE

John is driving his car at 40 miles an hour. How far will he go in four hours if he maintains that speed?

The formula is $d = rt$
$r = 40 \quad t = 4$
Then $d = 4 \times 40 = 160$

John will travel 160 miles in four hours, at 40 miles per hour.

1. How far will a car travel in six hours at an average speed of 55 miles per hour? 330 mi

2. A bus is scheduled to average 50 miles an hour. How far will it travel between 9:30 A.M. and 12 noon? 125 mi

3. An airplane in use in 1915 could travel 85 miles an hour and carried enough fuel for $4\frac{1}{2}$ hours flying time. How far could it travel on one fueling? 382.5 mi

4. When you want to know how long a trip will take, given the distance and rate, the formula is most convenient when solved for t.

$$d = rt \qquad p = f_1 \times f_2$$
$$t = \underline{\ ?\ } \qquad f_2 = p \div f_1$$

Write the formula solved for t. $t = \frac{d}{r}$

5. John is planning to visit his cousin in a city 240 miles away. He plans to drive and to average about 40 miles an hour. How long will the trip take at this rate of driving? 6 hr.

6. A truck driver is to transport a load of merchandise from San Francisco to San Diego, a distance of 540 miles. He plans to average 45 miles an hour. How long should the trip take him? 12 hr.

7. If you know the distance traveled and the time required, to find the rate, the formula is most convenient when solved for r.

$$d = rt \qquad p = f_1 \times f_2$$
$$r = \underline{\ ?\ }\ \frac{d}{t} \qquad f_1 = p \div f_2$$

8. The distance from Chicago to New York is 850 miles by air. A plane is scheduled to make the trip in 1 hour 40 minutes. What is the average speed of the plane in miles per hour? 510 m.p.h.

9. The high school record for the 100-yard dash is 9.4 seconds. How many feet is this per second? 31.9

10. It takes 8 minutes 20 seconds (in August and April) for the light from the sun to reach the earth. The distance to the sun is then 93 million miles. How far does light travel per second? (The distance from the earth to the sun varies somewhat; in January it is $91\frac{1}{2}$ million miles.) 186,000 mi.

11. How long does it take light from the sun to reach the earth in January? 8 min. 12 sec.

Using The Distance-Rate-Time Formula

1. The distance from Yellowstone National Park to the Grand Canyon is 730 miles. Traveling at an average rate of 50 miles per hour, how many hours will it take to make this trip? 14.6 hr.

2. How many hours driving time will it take to travel the 1700 miles from Omaha to Los Angeles at an average speed of 50 miles per hour? 34 hr.

3. The distance from Philadelphia to Miami, Florida, is 1250 miles. A bus makes the trip in 25 hours actual driving time. What is the average speed of the bus? 50 m.p.h.

4. The Great Northern streamliner takes 27 hours to travel from Minneapolis to Seattle, a distance of 1701 miles. What is the average speed of the train when traveling between these cities? 63 m.p.h.

5. A train and a bus leave Chicago at 10:00 A.M. for Omaha. The train averages 60 miles per hour and the bus 40 miles per hour. At 3:00 P.M., how much farther has the train traveled than the bus? 100 mi.

6. On their vacation trip the Caspers agreed to limit their driving time to $7\frac{1}{2}$ hours each day. Mr. Casper wished to travel 375 miles each day. What would be Mr. Casper's average rate of speed? 50 m.p.h.

7. An airliner flies at an average speed of 550 miles per hour. How far will it travel in $5\frac{1}{2}$ hours? 3025 mi.

8. A submarine can average 10 miles per hour submerged and 25 miles per hour on the surface. In a test a submarine cruised 300 miles, the first half of the trip submerged and the last half on the surface. How long did it take the submarine to make the trip? 21 hr.

9. A jet plane which can average 1500 miles per hour must refuel every 40 minutes. How far can the plane travel between fueling intervals? 1000 mi.

10. It takes about $1\frac{1}{3}$ seconds for light from the moon to reach the earth. About how far from the earth is the moon? (Speed of light is 186,000 miles per second.) 248,000 mi

11. The Jensens plan to travel 400 miles per day on their vacation trip. They will travel three hours in the morning and $4\frac{1}{2}$ hours in the afternoon. At what rate should they travel if they are to maintain this schedule? Give answer to the nearest whole mile. 53 m.p.h.

12. In planning their vacation trip this summer, the Smiths are planning to drive about six hours each day at an average rate of 50 miles per hour. About how far are they planning to drive each day? 300 mi.

13. A truck driver maintained an average speed of 45 miles an hour for $6\frac{1}{2}$ hours. How far did he travel in that time? 292.5 mi.

14. Mr. Adams drove from Centralia to Elmstown in five hours. The distance is 275 miles. What was his average speed? 55 m.p.h.

15. It is 528 miles from Chicago to Buffalo. The Johnsons plan to make the trip in 11 hours of driving. At what rate do they plan to travel? 48 m.p.h.

16. Mr. Black averaged 45 miles an hour in driving from Janesville to Springdale, a distance of 360 miles. How long did it take him? 8 hr.

17. A truck driver left San Jose for Santa Barbara, a city 275 miles away, at 7 A.M. He plans to drive at the rate of 50 miles per hour. At what time does he plan to get to Santa Barbara? 12:30 P.M.

18. The Smiths left Omaha at 8 A.M. on a trip to Denver. By 12 noon they had traveled 220 miles. At what rate had they been driving? 55 m.p.h.

19. A bus leaves Jamestown at 7 A.M. and arrives at Centerville, 300 miles away, at 5 P.M., with a half-hour stop for lunch. At what rate has it traveled? Give answer to the nearest mile per hour. 32 m.p.h.

20. Mr. Smith plans to drive to Elroy, which is 120 miles away, for a business conference at 4 P.M. If he plans to travel about 40 miles per hour, at what time should he leave? 1:00 P.M.

21. The Hansens are planning a transcontinental trip this summer and hope to travel an average of 400 miles per day. They plan to stop for an hour at noon and to find a place for the night at about 5:30 P.M. If they average 50 miles per hour in traveling, at about what time must they start driving in the morning? 8:30 A.M.

1. The Morrisons, who live in Kansas City, plan to take a trip in August to visit four national parks and a national monument. After studying the road maps, Agnes Morrison made a sketch to outline the trip and show the distances.

DISTANCES

Kansas City to:	Hays	Denver	Salt Lake City	Bryce Canyon	Grand Canyon No. Rim	Grand Canyon So. Rim
	253	606	1113	1380	1540	1744

South Rim of Grand Canyon to	Mesa Verde	Colorado Springs	Salinda	Kansas City
	280	650	1097	1250

What are the four national parks and the national monument? *See front.*

2. What is the total mileage for the trip? *2994 mi.*

3. Joe Morrison set up the itinerary for the trip, selected desirable stopping places for each day, and planned to drive less than 400 miles on any day, where practical. Calculate the mileage planned for each day. How many of the planned drives are over 400 miles? *2*

Day	*Stopping Place*	*Miles Driven*
1	Hays	☐ 253
2,3,4	Denver, Rocky Mountain Park	☐ 353
5,6,7	Salt Lake City	☐ 507
8	Bryce Canyon	☐ 267
9,10	North Rim	☐ 160
11,12,13	South Rim	☐ 204
14,15,16	Mesa Verde	☐ 280
17,18	Colorado Springs	☐ 370
19	Salina	☐ 447
20	Kansas City	☐ 153

4. The Morrisons plan to spend a few days in each of the parks in Salt Lake City and in Colorado Springs, where they will drive to the top of Pike's Peak. Since they can be away from home for four weeks, they also have several extra days that will allow them to spend extra time at any interesting spot on the trip. How many days do they plan to spend driving from one spot to another? 10 What do they plan as an average day's drive? 300 mi.

5. The planned route would take them over the Continental Divide on Trail Ridge Road in Rocky Mountain National Park at 12,183 feet above sea level. The altitude of Kansas City is 754 feet. How many feet is this below the Divide in Rocky Mountain National Park? 11,429 ft.

6. From Trail Ridge Road many of the high peaks of the Rocky Mountains are visible. The highest in the Park is 14,225 feet high. How many feet is this above the Trail Ridge Road? 2042 ft.

7. After leaving the Grand Canyon, they went to Mesa Verde National Park. There they saw cliff dwellings that were inhabited when William the Conqueror invaded England in the Eleventh Century and were abandoned by the Indians about 1200 A.D. Denver was born in the gold rush of 1859. How many years separate the latter two events? 659 yr.

8. One of the places where they might spend some extra time is at the North Rim of the Grand Canyon. This is less crowded than the South Rim, because it is not near a major transcontinental highway. The summer climate is cooler here because it is 2000 feet higher than the South Rim. Across the Canyon from the North Rim to the South Rim is 6 miles. By the highway it is 204 miles. This is how many times the direct distance across the canyon? 34

9. At Monument Valley there are no night lodgings, but a place for picnic lunches has been provided in a beautiful setting. The Morrisons planned to lunch there on the drive to Mesa Verde, even though it is 21 miles off the direct route and is 140 miles from the South Rim. How many miles will they drive after lunch to get to the Mesa Verde? *182 mi.*

10. Spanish explorers camped at Mesa Verde in 1776, and the ruins were discovered in 1874. It was established as a park in 1906. This was how many years after its discovery? *32 yr.*

11. The ruin named Cliff Palace had 200 living rooms and sheltered about 400 persons. A modern 18-story hotel has about 15 rooms per story. How many more rooms are in an 18-story hotel than in Cliff Palace? *70*

12. Mr. Morrison figures that the variable costs for driving his car which he should use in this case are 3.7¢ per mile. At this rate, what will it cost him to drive the car on this trip? *$ 110.78*

13. Mr. Morrison had considered purchasing a larger car that would cost 5.6¢ per mile for operation. How much more would the trip cost with the larger car? *$ 56.88*

14. It is possible to rent a car for $3.65 a day plus 6¢ a mile. If the Morrisons rented a car for their trip, what would the rental cost? *$ 281.84*

15. Another company charges $68 a week and the first 1000 miles free. Thereafter, 5¢ a mile is charged. The minimum rental time is 21 days. Compare the rental costs of the trip using these rental fees. *The cost is $ 371.70, which is $ 89.86 cheaper.*

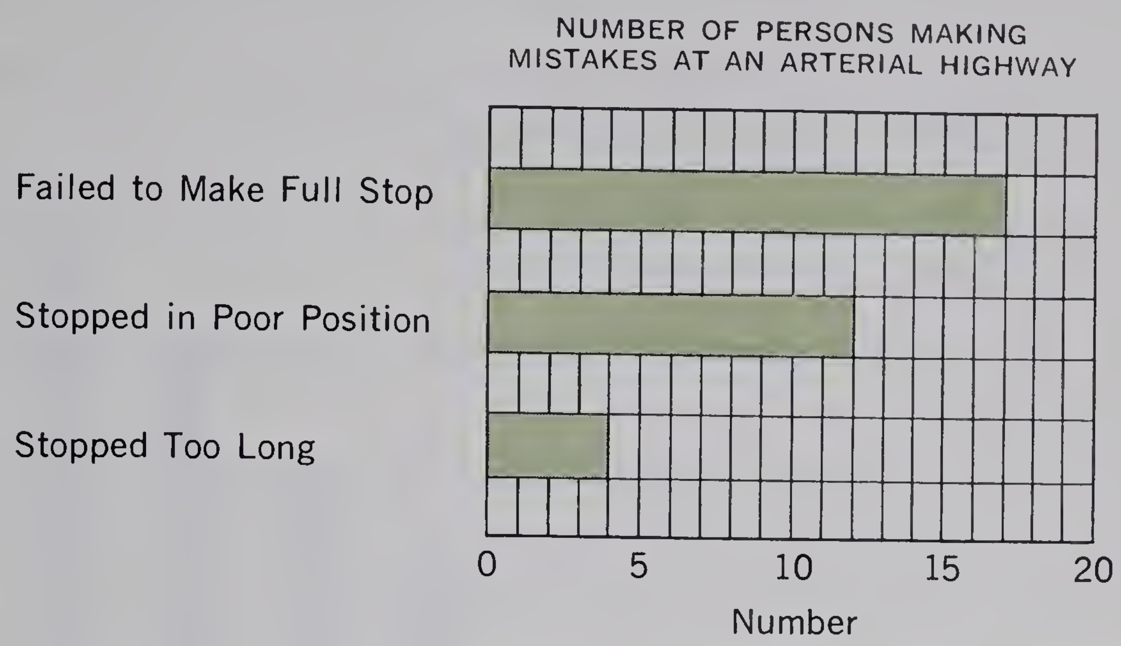
16. Investigate the rental fees in your area and calculate the costs of renting a car for the Morrison's trip at the rates specified for your area. *Answers will vary.*

17. The 500-mile Indianapolis Motor Speedway winner in 1965 was Jim Clark, with a time of 3 hours 19 minutes. What was his average speed in miles per hour to the nearest tenth of a mile? *150.8 m.p.h.*

18. In 1920 the winner at Indianapolis had an average speed of 88.5 miles per hour. How long did it take him to drive the 500 miles? *5 hr. 38 min.*

19. The Harmon family plans to make a tour of the national parks during the summer. On the days when they make a long drive, they will leave at 8:30 A.M. and drive until 5:30 P.M., with an hour stop at noon, and a half hour in the middle of the afternoon. They plan to drive at about 45 miles per hour. About how far can they travel in a day? *337.5 mi.*

1. The graph shows the number of drivers who made each kind of mistake when coming to an arterial highway where there was a stop sign. 220 drivers were checked and their actions recorded. How many made no mistakes? *187*

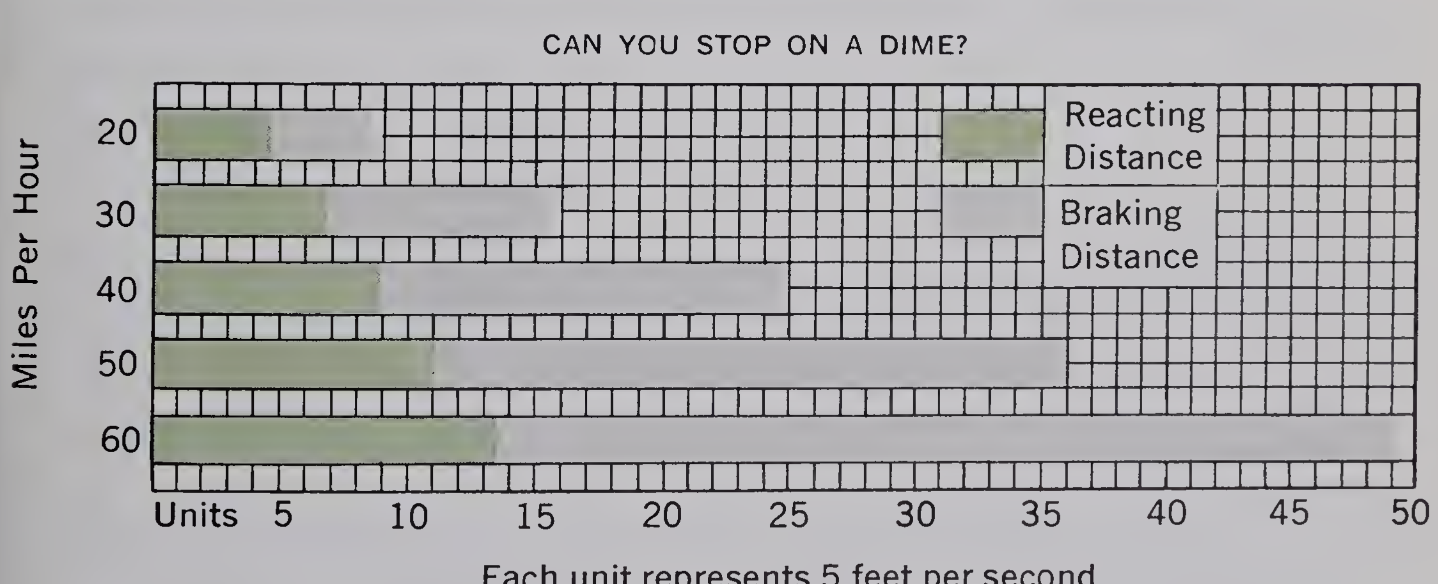


2. What per cent of the drivers, to the nearest tenth of 1%, failed to come to a full stop? *7.7 %*

3. Those who "Stopped in poor position" were not in position to see the street to the right or left. How many made this error? *12*

4. What per cent stopped too long? *1.8 %*

5. This graph shows how long it takes to stop a car after the driver sees the need to stop. How much more distance is needed to stop a car traveling 40 m.p.h. than at 20 m.p.h.? *77 ft.*



6. When a car is traveling 30 miles per hour, how many feet will it travel in one second? *44 ft.*

7. It takes 0.75 seconds for the average driver to take his foot from the accelerator and apply the brake. How far will a car travel during this action if the car is moving at 30 miles per hour? 33 ft.

8. How far will a car travel in that time at 60 miles per hour? 66 ft.

9. In the graph this distance is called the "reacting distance." Do your answers for Exercises 7 and 8 agree with the graph? yes

10. The braking distance at 40 miles per hour is how many times the braking distance at 20 miles per hour? 4

11. The braking distance at 60 miles per hour is how many times the braking distance at 30 miles per hour? 4

12. When the speed is doubled, the braking distance is multiplied by what number? 4

13. Calculate the reacting distance and braking distance at 80 miles per hour. Draw a divided-bar graph to show how the total distance required to stop at 80 miles per hour is divided. See front.

14. In passing a car, you should allow 200 feet as the *margin of safety*. If your car and the approaching car are both traveling at 50 m.p.h. on a two-lane highway, how many feet of clear highway are needed for passing a car traveling 45 m.p.h. (that is, the total distance traveled by your car and the approaching car plus 200 feet)? 1630 ft. (see p. 428)

15. How many feet of clear highway are needed if your car and the approaching car are traveling at 60 m.p.h. and the car you are passing is traveling at 50 m.p.h.? 1080 ft.

16. What kinds of motor vehicles are included in accidents? Prepare a suitable graph of this information. Have students draw large bar graphs.

Type	*Per Cent of Fatal Accidents*	*Per Cent of Non-Fatal Accidents*
Passenger car	79.5	82.1
Commercial car or truck	16.5	11.9
Taxi	.9	2.9
Bus	1.0	1.5
Motorcycle	2.1	1.6

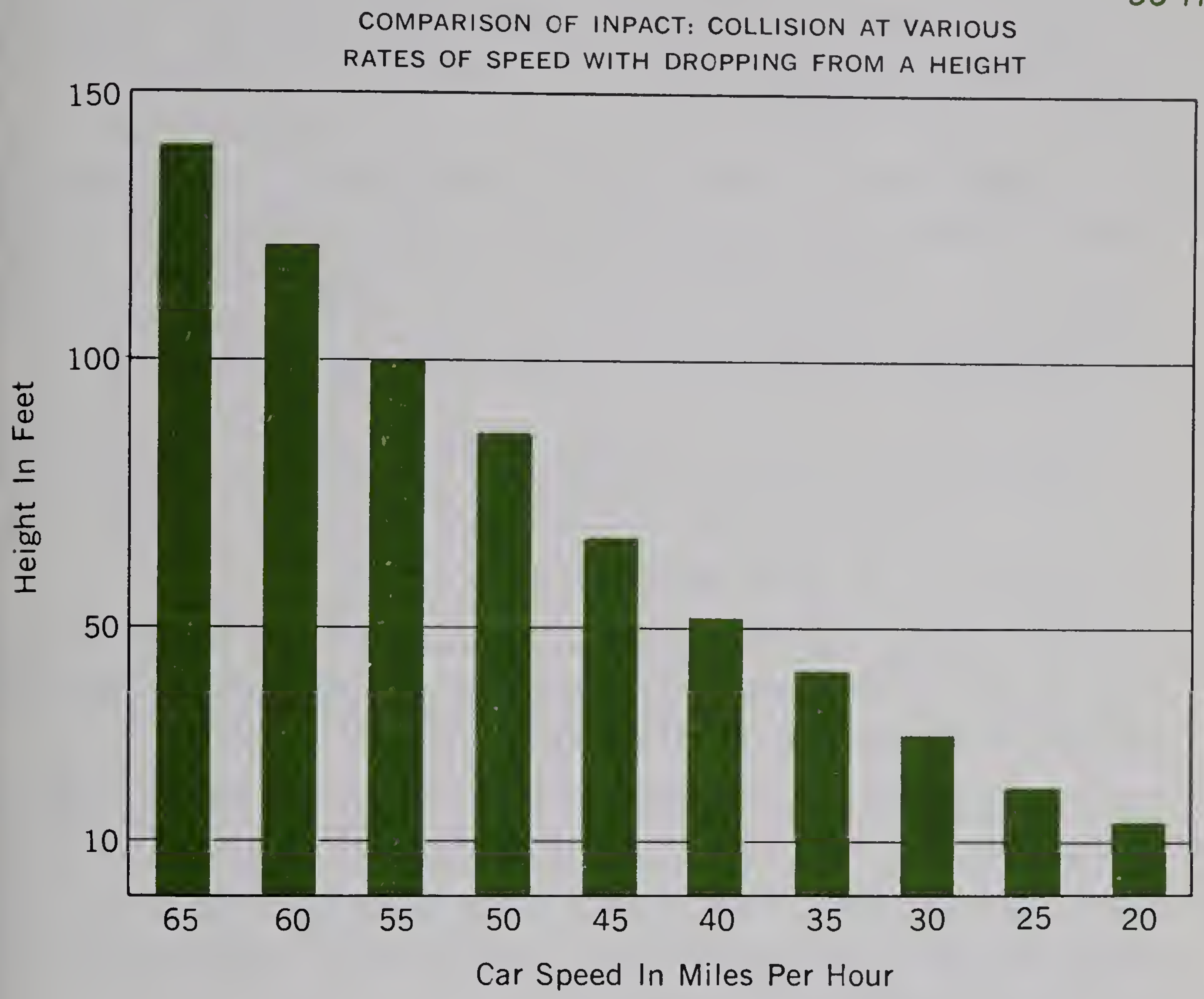
17. It is reported that highway construction is paid for as follows:

Gasoline taxes	53%	Federal aid	11%
Motor vehicle licenses	30%	Miscellaneous	6%

Suppose a mile of new road cost \$200,000. How much would be paid from each source, using the percentages listed? taxes, \$106,000; licenses, \$60,000; aid, \$22,000; miscellaneous, \$12,000

1. Did you know that if you run into a tree or wall, or some other solid object when driving at 25 miles an hour, the impact is the same as if you had driven the car off the side of a 20-foot cliff? Other comparisons may be read from the graph. If the collision were at 30 miles an hour, what is the height that would produce a similar impact?

30 ft.



2. The relationship between the rate the car is traveling, and the drop that will produce a comparable impact, is given (approximately) by:

$$h = \frac{r^2}{30}$$

where h is height of the cliff in feet and r is speed of the car in miles per hour. How high is the cliff if the car is traveling at 60 miles per hour? Make the calculation and compare your results with the graph. *120 ft.*

3. Speeds on the Indianapolis race track in the annual "500" are around 150 miles per hour. If a car runs into the wall at that speed, it is equivalent to dropping from what height? *750 ft.*

4. Prepare a line graph showing the relationships from the formula for rates from 0 to 75 miles per hour. *See front.*

Though the American automotive industry is only a little more than a half-century old, nothing before or since has so captured the fancy of Americans and become such a massive industry so rapidly. In 1948 the auto industry manufactured its 100 millionth vehicle. Each year thereafter several million more have been produced.

1. The first successful automobile in the United States was made by the Duryea Brothers in 1893. How many years later was the 100 millionth car built? 55
2. The automobile as we know it was made possible by the development of the gasoline engine, which is called an internal combustion engine. Dr. Nikolaus August Otto of Cologne, Germany, in 1878 worked out the design we utilize today. How many years after Dr. Otto's discovery did the Duryeas make their first car? 15
3. In the early days of the automobile, inventors also developed electric and steam powered cars. The Stanley Steamer set a world speed record of 127.66 miles per hour in 1906. How many miles is this per minute (to the nearest tenth)? 2.1 mi
4. The first Indianapolis 500-mile race was won by a Marmon Wasp in 6 hours and 42 minutes. What was the average speed in miles per minute to the nearest tenth? 1.2 mi. per min.
5. How much faster was the Stanley Steamer than the Marmon Wasp? 0.9 mi per min.
6. In 1901 Ransom Olds became the first mass producer of automobiles when he built 1500 Oldsmobiles. The average cost of an automobile at that time was $925.00. At this price what was the total cost for the 1500 cars? $ 1,387,500
7. Henry Ford built his first car in 1896. The first Model T appeared in 1908 and was the sales leader for many years thereafter. He changed to the Model A in 1929. How many years did Ford build Model T's? 21
8. Ford Motor Company was a great success. A friend invested $25,000 in 1903. He received $5,000,000 in dividends and sold his interest in 1919 for $30,000,000. How much did he receive in dividends each year per dollar invested? $ 12.50
9. Nearly 90% of all passenger cars made in 1919 were open cars. About 1,800,000 cars were manufactured in 1919. How many of these were closed cars? 180,000
10. By 1930, 80% of the cars were closed models. There were 3,787,455 cars manufactured in 1930. How many were open cars? 757,491

For Americans it seems that automobiles are here to stay. American families own a total of 70 million cars, and new ones are being produced at the rate of eight million a year. This figure does not include 15 million trucks and 300,000 buses on the highways. Last year 60 billion dollars were spent in automotive stores and gasoline stations alone to help keep the army of vehicles on the road. There are 45 million more cars today than there were in 1947. This is an increase of 180%. Two or more cars are owned by 25% of American families. Last year Americans drove their passenger cars more than 800 billion miles, and in the process burned 67 billion gallons of fuel. The average American citizen mutters about traffic but doesn't seem at all ready to give up his car, nor is he likely to keep off the crowded highway.

1. How many more cars are there today than there were in 1947?
a. 45 million **b.** 36 million **c.** 50 billion **d.** 30 million

2. What per cent of American families own two or more cars?
a. 78% **b.** 25% **c.** 14% **d.** 30%

3. American families now own how many cars?
a. 60 million **b.** 70 million **c.** 15 million **d.** 300,000

4. How many miles did Americans drive last year?
a. 50 million **b.** 660 billion **c.** 800 billion **d.** 67 billion

5. How many cars were owned by Americans in 1947?
a. 45 million **b.** 25 million **c.** 67 million **d.** 30 million

6. Last year motorists in this country used how many gallons of fuel?
a. 62 billion **b.** 51.5 billion **c.** 54.5 billion **d.** 67 billion

7. How many dollars were spent last year in automotive stores and gas stations?
a. 30 million **b.** 60 million **c.** 67 billion **d.** 60 billion

8. There are how many times as many automobiles as trucks in this country? **a.** 3 **b.** $5\frac{1}{3}$ **c.** $4\frac{2}{3}$ **d.** $3\frac{1}{3}$

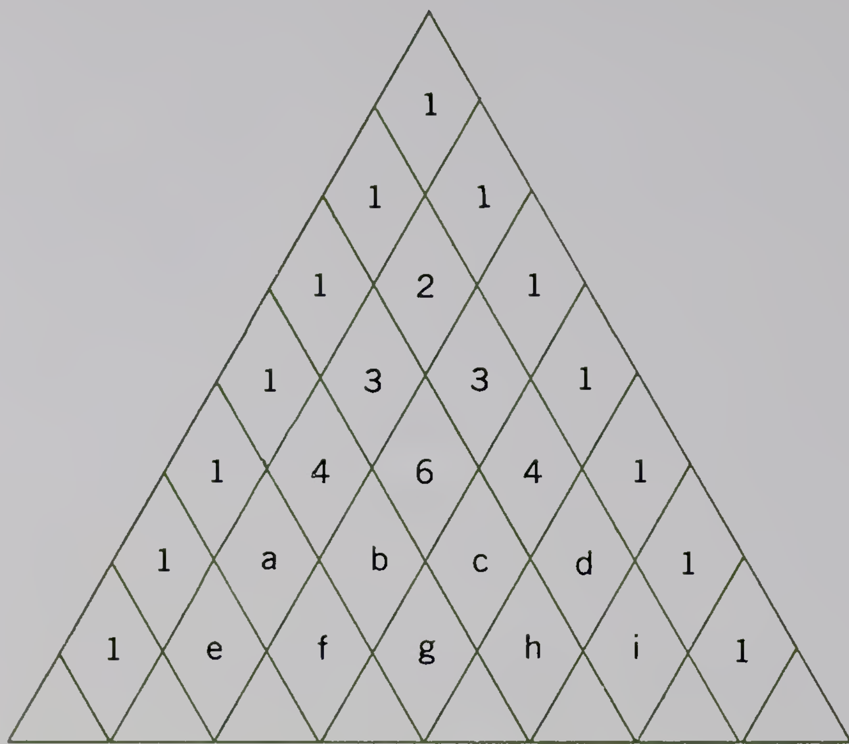
9. There are how many times as many trucks as buses?
a. 90 **b.** 50 **c.** 70 **d.** 80

10. The average passenger car traveled about how many miles?
a. 9000 **b.** 10,000 **c.** 12,000 **d.** 11,000

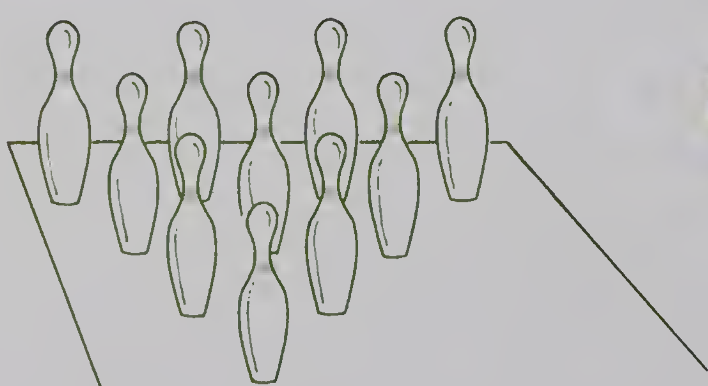
11. What was the number of gallons of gasoline consumed per automobile? **a.** 500 **b.** 750 **c.** 1000 **d.** 1200

12. About how many miles were traveled, on the average, per gallon of gasoline? **a.** 8 **b.** 10 **c.** 12 **d.** 14

The following arrangement of numbers is called Pascal's Triangle. It has a number of interesting applications.



1. Note that each row begins with 1. After that each number is determined by the two numbers above it. Examine the triangle and find the rule that determines the numbers. *Each numeral is determined by the sum of the two numbers above it.*
2. Find the missing numbers in the sixth row. *1, 5, 10, 10, 5, 1*
3. Write the numbers in the seventh row in the triangle. *1, 6, 15, 20, 15, 6, 1*
4. The pins in bowling are arranged in a triangle. This is illustrated in the arrangement of the top four rows of Pascal's Triangle. How many pins are there? *10*



5. Certain numbers are referred to as "triangular numbers," since they define numbers of objects that can be arranged in triangles. Refer to the pins and explain why 10 is a triangular number. *Ten objects can be arranged in a triangular shape.*
6. At the beginning of a game in pocket billiards, the 15 balls are racked racked up in a triangular frame. Is 15 a triangular number? *yes*

7. Is there any triangular number between 10 and 15? no You can test the possibility by making dots on a piece of paper.

8. Find three other triangular numbers by drawing diagrams. Answers will vary; however the three consecutive ones after 15 are 21, 28, and 36.

9. A systematic way to discover triangular numbers is provided in Pascal's Triangle. Use the triangle to identify the triangular numbers you found in the exercises above.

10. What are the next two triangular numbers after 15? 21 and 28

11. If you use marbles or small cubes, you can illustrate another kind of number called *pyramidal*. As suggested by the name, these numbers define the number of objects needed to form a pyramid. You can see that the first two pyramidal numbers are 1 and 4. There is one member in the top layer of the second pyramid, and there are three members in the next layer. What kind of numbers define the number of objects in each layer? triangular numbers

Pyramidal Numbers

12. If you add a third layer, how many members will it have? What is the third pyramidal number? 6; 10

13. Referring to Pascal's Triangle, what should be the fourth pyramidal number? Draw a sketch or construct a pyramid, and see if you are correct. 20

14. Refer to Pascal's Triangle and list the first seven pyramidal numbers.
1, 4, 10, 20, 35, 56, 84

RESEARCH AND REPORT

Pascal's Triangle has many important applications in algebra and statistics. Use the encyclopedia and the mathematics library to find out about its applications and prepare a report for the class on the subject.

The followers of Pythagoras in ancient Greece devoted much attention to triangular numbers. Find out and report on the reasons for the Greeks' interest in them and what properties were attributed to them. You may find mention of square numbers, too.

Pyramidal numbers do not have as long a history. Investigate to see who invented them and for what reason.

Part One

Find the products to the nearest cent:

1. (1.25) · ($18.50) $23.13
2. (21.75) · ($90) $1957.50
3. (.043) · ($175) $7.53
4. (5.19) · ($6.55) $33.99
5. (13.7) · ($25.95) $355.52
6. (6.15) · ($55.47) $341.14

Find the quotients to the nearest cent:

1. $19.75 ÷ 3.25 $6.08
2. $48.67 ÷ .47 $103.55
3. $325.50 ÷ 37.5 $8.68
4. $59.85 ÷ 42.8 $1.40
5. $9.75 ÷ 7.5 $1.30
6. $200.50 ÷ 3.82 $52.49

Find the value for n:

1. 18 is n% of 45 40
2. 17.5% of 56 is n 9.8
3. 15 is 30% of n 50
4. 185% of 60 is n 111
5. 25 is n% of 6.25 400
6. 26 is 200% of n 13
7. 16 is 25% of n 64
8. 85 is n% of 200 42.5
9. 0.3% of 250 is n 0.75
10. n is 17.5% of 60 10.5
11. 56 is 40% of n 140
12. n is 0.2% of 40 .08
13. 72 is n% of 60 120
14. 48 is 120% of n 40

Find the interest on the following:

1. $250 for 90 days at 5% $3.13
2. $175 for 72 days at 4% $1.40
3. $375 at $5\frac{1}{2}$% for 18 months $30.94
4. $500 at 6% for 2 years $60
5. $600 for 2 years at 6% $72
6. $150 for 75 days at 4% $1.26
7. $325 for $1\frac{1}{2}$ years at 3% $14.63
8. $750 for 90 days at 5% $9.38
9. $1800 for 120 days at $4\frac{1}{2}$% $27
10. $800 for 80 days at $5\frac{1}{2}$% $9.68

Copy each answer, and place the decimal point correctly.

1. (3.15) · (23.5) = 74025 74.025
2. (3.15) · (.0235) = 74025 .074025
3. (31.5) · (2.35) = 74025 74.025
4. (.315) · (235) = 74025 74.025
5. 899.64 ÷ 1.19 = 756 .0756
6. 8.9964 ÷ 1.19 = 756 .000756
7. .89964 ÷ 11.9 = 756 .000756
8. 899.64 ÷ 0.119 = 756 .00756

Part Two

Each of the following questions is answered by one of the words or phrases which follow it. Copy the number and next to it, write the letter for the word(s) which the statement defines. (circled)

1. Which kind of insurance protects the car owner from damages for injuries to someone else in a traffic accident?
   a. collision insurance
   b. property damage
   c. personal liability
   d. accident insurance

2. Which of these is not a variable cost of owning and operating a car?
   a. gasoline
   b. auto license
   c. car service
   d. oil and lubrication

3. Which of these is not a fixed cost of owning and operating a car?
   a. fire and theft insurance
   b. liability insurance
   c. depreciation
   d. tires

4. What is the term to identify the decrease in value of a car due to increasing age?
   a. assessed value
   b. depreciation
   c. liability
   d. property damage

5. If you know the distance to the next town and know how long it takes you to drive there, which formula would you use to find your average rate of travel?
   a. $d = rt$ b. $r = \frac{d}{t}$ c. $t = \frac{d}{r}$

6. Which of these units is used to measure the power of an automobile engine?
   a. kilowatts b. meters c. horsepower d. grams

7. What is the price paid for insurance for a stated period called?
   a. purchase price b. premium c. liability d. depreciation

8. If you know the distance to the next town and plan to drive at 50 miles per hour, which of these formulas would you use to find how long it will take you to get there?
   a. $d = rt$ b. $t = \frac{d}{r}$ c. $r = \frac{d}{t}$

9. Which of these is a fixed cost in owning and operating a car?
   a. insurance b. tires c. gasoline d. car service

10. Which of these is a variable cost in owning and operating a car?
    a. license
    b. depreciation
    c. lubrication
    d. insurance

On the left is a list of risks against which you can protect yourself with car insurance. List the numbers 1-4 on your paper, and after each number, write the name of the insurance that will protect you against that risk.

1. Against claims for injury to others. *personal liability*	collision insurance
2. Against loss if your car burns up. *fire insurance*	fire insurance
3. Against damage to your own car by a moving object *collision insurance*	personal liability insurance
4. Against loss if your car is stolen. *theft insurance*	theft insurance

Part Three

1. Mr. Henry purchased a new car for $3800. He paid half the price in cash and the rest in 24 monthly installments of $83.33 each. What rate of interest is he paying on the balance? *5.3 %*

2. How much is the depreciation on the car during the first year, according to standard figures (page 430)? *$ 1520*

3. Mr. Henry took out the following insurance policies: comprehensive; personal liability, $20,000-$40,000; property damage, $5000. The premium rates are those listed for new car *Y* (page 432). What were his total premiums? *$ 118.85*

4. The auto license cost $65. What were Mr. Henry's fixed expenses for the first year he owned the car? *$ 1703.85*

5. At 60 miles per hour, 63% of the power developed by the engine of a car is required to push the car through the air. If a car is developing 325 horsepower, what horsepower is used to overcome friction of the air? *204.75*

6. It takes 3% of the engine's horsepower to run the generator and fan. In a 325 horsepower engine, how many horsepower is this? *9.75*

7. The fuel-to-air ratio of a certain car is 1:15; that is, for each pound of fuel burned, 15 pounds of air are used to burn the fuel. How many pounds of fuel would be burned with 135 pounds of air? *9*

8. A gallon of gasoline weighs about 7.5 pounds. How many pounds of gasoline are in an 18-gallon tank? *135*

9. If the fuel-air ratio is 1:15, how many pounds of air will be used in burning the gasoline in an 18-gallon tank? *270*

10. How many pounds of air will be used in burning 300 gallons? *4500*

Part One

A. Find the value of n to make each of these conditional statements true.

1. $n + 18 = 23$ 5
2. $61 - n = 24$ 37
3. $69 - n = 31$ 38
4. $17n = 189$ $11\frac{2}{17}$
5. $\frac{1}{2}n = 27$ 54
6. $n \div 11 = 98$ 1078
7. $16n = 128$ 8
8. $29 + n = 85$ 56
9. $72 \div n = 3$ 24
10. $15n = 180$ 12

B. Solve each of these proportions:

1. $\frac{x}{7} = \frac{15}{35}$ 3
2. $\frac{17}{102} = \frac{9}{x}$ 54
3. $\frac{15}{x} = \frac{16}{2}$ 1.875
4. $\frac{x}{54} = \frac{15}{90}$ 9
5. $\frac{7}{18} = \frac{x}{36}$ 4
6. $\frac{6}{24} = \frac{x}{100}$ 25

C. Find the value for n:

1. 12 is $n\%$ of 20 60
2. 125% of n is 60 48
3. 40 is $n\%$ of 64 62.5
4. 112.5% of n is 72 64
5. 125 is $n\%$ of 200 62.5
6. n is 45% of 64 28.8
7. $\frac{1}{2}\%$ of 438 is n 2.19
8. 45% of n is 99 220
9. $\frac{3}{8}\%$ of n is 24 6400
10. 137.5% of 88 is n 12.1

D. Solve:

1. $x + 14 = 192$ 178
2. $\frac{x}{2} + 31 = 13$ $^-36$
3. $\frac{x}{12} = \frac{15}{360}$ 0.5
4. $2x + 15 = 9$ $^-3$
5. $\frac{x}{6} = \frac{240}{36}$ 40
6. $\frac{2x}{3} + 141 = 201$ 90
7. $3x = 164 - x$ 41
8. $4x + 31 = 65$ 8.5
9. $5x - 17 = 2x + 1$ 6
10. $3x + 84 = 79 + 2x$ $^-5$

E. Write each of the following as an equivalent fraction in lowest terms.

1. .63 $\frac{63}{100}$
2. .14 $\frac{7}{50}$
3. .07 $\frac{7}{100}$
4. .265 $\frac{53}{500}$
5. .024 $\frac{3}{125}$
6. .005 $\frac{1}{200}$
7. 2.43 $2\frac{43}{100}$
8. 6.5 $6\frac{1}{2}$

F. Write each of the following as an equivalent fraction in lowest terms.

1. 65% $\frac{13}{20}$	**3.** 2.5% $\frac{1}{40}$	**5.** 12.5% $\frac{1}{8}$	**7.** 7.8% $\frac{39}{500}$
2. 46.2% $\frac{231}{1000}$	**4.** 0.6% $\frac{3}{500}$	**6.** 216.4% $2\frac{41}{250}$	**8.** 350% $3\frac{1}{2}$

G. Express each of the following as a decimal and a per cent.

1. $\frac{1}{4}$ 0.25; 25%	**3.** $\frac{2}{5}$ 0.4; 40%	**5.** $\frac{7}{13}$ 0.538; 53.8%	**7.** $\frac{2}{15}$ 0.133; 13.3%
2. $\frac{3}{10}$ 0.3; 30%	**4.** $\frac{5}{8}$ 0.625; 62.5%	**6.** $\frac{7}{9}$ 0.778; 77.8%	**8.** $\frac{7}{8}$ 0.875; 87.5%

H. On the right is a list of formulas. On the left is a list of area formulas. Write the numbers 1 through 6 on your paper and after each number copy the formula which will give you the area desired. (There may be more than one.)

1. Area of a circle $A = \pi r^2$	$A = \pi r^2$
2. Area of a parallelogram $A = bh$	$A = bh$
3. Area of a rectangle $A = lw$ or $A = bh$	$A = 2\pi r$
4. Area of a square $A = s^2$	$A = h\dfrac{(a+b)}{2}$
5. Area of a trapezoid $A = h\dfrac{(a+b)}{2}$	$A = lw$
6. Area of a triangle $A = \frac{1}{2}ab$	$A = 4s$
	$A = \pi dl$
	$A = \frac{1}{2}ab$
	$A = s^2$

I. Using n to represent the number, express each of these relationships algebraically:

1. Five times the number $5n$

2. Five more than the number $n + 5$

3. Five more than $\frac{1}{5}$ of the number $\frac{n}{5} + 5$

4. Three less than twice the number $2n - 3$

5. Half the square of the number $\frac{n^2}{2}$

6. Five times the number increased by half the number $5n + \frac{n}{2}$

7. The number increased by 3, then multiplied by 3 $3(n+3)$

Part Two

1. A refrigerator whose cash price is $250 is sold on the installment plan for $15 down, and $15 a month for 18 months. How much does the customer pay on the principal each month? $13.06

2. How much interest does he pay each month? $ 1.94

3. How much interest is this per year? $ 23.28

4. Find the average amount owed. $ 124.03

5. What rate of interest is the customer paying? 19 %

6. The cash price of a radio is $165. When sold on the installment plan, a carrying charge of $30 is added. The terms are $15 down and the balance in 24 monthly payments. How much is each payment? $ 7.50

7. Deduct the down payment from the cash price, and find the first principal. How much does the customer pay on the principal each month? $ 6.25

8. How much interest does the customer pay each month? How much would this be per year? $ 1.25 ; $ 15.00

9. What is the average amount the customer owes? $ 78.13

10. Find the rate of interest the customer is paying. 19%

11. A gas stove is marked as follows: Cash price, $194.40. Budget plan, $236—terms, $20 down, the rest in 18 monthly payments. Deduct the down payment from the cash price and find the first principal. $ 174.40

12. Complete the following statement for the monthly payment:

Payment on principal	$? 9.69
Payment of interest	? 2.31
Total monthly payment	$? 12.00

13. How much interest would this be for a year? $ 27.72

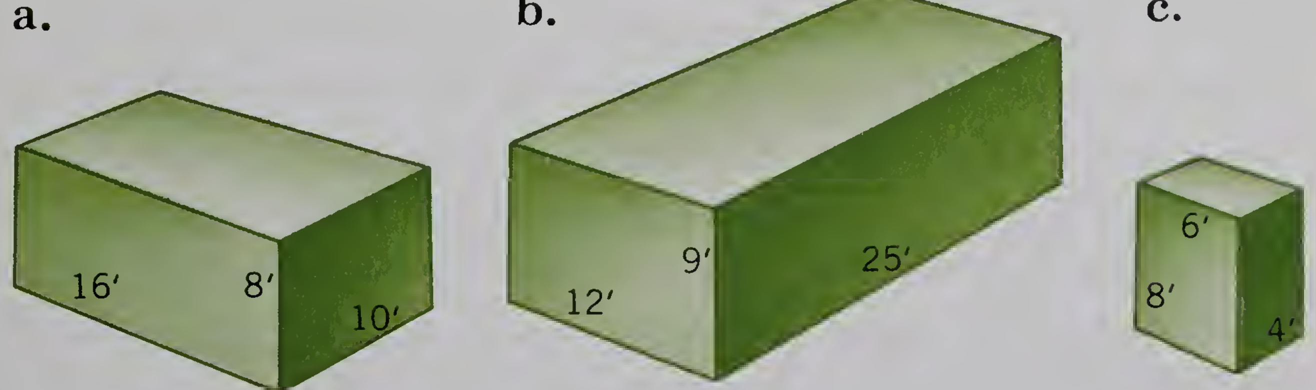
14. Find the rate of interest the customer is paying. 30%

Part Three

1. Find the volume and total surface area of the following solids.

a. 1280 cu. ft. b. 2700 cu. ft. c. 192 cu. ft.

a. b. c.



RECTANGULAR SOLIDS

2. A second cylindrical water tank in Elmtown is 45 ft. high and 35 ft. in diameter. How many gallons of water will it hold? approx. 324,548 gal. ($\pi \approx 3.14$)

3. Steel weighs .29 lb. per cubic inch. What is the weight of a steel ball bearing 1″ in diameter? 0.3 lb.

4. How much will a steel rod weigh that is 2″ in diameter and 10 ft. long? 109.3 lb.

5. A spherical water tank has a diameter of 18 feet. Water weighs about 62.4 pounds per cubic foot. How much will the water in the tank weigh when the tank is full? approx. 190,450 lb.

6. Aluminum weighs 2.7 times as much as an equal volume of water, and steel is 7.87 times as heavy as water. A beam is 4″ × 8″ × 16′. What will this beam weigh if it is made of aluminum? of steel? 599 lb; 1746 lb.

7. If a rectangular storage shed is 72 ft. long, 18 ft. wide, and 12 ft. high, find the number of cubic yards of storage space in this shed. 576 cu. yd.

8. Compare the area of a rectangle whose base is 28 in. and whose height is 32 in. with the area of a 30 in. square. Which is larger? How many square inches larger? rectangle; 4 sq. in.

9. What is the capacity of a cylindrical tank which is 28 feet long and 8 feet in diameter? 1406.7cu. ft.

Part Four

1. John has saved \$500 and wants to buy a car. He found a suitable car that he could buy for \$500 down and \$30 a month for 24 months or for \$1100 cash. The amount that the total installment price is greater than the cash price is called a *carrying charge.* How much was the carrying charge? \$ 120

2. John bought this car on installments and kept it for three years. He then sold it for \$150. How much did the car drop in value per year, or *depreciate,* from the cash price? \$ 316.67

3. Suppose, instead of paying out \$500 as a down payment for a car, John invested this money at simple interest of 4% for three years. What would his money have earned during that period? \$ 60

4. Doing most of the work himself, John was able to keep his car in running order at an average of \$8.00 per month for auto parts. What was his total cost for the three years that he kept the car? \$ 288

5. John drove an average of 600 miles per month. His car averaged 15 miles per gallon. The average cost of gasoline was 32¢ per gallon. What was John's gasoline bill per month? for three years? \$ 12.80; \$ 460.80

6. If his car was lubricated and the oil changed every 1800 miles, how many times did John have this done during three years? At an average cost of \$3.50, what did it cost him to lubricate his car and change the oil for three years? 12 ; \$ 42

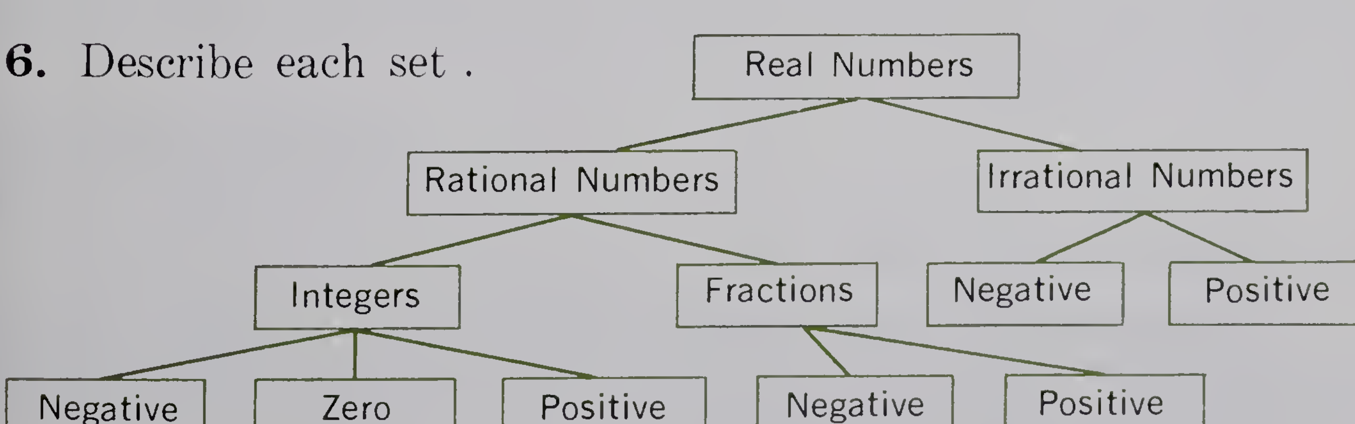
Part Five

Solve each of the following problems for x and y.

1. $2x + y = 9$; $3x - y = 11$ 4, 1
2. $4x - 2y = 6$; $2x + y = 5$ 2, 1
3. $6x + 2y = 9$; $3x + 2y = 3$ 2, $^-1.5$
4. $7x - 2y = 4$; $3x - 2y = 12$ $^-2$, $^-9$
5. $2x - 3y = 9$; $2x + 4y = {}^-5$ 1.5, $^-2$
6. $5x + 3y = 18$; $2x - y = 1\frac{1}{3}$ 2, $2\frac{2}{3}$
7. $3x - 4y = 15$; $x + 2y = 10$ 7, $1\frac{1}{2}$
8. $7x + 4y = 11$; $2x - y = 6$ $2\frac{1}{3}$, $^-1\frac{1}{3}$
9. $9x - 3y = 5$; $3x - 5y = 3$ $\frac{4}{9}$, $\frac{-1}{3}$

Part Six

1. John can do a task in 5 days alone while his brother would take 10 days alone. How long would it take to do the task working together? $3\frac{1}{3}$ days
2. Two planes leave the same airport at the same time and one is traveling east and the other west. If the plane traveling west is 60 m.p.h. faster than the other, what are their speeds if they are 2300 miles apart at the end of 2 hours? 545 m.p.h., 605 m.p.h.
3. A druggist has 72 ounces of a 3% medicine and wishes to reduce this to a 2% solution. How much water should be added? 36 oz.
4. A store mixed 200 pounds of candy to sell for 96 cents per pound. If candy worth 80 cents per pound were mixed with candy worth \$1.20 per pound, how much of each was used? \$1.20, 80 lb.; \$.80, 120 lb.
5. The sum of two numbers is 368 while the difference of the same two numbers is 124. Find each number. 246; 122
6. Describe each set.



See Glossary.

EXTRA PRACTICE

When you are having trouble with computation, you need the right kind of practice in order to correct your weaknesses. In mathematics as in basketball, tennis, or any activity requiring skill, practice is necessary to improve. These Practice Exercises will help you.

After taking an Inventory Test, identify the cause of your difficulty. Work on those Practice Exercises which are keyed to that test. If you had difficulty with an Inventory Test in Chapter 1, for example, the page reference at the end of that test guides you to the proper Practice Exercise.

Remember that you must practice the right procedures to correct your difficulty. The Examples or suggested review included with the Practice Exercises show you how the operations should be performed. Work carefully to develop proper skills as well as to get the right answers. You will develop speed after you have developed accuracy.

Do not waste time copying any exercises except when it is necessary. In addition, subtraction, and multiplication of whole numbers and numbers named by decimals, you can usually place a sheet of paper below the exercise, and write only the answer. After you have done the first row, you can fold the paper and do the second row, and so on.

Each set of Practice Exercises is followed by a Practice Test that will help you find out if you have mastered the computations. When you have completed work on the Practice Exercises, your teacher will give you directions for taking the Practice Test. When this test has been scored, examine those exercises you missed to determine the causes of your errors. If necessary, ask the teacher for help.

Remember! Practice leads to improvement only if you are learning correct procedures. Usually the correct procedures you need to learn are those illustrated in the EXAMPLES. You can then readily learn by following them. Your difficulty, however, may not be due to incorrect procedures, but to failure to check your answers, to write legible figures, or to carelessness in arranging your work. Time spent in arranging your work neatly, writing the numerals legibly, and in checking your work can pay big dividends.

PRACTICE EXERCISES

A. *Addition of whole numbers*

In practicing, try to develop a steady rate of adding. You can increase your rate with practice. Try to detect any errors in using number combinations especially using 9's.

EXAMPLE

23	20 + 3
39	30 + 9
60	60 + 0
59	50 + 9
12	10 + 2
193	170 + 23

Keep the columns straight. How many tens are carried to the tens column? 2 How many hundreds are carried to the hundreds column? 1

1.	2.	3.	4.
249	548	307	819
356	944	548	178
216	695	757	750
395	586	189	209
435	259	487	582
938	537	296	659
2589	3569	2584	3197

5.	6.	7.	8.
\$6.72	\$19.50	\$14.36	\$26.55
4.95	74.56	27.80	50.73
7.73	58.28	13.03	10.24
2.54	28.73	50.48	88.47
8.49	32.54	89.75	69.21
5.93	68.37	37.39	10.16
\$36.36	\$281.98	\$232.81	\$255.36

B. *Subtraction*

Think of 85 as 8 tens and 5 ones. Think of 56 as 5 tens and 6 ones. Since you cannot subtract 6 from 5, you may think of 85 as 7 tens and 15 ones. (In other words, "borrow" ten from the 8 tens, leaving 7 tens.) Then subtract 6 from 15, and the remainder is 9. Subtract 5 from 7 and the remainder is 2.

EXAMPLE

85	=	70 + 15
−56	=	50 + 6
29	=	20 + 9

1.	2.	3.	4.	5.	6.
12	22	51	27	52	74
6	13	13	18	33	45
6	9	38	9	19	29

7.	8.	9.	10.	11.	12.
35	78	32	85	93	38
27	39	23	48	79	29
8	39	9	37	14	9

13.	14.	15.	16.	17.	18.
62	73	87	91	56	42
48	66	58	19	37	35
14	7	29	72	19	7

19.	20.	21.	22.	23.	24.
37	51	68	83	61	43
18	13	59	38	25	18
19	38	9	45	36	25

C. *Multiplication*

Rule: The right-hand figure of each partial product should be placed in the same column as the digit being used as a multiplier, and directly below it.

First, write the product of 5 × 389, or 1945. Then on the next line, but in the same column as the 0 in the multiplier, write 0 to show 0 × 389. Then write the product of 2 × 389, or 778, so that the 8 is in the same column as 2, the multiplier. As you see, the 778 is written to the left of the 0. Then add the partial products to get 79745.

EXAMPLE

$$\begin{array}{r} 389 \\ \underline{205} \\ 1945 \\ \underline{7780} \\ 79745 \end{array}$$

Multiply:

1. 973	**3.** 901	**5.** 603	**7.** 123	**9.** 579	**11.** 613
201 *195,573*	302 *272,102*	270 *162,810*	400 *49,200*	802 *464,358*	400 *245,200*
2. 601	**4.** 473	**6.** 701	**8.** 805	**10.** 475	**12.** 331
79 *47,479*	206 *97,438*	302 *211,702*	240 *193,200*	903 *428,925*	802 *265,462*

D. *Division*

Divide the first one or two digits in the dividend by the first digit of the divisor, rounded off. This gives you a trial quotient. Test it.

EXAMPLE

$$\begin{array}{r} 235 \\ 41\overline{)9635} \\ \underline{82} \\ 143 \\ \underline{123} \\ 205 \\ \underline{205} \end{array}$$

In 9 there are 2 fours, so write 2 as the first number of the quotient. Notice that 2 is placed directly above the 6 in 96. Subtract 82 from 96 and the remainder is 14. Then bring down the 3, the next figure in the dividend. You now have a new dividend, 143. Now, think "41 goes into 143." Finish the explanation.

1. 3515 ÷ 19 *185*	**6.** 2556 ÷ 71 *36*	**11.** 83,424 ÷ 48 *1738*
2. 9875 ÷ 25 *395*	**7.** 10,795 ÷ 17 *635*	**12.** 141,185 ÷ 55 *2567*
3. 2888 ÷ 38 *76*	**8.** 23,595 ÷ 33 *715*	**13.** 117,425 ÷ 77 *1525*
4. 10,050 ÷ 25 *402*	**9.** 22,736 ÷ 28 *812*	**14.** 126,665 ÷ 49 *2585*
5. 27,360 ÷ 608 *45*	**10.** 81,545 ÷ 47 *1735*	**15.** 116,643 ÷ 59 *1977*

PRACTICE TEST

A. *Add:*

1. 38	**2.** 940	**3.** 305	**4.** 300	**5.** $3.07	**6.** $15.07
27	428	74	912	.03	.38
54	358	680	56	.37	2.19
93	870	15	48	4.23	10.27
43	646	502	207	6.10	50.20
60	302	190	323	2.50	.80
315	*3544*	*1766*	15	.72	.79
			1861	*$17.02*	*$79.70*

B. *Subtract:*

1. 14505	**2.** 35031	**3.** 61900	**4.** 50684	**5.** 40360
12283 *2222*	25030 *10001*	30480 *31420*	25047 *25637*	30366 *9994*

6. $1850.75	**7.** $1050.20	**8.** $2007.35	**9.** $4050.75	**10.** $7000.00
943.65	89.19	430.17	1602.38	115.43
$907.10	*$961.01*	*$1577.18*	*$2448.37*	*$6884.57*

C. *Multiply:*

1. $27.30	**2.** $90.50	**3.** $48.85	**4.** $39.47
120	215	307	573
$3276.00	*$19,457.50*	*$14,996.95*	*$22,616.31*

D. *Divide:*

.65	*.96*	*.89*	*.93*
1. 71)$46.15	**2.** 38)$36.48	**3.** 56)$49.84	**4.** 703)$653.79

PRACTICE EXERCISES

A. *Addition of whole numbers* (Review page 469)

1. 36	**2.** 88	**3.** 50	**4.** 35	**5.** 87	**6.** 38
71	50	38	74	45	75
54	71	79	34	84	47
39	48	84	16	92	66
20	10	40	31	47	38
220	*267*	*291*	*190*	*355*	*264*

B. *Subtraction* (Review page 469)

1. 25	**2.** 33	**3.** 52	**4.** 44	**5.** 72
18 *7*	19 *14*	43 *9*	29 *15*	55 *17*
6. 61	**7.** 46	**8.** 74	**9.** 36	**10.** 78
45 *16*	28 *18*	66 *8*	17 *19*	39 *39*

C. *Multiplication* (Review page 470)

1. 241	**4.** $410.85	**7.** $3110.75	**10.** $90.80
40	87	413	150
9640	$ 35,743.95	$ 1,284,739.95	$ 13,620.00
2. 342	**5.** $150.05	**8.** 312	**11.** $210.30
30	206	50	58
10,260	$ 30,910.30	15,600	$ 12,197.40
3. $81.90	**6.** $106.06	**9.** 212	**12.** $186.00
305	310	50	109
$ 24,979.50	32,878.60	10,600	$ 20,274.00

D. *Division* (Review page 470)

Find the quotient. If there is a remainder, express it as a fraction in lowest terms.

1. 2257 ÷ 60 $37\frac{37}{60}$	**5.** 4514 ÷ 44 $102\frac{13}{22}$	**9.** 2944 ÷ 72 $40\frac{8}{9}$
2. 9848 ÷ 48 $205\frac{1}{6}$	**6.** 10,944 ÷ 252 $43\frac{3}{9}$	**10.** 5544 ÷ 66 84
3. 1632 ÷ 54 $30\frac{2}{9}$	**7.** 8652 ÷ 36 $240\frac{1}{3}$	**11.** 22,048 ÷ 240 $91\frac{13}{15}$
4. 8275 ÷ 95 $87\frac{2}{19}$	**8.** 1104 ÷ 42 $26\frac{2}{7}$	**12.** 22,376 ÷ 124 $180\frac{14}{31}$

PRACTICE TEST

A. *Add:*

1. 46	**2.** 511	**3.** 626	**4.** 333	**5.** $307.21	**6.** $19.99
72	824	47	219	1.51	.33
45	853	186	22	1027.33	11.01
39	78	51	3	43.37	20.50
34	464	205	109	206.17	.03
77	205	7	777	1.57	107.08
313	2935	1122	1463	$ 1587.16	$ 158.94

B. *Subtract:*

1. 15404	**2.** 10353	**3.** 90,601	**4.** 48,605	**5.** 60,304
9600	7836	89,701	38,307	55,555
5804	2517	900	10,298	4,749
6. $7350.75	**7.** $2060.33	**8.** $3005.60	**9.** $19635.22	**10.** $9950.50
1951.76	991.49	2995.61	18953.33	8735.85
$ 5398.99	$ 1068.84	$ 9.99	$ 681.89	$ 1214.65

C. *Multiply:*

1. $26.56	**2.** $75.75	**3.** $56.65	**4.** $53.27
160	318	605	822
$ 4249.60	$ 24,088.50	$ 34,273.25	$ 43,787.94

D. *Divide:*

$ .83	$ 1.67	$ 2.55	$ 1.45
1. 69)$57.27	**2.** 43)$71.81	**3.** 83)$211.65	**4.** 519)$752.55

A. *Finding an equivalent fraction*

EXAMPLE

$\frac{3}{16} = \frac{12}{n}$ $\quad 3 \times n = 16 \times 12$

Then: $3n = 192$

$n = 64$

$\frac{3}{16} = \frac{12}{64}$

To find the unknown term of a fraction equivalent to a given fraction, multiply each numerator by the denominator of the other fraction. Then solve the equation.

Find the variable in the equivalent fraction:

1. $\frac{3}{8} = \frac{n}{24}$ *9*

2. $\frac{5}{9} = \frac{25}{n}$ *45*

3. $\frac{2}{3} = \frac{14}{n}$ *21*

4. $\frac{7}{8} = \frac{n}{32}$ *28*

5. $\frac{3}{4} = \frac{12}{n}$ *16*

6. $\frac{3}{16} = \frac{n}{48}$ *9*

7. $\frac{7}{15} = \frac{n}{60}$ *28*

8. $\frac{9}{25} = \frac{36}{n}$ *100*

9. $\frac{4}{5} = \frac{n}{25}$ *20*

10. $\frac{5}{16} = \frac{n}{48}$ *15*

11. $\frac{4}{7} = \frac{20}{n}$ *35*

12. $\frac{7}{12} = \frac{n}{60}$ *35*

13. $\frac{4}{9} = \frac{20}{n}$ *45*

14. $\frac{5}{6} = \frac{n}{36}$ *30*

15. $\frac{2}{7} = \frac{14}{n}$ *49*

16. $\frac{7}{9} = \frac{35}{n}$ *45*

B. *Finding a common denominator*

First check to see if the largest denominator is the common denominator. If not, multiply it by 2, then by 3, by 4, etc. until it is divisible by each of the other denominators.

EXAMPLE

Find the common denominator: $\frac{3}{8}, \frac{2}{3}, \frac{1}{6}, \frac{1}{2}$.
8 is not the common denominator. Is $2 \cdot 8$? No. Is $3 \cdot 8$? Yes.

Find the common denominator for each group.

1. $\frac{2}{3}, \frac{1}{6}, \frac{5}{12}$ *12*

2. $\frac{3}{8}, \frac{5}{6}, \frac{17}{24}$ *24*

3. $\frac{7}{9}, \frac{2}{3}, \frac{1}{2}$ *18*

4. $\frac{4}{5}, \frac{7}{15}, \frac{3}{10}$ *30*

5. $\frac{4}{7}, \frac{1}{2}, \frac{3}{4}$ *28*

6. $\frac{3}{5}, \frac{5}{6}, \frac{3}{10}$ *30*

7. $\frac{3}{4}, \frac{2}{3}, \frac{1}{2}$ *12*

8. $\frac{7}{12}, \frac{1}{2}, \frac{5}{6}, \frac{2}{3}$ *12*

9. $\frac{3}{4}, \frac{5}{8}, \frac{7}{16}$ *16*

10. $\frac{1}{9}, \frac{2}{3}, \frac{3}{4}$ *36*

11. $\frac{4}{15}, \frac{3}{10}, \frac{2}{5}$ *30*

12. $\frac{5}{8}, \frac{3}{16}, \frac{3}{4}, \frac{2}{3}$ *48*

C. *Addition of fractions*

EXAMPLE

$$\text{Add}\quad \begin{aligned} 1\tfrac{4}{5} &= 1\tfrac{24}{30}\\ 3\tfrac{5}{6} &= 3\tfrac{25}{30}\\ 4\tfrac{2}{3} &= 4\tfrac{20}{30}\\ \hline & \, 8\tfrac{69}{30} = 10\tfrac{9}{30} = 10\tfrac{3}{10} \end{aligned}$$

If the fractions are not like-fractions, first write them as like-fractions using the lowest common denominator. Then write the sum of the numerators over the common denominator. Reduce the sum to simplest form.

1. $\frac{3}{8} + \frac{2}{3} + \frac{5}{6}$ $\quad 1\frac{7}{8}$
2. $\frac{3}{4} + \frac{7}{8} + \frac{1}{2}$ $\quad 2\frac{1}{8}$
3. $\frac{1}{6} + \frac{2}{3} + \frac{3}{4}$ $\quad 1\frac{7}{12}$
4. $\frac{9}{16} + \frac{1}{2} + \frac{1}{4}$ $\quad 1\frac{5}{16}$
5. $\frac{7}{9} + \frac{5}{12} + \frac{3}{4}$ $\quad 1\frac{17}{18}$
6. $\frac{15}{16} + \frac{3}{4} + \frac{1}{2}$ $\quad 2\frac{3}{16}$
7. $6\frac{3}{8} + 9\frac{1}{3} + 7\frac{11}{18}$ $\quad 23\frac{23}{72}$
8. $14\frac{2}{3} + 6\frac{7}{8} + 9\frac{5}{16}$ $\quad 30\frac{41}{48}$
9. $8\frac{1}{2} + 5\frac{5}{16} + \frac{7}{12}$ $\quad 14\frac{19}{48}$

D. *Subtraction of fractions*

EXAMPLE

Subtract: $4\frac{5}{8} - 2\frac{5}{6}$

$$\begin{aligned} 4\tfrac{5}{8} &= 4\tfrac{15}{24} = 3\tfrac{39}{24}\\ 2\tfrac{5}{6} &= 2\tfrac{20}{24} = 2\tfrac{20}{24}\\ \hline & \phantom{= 2\tfrac{20}{24} =}\ 1\tfrac{19}{24} \end{aligned}$$

Study the example; explain why 24 is the common denominator. Explain the subtraction.

1. $\frac{3}{4} - \frac{1}{2}$ $\quad \frac{1}{4}$
2. $\frac{11}{16} - \frac{3}{24}$ $\quad \frac{9}{16}$
3. $13\frac{1}{3} - 7\frac{1}{6}$ $\quad 6\frac{1}{6}$
4. $7\frac{1}{2} - 5\frac{5}{6}$ $\quad 1\frac{2}{3}$
5. $6\frac{5}{16} - 3\frac{7}{8}$ $\quad 2\frac{7}{16}$
6. $2\frac{5}{8} - 1\frac{15}{16}$ $\quad \frac{11}{16}$
7. $3\frac{1}{4} - 1\frac{7}{8}$ $\quad 1\frac{3}{8}$
8. $6\frac{5}{8} - 4\frac{3}{4}$ $\quad 1\frac{7}{8}$
9. $14\frac{1}{6} - 11\frac{1}{3}$ $\quad 2\frac{5}{16}$
10. $16\frac{5}{16} - 15\frac{3}{8}$ $\quad \frac{15}{16}$
11. $7\frac{11}{12} - 5\frac{2}{3}$ $\quad 2\frac{1}{4}$
12. $8\frac{1}{5} - 6\frac{7}{10}$ $\quad 1\frac{1}{2}$
13. $8\frac{3}{5} - 7\frac{11}{16}$ $\quad \frac{73}{80}$
14. $4\frac{5}{8} - 3\frac{3}{4}$ $\quad \frac{7}{8}$
15. $8\frac{3}{4} - 6\frac{2}{3}$ $\quad 2\frac{1}{12}$
16. $12\frac{11}{12} - 11\frac{3}{4}$ $\quad 1\frac{1}{6}$
17. $4\frac{5}{16} - 3\frac{3}{4}$ $\quad \frac{9}{16}$
18. $14\frac{3}{8} - 13\frac{1}{2}$ $\quad \frac{7}{8}$
19. $12\frac{3}{4} - 10\frac{5}{6}$ $\quad 1\frac{11}{12}$
20. $6\frac{7}{10} - 3\frac{3}{5}$ $\quad 3\frac{1}{10}$
21. $13\frac{1}{2} - 2\frac{5}{12}$ $\quad 11\frac{1}{12}$

E. *Fractional equations*

Solve for x and check:

1. $1\frac{5}{8} + x = 3\frac{1}{2}$ $1\frac{7}{8}$
2. $x - 2\frac{2}{3} = 5\frac{5}{12}$ $8\frac{1}{12}$
3. $8\frac{3}{5} - x = 6$ $2\frac{3}{5}$
4. $x + 7\frac{1}{2} = 12\frac{15}{16}$ $5\frac{7}{16}$
5. $2\frac{3}{4} + x = 7\frac{1}{5}$ $4\frac{9}{20}$
6. $9\frac{5}{12} - x = 2\frac{1}{2}$ $6\frac{11}{12}$
7. $x + 5\frac{2}{3} = 8\frac{5}{9}$ $2\frac{8}{9}$
8. $3\frac{2}{3} + x = 8\frac{1}{2}$ $4\frac{5}{6}$
9. $x - 3\frac{5}{8} = 2\frac{2}{3}$ $6\frac{7}{24}$
10. $5\frac{1}{4} + x = 8\frac{1}{8}$ $2\frac{7}{8}$
11. $16\frac{5}{6} - x = 4\frac{2}{3}$ $12\frac{1}{6}$
12. $5\frac{13}{16} + x = 9$ $3\frac{3}{16}$
13. $6\frac{1}{2} + x = 13$ $6\frac{1}{2}$
14. $x + 9\frac{13}{16} = 16$ $6\frac{3}{16}$
15. $x - 1\frac{5}{9} = 3\frac{1}{3}$ $4\frac{8}{9}$
16. $18\frac{1}{2} - x = 6\frac{2}{3}$ $11\frac{5}{6}$

PRACTICE TEST

A. Find the missing term in the equivalent fraction:

1. $\frac{5}{16} = \frac{n}{48}$ 15
2. $\frac{3}{8} = \frac{9}{n}$ 24
3. $\frac{9}{25} = \frac{36}{n}$ 100
4. $\frac{2}{7} = \frac{n}{35}$ 10
5. $\frac{7}{15} = \frac{n}{75}$ 35
6. $\frac{7}{9} = \frac{21}{n}$ 27

B. Find the common denominator for each set of fractions:

1. $\frac{4}{5}, \frac{7}{15}, \frac{1}{45}$ 45
2. $\frac{8}{15}, \frac{2}{3}, \frac{3}{5}$ 15
3. $\frac{5}{12}, \frac{3}{8}, \frac{5}{6}$ 24
4. $\frac{2}{9}, \frac{1}{3}, \frac{7}{12}$ 36
5. $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}$ 8
6. $\frac{7}{10}, \frac{11}{15}, \frac{4}{5}$ 30

C. *Add:* Write all answers in simplest form.

1. $\frac{1}{2} + \frac{3}{4} + \frac{7}{8}$ $2\frac{1}{8}$
2. $\frac{5}{16} + \frac{7}{8} + \frac{1}{4}$ $1\frac{7}{16}$
3. $9\frac{2}{3} + 4\frac{5}{12} + 6\frac{1}{6}$ $20\frac{1}{4}$
4. $4\frac{1}{3} + 5\frac{7}{12} + 8\frac{3}{4}$ $18\frac{2}{3}$
5. $8\frac{1}{2} + 6\frac{3}{5} + 7\frac{9}{10}$ 23
6. $5\frac{1}{2} + 2\frac{3}{8} + 9\frac{3}{4}$ $17\frac{5}{8}$

D. *Subtract:* Write all answers in simplest form.

1. $7\frac{1}{2} - 2\frac{3}{4}$ $4\frac{3}{4}$
2. $5\frac{3}{4} - 2\frac{7}{8}$ $2\frac{7}{8}$
3. $13\frac{1}{8} - 11\frac{2}{3}$ $1\frac{11}{24}$
4. $8\frac{5}{6} - 4\frac{2}{3}$ $4\frac{1}{6}$
5. $9\frac{3}{5} - 6\frac{7}{10}$ $2\frac{9}{10}$
6. $8\frac{3}{4} - 4\frac{2}{3}$ $4\frac{1}{12}$

E. Solve for x, and check:

1. $x + 11\frac{1}{2} = 18\frac{1}{3}$ $6\frac{5}{6}$
2. $18 - x = 3\frac{1}{3}$ $14\frac{2}{3}$
3. $x - 15\frac{3}{5} = 17\frac{1}{10}$ $32\frac{7}{10}$
4. $x - 7\frac{5}{8} = 3\frac{1}{4}$ $10\frac{7}{8}$
5. $15 + x = 20\frac{1}{7}$ $5\frac{1}{7}$
6. $17\frac{3}{5} - x = 5\frac{2}{5}$ $12\frac{1}{5}$

A. *Multiplying fractional numbers*

Express mixed numbers and whole numbers as fractions. Write the product of the numerators over the product of the denominators. Write all answers in simplest form.

Multiply:

1. $4\frac{2}{5} \cdot 3\frac{3}{4}$ $16\frac{1}{2}$
2. $7\frac{1}{2} \cdot 5\frac{3}{5}$ 42
3. $4\frac{1}{5} \cdot 7\frac{1}{2}$ $31\frac{1}{2}$
4. $3\frac{3}{5} \cdot 3\frac{1}{3}$ 12
5. $16\frac{2}{3} \cdot 7\frac{1}{7}$ $119\frac{1}{4}$
6. $6 \cdot 4\frac{2}{3}$ 28
7. $12 \cdot 6\frac{3}{4}$ 81
8. $9\frac{5}{8} \cdot 32$ 308
9. $36 \cdot 14\frac{3}{16}$ $510\frac{3}{4}$
10. $3\frac{3}{5} \cdot 25$ 90
11. $18\frac{1}{3} \cdot 16\frac{2}{5}$ $300\frac{2}{3}$
12. $14\frac{2}{7} \cdot 6\frac{1}{4}$ $89\frac{2}{7}$
13. $11\frac{3}{10} \cdot 11\frac{1}{9}$
14. $20\frac{2}{3} \cdot 11\frac{2}{7}$ $233\frac{5}{21}$
15. $17\frac{1}{2} \cdot 4\frac{3}{5}$ $80\frac{1}{2}$
16. $18 \cdot 3\frac{3}{4}$ $67\frac{1}{2}$
17. $6\frac{3}{4} \cdot 36$ 243
18. $3\frac{3}{16} \cdot 18$ $57\frac{3}{8}$
19. $16 \cdot 19\frac{3}{8}$ 310
20. $6\frac{4}{7} \cdot 56$ 368

(ex. 13) $125\frac{5}{9}$

B. *Dividing fractional numbers*

EXAMPLE

Divide: $14\frac{2}{7} \div 6\frac{1}{4}$

$$\frac{100}{7} \div \frac{25}{4} = \frac{100}{7} \cdot \frac{4}{25} = \frac{16}{7} = 2\frac{2}{7}$$

Express mixed numbers and whole numbers in the form $\frac{a}{b}$. Multiply the dividend by the multiplicative inverse of the divisor. Write all answers in simplest form.

Divide:

1. $\frac{4}{5} \div \frac{5}{6}$ $\frac{24}{25}$
2. $\frac{2}{5} \div \frac{14}{15}$ $\frac{3}{7}$
3. $\frac{11}{24} \div \frac{7}{12}$ $\frac{11}{24}$
4. $\frac{2}{5} \div \frac{4}{15}$ $1\frac{1}{2}$
5. $\frac{8}{15} \div \frac{6}{7}$ $\frac{28}{45}$
6. $\frac{7}{8} \div \frac{3}{4}$ $1\frac{1}{6}$
7. $\frac{3}{4} \div \frac{15}{16}$ $\frac{4}{5}$
8. $\frac{2}{5} \div \frac{5}{9}$ $\frac{18}{25}$
9. $\frac{3}{4} \div \frac{5}{8}$ $1\frac{1}{5}$
10. $\frac{14}{15} \div \frac{2}{5}$ $2\frac{1}{3}$
11. $\frac{2}{3} \div 5$ $\frac{2}{15}$
12. $\frac{3}{8} \div 6$ $\frac{1}{16}$
13. $4\frac{4}{9} \div 8$ $\frac{5}{9}$
14. $3\frac{3}{5} \div 9$ $\frac{2}{5}$
15. $20\frac{5}{8} \div 15$ $1\frac{3}{8}$
16. $\frac{3}{5} \div 12$ $\frac{1}{20}$
17. $\frac{5}{9} \div 15$ $\frac{1}{27}$
18. $1\frac{13}{18} \div 10$ $\frac{31}{180}$
19. $4\frac{3}{8} \div 12$ $\frac{35}{96}$
20. $12\frac{3}{4} \div 17$ $\frac{3}{4}$

C. *Factor-factor product relationship*

In solving equations dealing with the factor-factor-product relationship, first identify the terms in the equations as factors and product. Then write an equivalent equation that has the variable alone on one side. Solve. Check your solution in the original equation.

EXAMPLE

Solve: $\frac{15x}{7} = 60$. Note that $\frac{15x}{7}$ is equivalent to $\frac{15}{7} \cdot x$

$$\frac{15x}{7} = 60. \qquad f_1 \cdot f_2 = p$$

$$x = 60 \div \frac{15}{7} = 60 \cdot \frac{7}{15} \qquad f_2 = p \div f_1$$

$$x = 28$$

Check: $\frac{15}{7} \times 28 = 60,\ 60 = 60$

1. $\frac{8x}{5} = 72$ 45
2. $6\frac{1}{4} \div x = \frac{1}{4}$ 25
3. $x \div 3\frac{1}{3} = 10$ $33\frac{1}{3}$
4. $\frac{17x}{4} = 68$ 16
5. $27\frac{1}{2} \div x = 55$ $\frac{1}{2}$
6. $x \div 4\frac{1}{2} = 3\frac{1}{3}$ 15
7. $\frac{7x}{3} = 28$ 12
8. $18\frac{1}{3} \div x = 11$ $1\frac{2}{3}$
9. $x \div 16\frac{2}{3} = 6$ 100
10. $\frac{5x}{9} = 45$ 81
11. $19\frac{1}{2} \div x = 9\frac{3}{4}$ 2
12. $x \div 6\frac{2}{3} = 7\frac{1}{2}$ 50

PRACTICE TEST

A. *Multiply:* Write all answers in simplest form.

1. $2\frac{2}{5} \cdot \frac{3}{4}$ $1\frac{4}{5}$
2. $16 \cdot 4\frac{3}{4}$ 76
3. $5\frac{2}{3} \cdot 4\frac{2}{5}$ $24\frac{14}{15}$
4. $2\frac{5}{9} \cdot 18$ 46
5. $14\frac{2}{7} \cdot \frac{9}{10}$ $12\frac{6}{7}$
6. $12 \cdot 3\frac{1}{3}$ 40

B. *Divide:* Write all answers in simplest form.

1. $\frac{3}{8} \div \frac{5}{16}$ $1\frac{1}{5}$
2. $\frac{14}{15} \div \frac{7}{12}$ $1\frac{3}{5}$
3. $\frac{7}{9} \div 2\frac{1}{3}$ $\frac{1}{3}$
4. $\frac{5}{18} \div \frac{1}{6}$ $1\frac{2}{3}$
5. $\frac{15}{16} \div 6$ $\frac{5}{32}$
6. $\frac{18}{19} \div 9$ $\frac{2}{19}$

C. Solve and check. Write all answers in simplest form.

1. $\frac{7x}{3} = 35$ 15
2. $x \div 6\frac{2}{3} = 15$ 100
3. $100 \div x = 8\frac{1}{3}$ 12
4. $14\frac{2}{7} \div x = 10$ $1\frac{3}{7}$
5. $\frac{5x}{6} = 25$ 30
6. $x \div 12\frac{1}{2} = 16$ 200

A. *Add:* Write all answers in simplest form.

1. $\begin{array}{r} 7\frac{1}{3} \\ \underline{5\frac{2}{3}} \\ 13 \end{array}$	**3.** $\begin{array}{r} 2\frac{1}{2} \\ \underline{3\frac{1}{3}} \end{array}$ $5\frac{5}{6}$	**5.** $\begin{array}{r} 4\frac{1}{5} \\ \underline{8\frac{2}{3}} \end{array}$ $12\frac{13}{15}$	**7.** $\begin{array}{r} 2\frac{1}{2} \\ \underline{5\frac{1}{5}} \end{array}$ $7\frac{7}{10}$	**9.** $\begin{array}{r} 8\frac{1}{4} \\ \underline{6\frac{1}{2}} \end{array}$ $14\frac{3}{4}$
2. $\begin{array}{r} 2\frac{1}{2} \\ 3\frac{3}{14} \\ 1\frac{2}{7} \\ \underline{2\frac{3}{7}} \\ 9\frac{3}{7} \end{array}$	**4.** $\begin{array}{r} 7\frac{3}{8} \\ 6\frac{1}{8} \\ \frac{7}{16} \\ \underline{1\frac{1}{2}} \\ 15\frac{7}{16} \end{array}$	**6.** $\begin{array}{r} 4\frac{1}{3} \\ 3\frac{1}{2} \\ 2 \\ \underline{4\frac{2}{3}} \\ 14\frac{1}{2} \end{array}$	**8.** $\begin{array}{r} 5\frac{1}{2} \\ 7 \\ 6\frac{3}{8} \\ \underline{3\frac{3}{4}} \\ 22\frac{5}{8} \end{array}$	**10.** $\begin{array}{r} 6\frac{3}{4} \\ 3\frac{1}{8} \\ 1\frac{1}{16} \\ \underline{1\frac{1}{2}} \\ 12\frac{7}{16} \end{array}$

B. *Subtract:* Write all answers in simplest form.

1. $\begin{array}{r} 5\frac{3}{10} \\ \underline{4\frac{2}{15}} \end{array}$ $1\frac{1}{6}$	**3.** $\begin{array}{r} 7\frac{2}{5} \\ \underline{3\frac{1}{4}} \end{array}$ $4\frac{3}{20}$	**5.** $\begin{array}{r} 5\frac{7}{8} \\ \underline{2\frac{1}{3}} \end{array}$ $3\frac{13}{24}$	**7.** $\begin{array}{r} 7\frac{1}{6} \\ \underline{3\frac{1}{9}} \end{array}$ $4\frac{1}{18}$	**9.** $\begin{array}{r} 6\frac{2}{5} \\ \underline{2\frac{3}{4}} \end{array}$ $3\frac{13}{20}$
2. $\begin{array}{r} 4\frac{2}{3}\ 1 \\ \underline{1\frac{5}{6}} \\ 2\frac{5}{6} \end{array}$	**4.** $\begin{array}{r} 13\frac{1}{6} \\ \underline{8\frac{5}{8}} \\ 4\frac{13}{24} \end{array}$	**6.** $\begin{array}{r} 5\frac{5}{6} \\ \underline{2\frac{2}{3}} \\ 3\frac{1}{6} \end{array}$	**8.** $\begin{array}{r} 33\frac{3}{8} \\ \underline{21\frac{3}{4}} \\ 11\frac{5}{8} \end{array}$	**10.** $\begin{array}{r} 7\frac{1}{10} \\ \underline{3\frac{1}{5}} \\ 3\frac{9}{10} \end{array}$

C. *Multiply:* Write all answers in simplest form.

1. $14 \cdot 6\frac{2}{3}$ $93\frac{1}{3}$	**4.** $11\frac{1}{2} \cdot 9\frac{3}{5}$ $110\frac{2}{5}$	**7.** $5\frac{1}{2} \cdot 14\frac{2}{3}$ $80\frac{2}{3}$
2. $4\frac{2}{5} \cdot \frac{3}{10}$ $1\frac{8}{25}$	**5.** $3\frac{11}{12} \cdot 12$ 47	**8.** $13\frac{5}{8} \cdot 12\frac{7}{10}$ $173\frac{3}{80}$
3. $7\frac{2}{5} \cdot 5\frac{1}{4}$ $38\frac{17}{20}$	**6.** $23\frac{1}{2} \cdot 2\frac{4}{5}$ $65\frac{4}{5}$	**9.** $3\frac{5}{6} \cdot 7\frac{5}{6}$ $30\frac{1}{36}$

D. *Divide:* Write all answers in simplest form.

1. $14 \div 1\frac{1}{2}$ $9\frac{1}{3}$	**3.** $4\frac{1}{2} \div 5\frac{4}{5}$ $\frac{45}{58}$	**5.** $1\frac{3}{4} \div 12$ $\frac{7}{48}$
2. $6 \div 10\frac{1}{2}$ $\frac{4}{7}$	**4.** $\frac{7}{8} \div 5\frac{5}{6}$ $\frac{3}{20}$	**6.** $12\frac{1}{6} \div 7\frac{1}{2}$ $1\frac{28}{45}$

E. *Addition of integers* (Review page 489)

1. $^{+}15 + (^{+}6)$ 21	**4.** $^{+}26 + (^{-}19)$ 7	**7.** $^{-}26 + (^{+}30)$ 4
2. $^{+}21 + (^{-}9)$ 12	**5.** $^{-}29 + (^{-}20)$ $^{-}49$	**8.** $^{+}22 + (^{+}25)$ 47
3. $^{-}27 + (^{-}18)$ $^{-}45$	**6.** $^{+}32 + (^{+}6)$ 38	**9.** $^{+}41 + (^{-}41)$ 0

F. *Subtraction of integers* (Review page 490)

1. $^{+}25 - (^{+}17)$ 8	**6.** $^{-}17 - (^{-}26)$ 9	**11.** $^{-}40 - (^{-}40)$ 0
2. $^{+}18 - (^{-}19)$ 37	**7.** $^{-}24 - (^{+}24)$ $^{-}48$	**12.** $^{+}16 - (^{-}16)$ 32
3. $^{-}19 - (^{-}25)$ 6	**8.** $^{+}26 - (^{+}20)$ 6	**13.** $^{-}27 - (^{+}17)$ $^{-}44$
4. $^{-}26 - (^{-}16)$ $^{-}10$	**9.** $^{-}18 - (^{-}20)$ 2	**14.** $^{-}34 - (^{-}25)$ $^{-}9$
5. $^{+}36 - (^{+}28)$ 8	**10.** $^{+}27 - (^{-}37)$ 64	**15.** $^{+}50 - (^{+}60)$ $^{-}10$

Write all answers in simplest form.

A. *Add:*

1. $\begin{array}{r} 19\frac{3}{4} \\ 7\frac{11}{12} \\ \frac{5}{8} \\ \underline{13\frac{5}{16}} \end{array}$ $41\frac{29}{48}$

2. $\begin{array}{r} 17\frac{7}{8} \\ 5\frac{7}{12} \\ 3\frac{2}{3} \\ \underline{23\frac{3}{4}} \end{array}$ $50\frac{7}{8}$

3. $\begin{array}{r} 11\frac{1}{6} \\ 9\frac{1}{2} \\ 7\frac{5}{18} \\ \underline{13\frac{5}{12}} \end{array}$ $41\frac{13}{36}$

4. $\begin{array}{r} \frac{3}{4} \\ 8\frac{1}{3} \\ 17\frac{2}{3} \\ \underline{31\frac{3}{16}} \end{array}$ $57\frac{15}{16}$

5. $\begin{array}{r} 13\frac{5}{12} \\ 9\frac{3}{5} \\ 13\frac{1}{4} \\ \underline{29\frac{2}{3}} \end{array}$ $65\frac{14}{15}$

6. $\begin{array}{r} 53\frac{1}{4} \\ 17\frac{1}{6} \\ 22\frac{3}{8} \\ 9\frac{1}{2} \\ 19\frac{2}{3} \\ \underline{57\frac{5}{6}} \end{array}$ $179\frac{19}{24}$

7. $\begin{array}{r} 78\frac{5}{6} \\ 19\frac{2}{3} \\ 34\frac{3}{4} \\ 17\frac{5}{8} \\ 39\frac{1}{2} \\ \underline{\frac{7}{8}} \end{array}$ $191\frac{1}{4}$

8. $\begin{array}{r} 34\frac{3}{5} \\ 35\frac{1}{2} \\ 17\frac{7}{10} \\ 9\frac{1}{3} \\ 38\frac{3}{4} \\ \underline{8\frac{9}{10}} \end{array}$ $144\frac{47}{60}$

9. $\begin{array}{r} 51\frac{2}{9} \\ 49\frac{7}{12} \\ 13\frac{2}{3} \\ 18\frac{5}{6} \\ 55\frac{5}{9} \\ \underline{\frac{2}{3}} \end{array}$ $189\frac{19}{36}$

10. $\begin{array}{r} 20\frac{3}{4} \\ 15\frac{1}{2} \\ 19\frac{5}{6} \\ 26\frac{1}{5} \\ 9\frac{1}{3} \\ \underline{1\frac{1}{4}} \end{array}$ $92\frac{13}{15}$

B. *Subtract:*

1. $\begin{array}{r} 37\frac{1}{6} \\ \underline{19\frac{1}{4}} \end{array}$ $17\frac{11}{12}$

2. $\begin{array}{r} 119\frac{2}{3} \\ \underline{91\frac{3}{4}} \end{array}$ $27\frac{11}{12}$

3. $\begin{array}{r} 305\frac{7}{12} \\ \underline{117\frac{5}{9}} \end{array}$ $188\frac{1}{36}$

4. $\begin{array}{r} 131\frac{1}{2} \\ \underline{95\frac{3}{5}} \end{array}$ $35\frac{9}{10}$

5. $\begin{array}{r} 115\frac{3}{4} \\ \underline{70\frac{5}{6}} \end{array}$ $44\frac{11}{12}$

6. $\begin{array}{r} 211\frac{3}{4} \\ \underline{89\frac{5}{12}} \end{array}$ $122\frac{1}{3}$

7. $\begin{array}{r} 108\frac{2}{3} \\ \underline{98\frac{5}{6}} \end{array}$ $9\frac{5}{6}$

8. $\begin{array}{r} 132\frac{4}{5} \\ \underline{103\frac{7}{10}} \end{array}$ $29\frac{1}{10}$

9. $\begin{array}{r} 116\frac{7}{8} \\ \underline{87\frac{3}{4}} \end{array}$ $29\frac{1}{8}$

10. $\begin{array}{r} 309\frac{7}{12} \\ \underline{245\frac{2}{15}} \end{array}$ $64\frac{9}{20}$

C. *Multiply:*

1. $15 \times \frac{2}{5}$ 6

2. $\frac{3}{8} \times 24$ 9

3. $\frac{15}{16} \times 88$ $82\frac{1}{2}$

4. $\frac{5}{8} \times \frac{4}{5}$ $\frac{1}{2}$

5. $4\frac{3}{16} \times 8\frac{4}{9}$ $35\frac{13}{36}$

6. $9\frac{4}{5} \times \frac{5}{7}$ 7

7. $4\frac{1}{2} \times 7\frac{2}{3}$ $34\frac{1}{2}$

8. $9\frac{5}{7} \times 3\frac{1}{2}$ 34

9. $6\frac{7}{8} \times 1\frac{3}{5}$ 11

D. *Divide:*

1. $3\frac{3}{5} \div 15$ $\frac{6}{25}$

2. $29\frac{5}{8} \div 79$ $\frac{3}{8}$

3. $12\frac{3}{4} \div 25$ $\frac{51}{100}$

4. $156\frac{1}{4} \div 25$ $6\frac{1}{4}$

5. $31\frac{3}{4} \div 65$ $\frac{127}{260}$

6. $\frac{31}{40} \div 1\frac{7}{16}$ $\frac{62}{115}$

7. $3\frac{9}{11} \div \frac{6}{7}$ $4\frac{5}{11}$

8. $\frac{14}{15} \div 1\frac{2}{5}$ $\frac{2}{3}$

9. $\frac{15}{16} \div 1\frac{1}{8}$ $\frac{5}{6}$

E. *Add:*

1. $^{+}15 + (^{+}6)$ 21

2. $^{+}18 + (^{-}19)$ $^{-}1$

3. $^{+}36 + (^{-}60)$ $^{-}24$

4. $^{-}27 + (^{+}17)$ $^{-}10$

5. $^{-}17 + (^{+}17)$ 0

6. $^{-}19 + (^{-}23)$ $^{-}42$

F. *Subtract:*

1. $^{+}15 - (^{+}35)$ $^{-}20$

2. $^{+}25 - (^{-}15)$ 40

3. $^{-}18 - (^{-}19)$ 1

4. $^{-}29 - (^{+}6)$ $^{-}35$

5. $^{+}27 - (^{+}27)$ 0

6. $^{-}15 - (^{-}25)$ 10

A. *Locating the decimal point in a product*

Rule: Add the number of places to the right of the decimal point in each of the factors. The sum represents the number of places to the right of the decimal point in the product.

EXAMPLE

$$\begin{array}{r} 28.7 \\ \underline{1.63} \\ 861 \\ 1722 \\ \underline{287} \\ 46.781 \end{array}$$

Find (28.7) · (1.63)
There is 1 digit to the right of the decimal point in 28.7, and 2 digits to the right in 1.63. Then there should be 3 digits to the right of the decimal point in the product, 46.781.

Check: 28.7 is about 30; 1.63 is about 1.5.
(30) · (1.5) = 45, which is close enough to 46.781 to assure that the answer is *probably* right and is pointed off correctly.

Copy the products, and place the decimal point, annex zeros when needed. Check by estimation.

1. (315) · (2.5) = 7875 787.5
2. (31.5) · (0.25) = 7875 7.875
3. (3.15) · (.025) = 7875 .07875
4. (.0315) · (0.25) = 7875 .007875
5. (0.417) · (0.57) = 23769 0.23769
6. (41.7) · (0.57) = 23769 23.769
7. (4.17) · (.57) = 23769 2.3769
8. (4.17) · (5.7) = 23769 23.769

B. *Finding the product: decimals*

Rule: First find the product, as with whole numbers. Then locate the decimal point.

Find each product. Place the decimal point according to rule, and by estimation.

1. (20.3) · (.05) 1.015
2. (3.3) · (0.7) 2.31
3. (1.21) · (.08) .0968
4. (49) · (.009) 0.441
5. (2.4) · (.07) 0.168
6. (27.8) · (0.35) 9.73
7. (8.06) · (0.725) 5.8435
8. (60.4) · (0.235) 14.194
9. (80.2) · (0.705) 56.541
10. (3.12) · (0.35) 1.092
11. (1.82) · (20) 36.4
12. (41) · (.09) 3.69
13. (0.77) · (5.1) 3.927
14. (0.31) · (.006) .00186
15. (9.1) · (.02) 0.182

C. *Locating the decimal point in division*

To find the location of the decimal point by estimation, first determine the size of the quotient in powers of 10.

EXAMPLE

$$\begin{array}{r} 87.3 \\ .032_\wedge\overline{)2.793_\wedge 6} \end{array}$$

2793.6 is about 2700; 32 is about 30. $2700 \div 30 = 90$. Then $100 > q > 10$; that is, the quotient is between 10 and 100, so the decimal point belongs after 7 in the quotient.

Use this method for locating the decimal point in each of these exercises. Copy the quotient and locate the decimal in the proper place.

1. $3.56 \div 4 = 89$ 0.89
2. $16.48 \div 4 = 412$ 4.12
3. $2.679 \div 1.9 = 141$ 1.41
4. $0.885 \div 15.0 = 59$ .059
5. $64.0 \div .08 = 8$ 800
6. $25.2 \div 0.3 = 84$ 84.0
7. $74.4 \div 0.8 = 93$ 93.0
8. $6.27 \div .03 = 209$ 209.0
9. $26.88 \div 1.2 = 224$ 22.4
10. $40.32 \div 0.45 = 896$ 89.6

D. *Division and decimals*

When the divisor is a decimal, the first thing to do is to multiply both divisor and dividend by a power of 10 that will make the divisor a whole number. As shown in the example above, we indicate with a caret ($\wedge$) the new location of the decimal point, both in dividend and divisor.

Find the quotients. Check the location of the decimal point by estimation.

1. $56 \div 0.8$ 70
2. $0.63 \div 0.7$ 0.9
3. $0.63 \div 9$ .07
4. $0.45 \div 0.9$ 0.5
5. $.056 \div 0.8$ .07
6. $.042 \div .06$ 0.7
7. $4.032 \div 0.8$ 5.04
8. $28.14 \div 0.7$ 40.2
9. $42.0 \div 0.7$ 60
10. $162.0 \div .018$ 9000
11. $376.0 \div .08$ 4700
12. $246.0 \div 1.2$ 205
13. $49.28 \div .08$ 616
14. $4.44 \div .006$ 740
15. $0.927 \div .009$ 103
16. $5.94 \div 6$ 0.99
17. $44.1 \div 0.21$ 210
18. $356.0 \div 0.4$ 890
19. $97.2 \div .09$ 1080
20. $0.732 \div .06$ 12.2
21. $496.0 \div .008$ 62,000
22. $67.2 \div .07$ 960
23. $36.936 \div 8.55$ 4.32
24. $356.15 \div 41.9$ 8.5

If the division is to be carried to a specified number of places, and there is a remainder, the division should be carried to an additional place, and the quotient rounded to the specified place.

Find the quotients to the nearest thousandth:

1. $6.4 \div 0.36$ 17.778
2. $197 \div 28.2$ 6.986
3. $17.5 \div 36.6$ 0.478
4. $5.95 \div 3.7$ 1.608
5. $62.5 \div 8.17$ 7.650
6. $9.23 \div 8.75$ 1.055

Find the quotients to the nearest cent:

1. \$16.58 ÷ .09 \$ 184.22
2. \$36.73 ÷ 4.8 \$ 7.65
3. \$47.85 ÷ 0.31 \$ 154.35
4. \$27.90 ÷ 7.8 \$ 3.58
5. \$84.80 ÷ 5.02 \$ 16.89
6. \$40.92 ÷ 42.8 \$ .96
7. \$600.82 ÷ 3.71 \$ 161.95
8. \$56.85 ÷ .025 \$ 2, 274 .00

PRACTICE TEST

A. Copy each product, inserting the decimal point.

1. $(11.9) \cdot (59) = 7021$ 702.1
2. $(1.19) \cdot (0.59) = 7021$ 0.7021
3. $(1.19) \cdot (.059) = 7021$ .07021
4. $(3.85) \cdot (7.29) = 280665$ 28.0665
5. $(0.385) \cdot (0.729) = 280665$ .0280665
6. $(385) \cdot (.0729) = 280665$ 28.0665

B. Find the products.

1. $(7.6) \cdot (1.83)$ 13.908
2. $(45) \cdot (1.06)$ 47.7
3. $(1.29) \cdot (.055)$ .07095
4. $(3.75) \cdot (600)$ 2250
5. $(25.3) \cdot (4.51)$ 114.103
6. $(1.95) \cdot (0.87)$ 1.6965

C. Copy each quotient, inserting the decimal point.

1. $112.45 \div 6.5 = 173$ 17.3
2. $11.245 \div 0.65 = 173$ 17.3
3. $1.1245 \div 65 = 173$ .0173
4. $25.0125 \div 1.15 = 2175$ 21.75
5. $2.50125 \div .0115 = 2175$ 217.5
6. $250.125 \div 11.5 = 2175$ 21.75

D. Find the quotients.

1. $18.275 \div 0.17$ 107.5
2. $5.321 \div 0.17$ 31.3
3. $20.71 \div 21.8$ 0.95
4. $677.16 \div 13.2$ 51.3
5. $1.406 \div 7.6$ 0.185
6. $55.375 \div 44.3$ 1.25

A. *Addition of integers* (Review page 489)

1. $^{+}18 + (^{+}3)$ 21
2. $^{+}20 + (^{-}7)$ 13
3. $^{-}17 + (^{-}8)$ $^{-}25$
4. $^{+}36 + (^{-}9)$ 27
5. $^{-}19 + (^{-}10)$ $^{-}29$
6. $^{+}42 + (^{+}6)$ 48
7. $^{-}15 + (^{+}20)$ 5
8. $^{+}31 + (^{+}15)$ 46
9. $^{+}60 + (^{-}18)$ 42

B. *Subtraction of integers* (Review page 490)

1. $^{+}5 - (^{+}7)$ $^{-}2$
2. $^{+}8 - (^{-}9)$ 17
3. $^{-}7 - (^{-}15)$ 8
4. $^{-}9 - (^{-}20)$ $^{-}11$
5. $^{+}38 - (^{-}17)$ 55
6. $^{-}50 - (^{-}25)$ $^{-}25$
7. $^{-}12 - (^{+}6)$ $^{-}18$
8. $^{-}40 - (^{-}17)$ $^{-}23$
9. $^{+}38 - (^{+}17)$ 21

C. *Solving proportions* (Review page 487)

1. $\frac{n}{8} = \frac{25}{100}$ 2
2. $\frac{5}{x} = \frac{8}{40}$ 25
3. $\frac{8}{25} = \frac{32}{y}$ 100
4. $\frac{x}{100} = \frac{5}{8}$ 62.5
5. $\frac{43}{100} = \frac{132}{n}$ $306\frac{42}{43}$
6. $\frac{6}{a} = \frac{48}{100}$ 12.5
7. $\frac{y}{9} = \frac{5}{45}$ 1
8. $\frac{16}{64} = \frac{35}{n}$ 140
9. $\frac{76}{100} = \frac{x}{20}$ 15.2
10. $\frac{17}{n} = \frac{20}{100}$ 85
11. $\frac{a}{16} = \frac{3}{24}$ 2
12. $\frac{18}{75} = \frac{y}{100}$ 24

D. *Finding a square root (without tables)*

If the number is not a perfect square, average the perfect squares next above and below it. Use the average as the divisor and the given number as the dividend, and carry the quotient to one decimal place.

Dividing again, take the average of the divisor and quotient as a new divisor, with the given number as dividend.

EXAMPLE

Find $\sqrt{75}$

$\sqrt{81} = 9$ $\quad \sqrt{64} = 8$ Average of 8 and 9 is 8.5

$75 \div 8.5 = 8.8$ Average of 8.5 and 8.8 is 8.65

$75 \div 8.65 = 8.67$ Average of 8.65 and 8.67 is 8.66

8.66 is the square root of 75 to two decimal places.

Find the square roots to two decimal places:

1. 90 9.49
2. 21 4.58
3. 56 7.48
4. 92 9.59
5. 73 8.54
6. 13 3.61
7. 31 5.57
8. 89 9.43
9. 57 7.55
10. 43 6.56
11. 19 4.36
12. 41 6.40
13. 72 8.49
14. 30 5.48
15. 91 9.54

E. *Solving conditional statements with radicals*

First, write an equivalent expression containing the $\sqrt{\ }$ alone on one side of the equation. Then, if $\sqrt{n} = a$, $n = a^2$

EXAMPLE

Solve:

$$\frac{4\sqrt{n}}{5} = 8 \qquad \sqrt{n} = 8 \times \frac{5}{4} \text{ or } \sqrt{n} = 10 \quad \text{Then } n = 100$$

Find the value for n:

1. $2\sqrt{n} = 4$ 4	**8.** $\sqrt{n} - 2 = 3$ 25	**15.** $20 = 5\sqrt{n}$ 16
2. $18 \div \sqrt{n} = 9$ 4	**9.** $\sqrt{n} + 8 = 12$ 16	**16.** $\sqrt{n} - 8 = 2$ 100
3. $\sqrt{n} - 6 = 8$ 196	**10.** $\sqrt{n} - 4 = 1$ 25	**17.** $\sqrt{n} + 5 = 7$ 4
4. $\frac{3\sqrt{n}}{5} = 6$ 100	**11.** $4 = \frac{2\sqrt{n}}{3}$ 36	**18.** $17 - \sqrt{n} = 15$ 4
5. $15 = 5\sqrt{n}$ 9	**12.** $25 - \sqrt{n} = 22$ 9	**19.** $\frac{\sqrt{n}}{2} = 2$ 16
6. $15 \div \sqrt{n} = 5$ 9	**13.** $\sqrt{n} + 15 = 17$ 4	**20.** $19 - \sqrt{n} = 16$ 9
7. $21 - \sqrt{n} = 19$ 4	**14.** $6 = 18 \div \sqrt{n}$ 9	**21.** $\sqrt{n} + 14 = 17$ 9

F. *Simplifying a radical*

If the number under the radical sign has a factor that is a perfect square, the factor may be removed, and its square root written as coefficient of the radical.

EXAMPLE

Simplify: $\sqrt{175}$

$$\sqrt{175} = \sqrt{5 \cdot 5 \cdot 7}$$

Since $\sqrt{5 \cdot 5} = \sqrt{5^2}$, we may write the radical as $5\sqrt{7}$

Simplify:

1. $\sqrt{45}$ $3\sqrt{5}$	**5.** $\sqrt{12}$ $2\sqrt{3}$	**9.** $\sqrt{32}$ $4\sqrt{2}$	**13.** $\sqrt{8}$ $2\sqrt{2}$
2. $\sqrt{18}$ $3\sqrt{2}$	**6.** $\sqrt{75}$ $5\sqrt{3}$	**10.** $\sqrt{80}$ $4\sqrt{5}$	**14.** $\sqrt{72}$ $6\sqrt{2}$
3. $\sqrt{108}$ $6\sqrt{3}$	**7.** $\sqrt{20}$ $2\sqrt{5}$	**11.** $\sqrt{50}$ $5\sqrt{2}$	**15.** $\sqrt{27}$ $3\sqrt{3}$
4. $\sqrt{125}$ $5\sqrt{5}$	**8.** $\sqrt{162}$ $9\sqrt{2}$	**12.** $\sqrt{48}$ $4\sqrt{3}$	**16.** $\sqrt{98}$ $7\sqrt{2}$

G. *Multiplication of decimals* (Review page 480)

1. 67.1 × .293 = 19.6603	**3.** .562 × 9.45 = 5.3109	**5.** 6.04 × .11 = 0.6644	**7.** 33.3 × .0047 = 0.15651	**9.** 8.06 × .075 = 0.6045
2. 28.4 × 2.79 = 79.236	**4.** 80.8 × .073 = 5.8984	**6.** 3.09 × 11.3 = 34.917	**8.** 60.7 × .475 = 28.8325	**10.** 420.9 × 38.05 = 16,015.245

PRACTICE TEST

A. *Add:*

1. $^{+}9 + (^{-}8)$ 1 **3.** $^{-}15 + (^{-}14)$ -29 **5.** $^{+}19 + (^{-}16)$ 3

2. $^{+}16 + (^{+}9)$ 25 **4.** $^{-}15 + (^{+}14)$ $^{-}1$ **6.** $^{-}7 + (^{+}7)$ 0

B. *Subtract:*

1. $^{+}19 - (^{+}9)$ 10 **3.** $^{+}25 - (^{-}18)$ 43 **5.** $^{-}17 - (^{-}17)$ 0

2. $^{-}21 - (^{+}16)$ $^{-}37$ **4.** $^{+}18 - (^{+}18)$ 0 **6.** $^{-}14 - (^{+}18)$ $^{-}32$

C. Find the value for n:

1. $\frac{n}{8} = \frac{15}{20}$ 6 **3.** $\frac{9}{12} = \frac{24}{n}$ 32 **5.** $\frac{n}{12} = \frac{25}{30}$ 10

2. $\frac{n}{100} = \frac{28}{35}$ 80 **4.** $\frac{5}{9} = \frac{n}{81}$ 45 **6.** $\frac{n}{100} = \frac{24}{32}$ 75

D. Without reference to tables, find the square root of each number to two decimal places:

1. 21 4.58 **2.** 46 6.78 **3.** 40 6.32 **4.** 69 8.31 **5.** 89 9.43

E. *Solve:*

1. $\sqrt{n} - 5 = 7$ 144 **3.** $\sqrt{n} - 8 = 10$ 324 **5.** $\frac{\sqrt{n}}{3} = 6$ 324

2. $15\sqrt{n} = 45$ 9 **4.** $17 - \sqrt{n} = 13$ 16 **6.** $18 \div \sqrt{n} = 9$ 4

F. Simplify by removing a factor that is a perfect square:

1. $\sqrt{50}$ $5\sqrt{2}$ **2.** $\sqrt{162}$ $9\sqrt{2}$ **3.** $\sqrt{108}$ $6\sqrt{3}$

G. Find the products:

1. $(3.05) \cdot (1.73)$ 5.2765 **2.** $(1.95) \cdot (14.7)$ 28.665 **3.** $(19.35) \cdot (1.005)$ 19.44675

Find the quotients:

1. $72.9 \div 1.62$ 45.0 **2.** $6.66 \div 0.037$ 180.0 **3.** $38.4 \div 96$ 0.4

A. *Solving Equations*

1. $x + 18 = 35$ 17
2. $\frac{15x}{8} = 30$ 16
3. $32 \div x = \frac{1}{2}$ 64
4. $x + 7\frac{1}{2} = 22$ $14\frac{1}{2}$
5. $x \div 17 = 2\frac{1}{2}$ $42\frac{1}{2}$
6. $\frac{16x}{3} = 8$ 1.5
7. $x + 15\frac{2}{3} = 28\frac{1}{2}$ $12\frac{5}{6}$
8. $18\frac{7}{8} - x = 7\frac{1}{2}$ $11\frac{3}{8}$
9. $46 \div x = 6\frac{1}{2}$ $7\frac{1}{13}$
10. $x + 16\frac{2}{3} = 30\frac{1}{2}$ $13\frac{5}{6}$
11. $x - 13\frac{1}{3} = 5\frac{2}{3}$ 19
12. $\frac{9x}{4} = 18$ 8

B. *Solving proportions* (Review page 487)

1. $\frac{5}{9} = \frac{x}{27}$ 15
2. $\frac{6}{x} = \frac{3}{4}$ 8
3. $\frac{7}{16} = \frac{x}{24}$ 10.5
4. $\frac{11}{12} = \frac{33}{x}$ 36
5. $\frac{2}{x} = \frac{6}{9}$ 3
6. $\frac{x}{16} = \frac{5}{8}$ 10
7. $\frac{9}{13} = \frac{27}{x}$ 39
8. $\frac{15}{9} = \frac{x}{36}$ 60
9. $\frac{17}{x} = \frac{85}{5}$ 1
10. $\frac{x}{13} = \frac{4}{52}$ 1
11. $\frac{3}{8} = \frac{18}{x}$ 48
12. $\frac{x}{7} = \frac{3}{14}$ 1.5

C. *Solving Percentage Statements* (Review page 488)

1. 48 is n% of 36 133.3
2. n is 45% of 50 22.5
3. 48 is 30% of n 160
4. n is 0.2% of 68 0.136
5. 28 is n% of 35 80
6. n is 130% of 45 58.5
7. 56 is 14% of n 400
8. n is 1.3% of 350 4.55
9. 46 is n% of 40 115

D. *Calculating Interest* (Review page 156)

Find the interest on the following:

		Principal	*Rate*	*Time*			*Principal*	*Rate*	*Time*
\$7.50	**1.**	\$500	6%	90 days	\$2.05	**7.**	\$175.50	6%	70 days
\$25.31	**2.**	750	4.5%	270 days	\$7.29	**8.**	250	7%	5 mo.
\$14.00	**3.**	400	7%	6 mo.	\$10.11	**9.**	266.75	6.5%	7 mo.
\$3.13	**4.**	250	5%	3 mo.	\$8.79	**10.**	180.40	6.5%	9 mo.
\$2.00	**5.**	200	6%	2 mo.	\$1.27	**11.**	190	4%	60 days
\$27.00	**6.**	600	3%	18 mo.	\$3.50	**12.**	525	3%	80 days

A. Solve and check:

1. $x - 7\frac{1}{3} = 12\frac{5}{6}$ $20\frac{1}{6}$ **3.** $48 \div x = 16$ 3 **5.** $x + 17\frac{5}{9} = 28\frac{2}{3}$ $11\frac{1}{9}$

2. $36 \div x = \frac{2}{3}$ 54 **4.** $\frac{8x}{7} = 24$ 21 **6.** $x \div 17\frac{1}{2} = 4$ 70

B. Solve and check:

1. $\frac{3}{5} = \frac{x}{20}$ 12 **3.** $\frac{8}{12} = \frac{20}{x}$ 30 **5.** $\frac{x}{100} = \frac{36}{48}$ 75

2. $\frac{14}{21} = \frac{x}{12}$ 8 **4.** $\frac{24}{16} = \frac{18}{x}$ 12 **6.** $\frac{20}{x} = \frac{8}{6}$ 15

C. Find the value for n.

1. n is 150% of 36 54 **3.** 46% of n is 2.3 5 **5.** 2.5% of 60 is n 1.5

2. 80% of 35 is n 28 **4.** 3.8% of 80 is n 3.04 **6.** 1.7% of 70 is n 1.19

D. Find the interest:

1. \$400 at 6% for 2 months \$ 4.00 **4.** \$17.45 at 4.5% for 6 months \$.39

2. \$750 at 5% for 160 days \$16.00 **5.** \$107.45 at 7% for 90 days \$1.88

3. \$300 at 4.5% for 3 months \$3.38 **6.** \$95.75 at 5.5% for 4 months \$1.76

PRACTICE EXERCISES

A. *Solving proportions*

Use the relationship: *the product of the means is equal to the product of the extremes.*

EXAMPLE

Find the value for n: $\frac{n}{16} = \frac{12}{64}$ or $64n = 192$ Then $n = 3$.

1. $\frac{n}{12} = \frac{2}{3}$ 8 **4.** $\frac{6}{n} = \frac{3}{18}$ 36 **7.** $\frac{n}{15} = \frac{2}{3}$ 10 **10.** $\frac{n}{11} = \frac{35}{75}$ $5\frac{2}{15}$

2. $\frac{3}{4} = \frac{n}{32}$ 24 **5.** $\frac{n}{3} = \frac{4}{6}$ 2 **8.** $\frac{45}{21} = \frac{n}{7}$ 15 **11.** $\frac{1}{n} = \frac{1}{4}$ 4

3. $\frac{n}{5} = \frac{30}{75}$ 2 **6.** $\frac{4}{n} = \frac{11}{22}$ 8 **9.** $\frac{63}{6} = \frac{9}{n}$ $\frac{6}{7}$ **12.** $\frac{12}{3} = \frac{n}{6}$ 24

B. *Expressing a fraction as per cent*

To express the fraction $\frac{a}{b}$ as per cent, first solve the proportion: $\frac{a}{b} = \frac{n}{100}\cdot$ Then affix the per cent symbol to your answer for n.

EXAMPLE

$$\frac{5}{6} = n\%. \quad \frac{5}{6} = \frac{n}{100}.$$

$$6n = 500. \quad n = 83.\overline{3}\cdots \quad \frac{5}{6} \approx 83.3\%$$

Express as per cent, to the nearest tenth of 1%:

1. $\frac{13}{20}$ 65.0% **5.** $\frac{9}{16}$ 56.3% **9.** $\frac{7}{8}$ 87.5% **13.** $\frac{19}{20}$ 95.0% **17.** $\frac{3}{5}$ 60.0%

2. $\frac{3}{8}$ 37.5% **6.** $\frac{19}{50}$ 38.0% **10.** $\frac{1}{6}$ 16.7% **14.** $\frac{7}{10}$ 70.0% **18.** $\frac{3}{4}$ 75.0%

3. $\frac{1}{4}$ 25.0% **7.** $\frac{15}{16}$ 93.8% **11.** $\frac{3}{16}$ 18.8% **15.** $\frac{3}{20}$ 15.0% **19.** $\frac{5}{8}$ 62.5%

4. $\frac{9}{10}$ 90.0% **8.** $\frac{5}{9}$ 55.6% **12.** $\frac{24}{25}$ 96.0% **16.** $\frac{1}{8}$ 12.5% **20.** $\frac{2}{3}$ 66.7%

C. *Solving a percentage statement*

Set up and solve the proportion expressed in the percentage statement. One ratio is the per cent expressed as a common fraction. The other is the ratio between the two numbers being compared.

EXAMPLES

1. 35 is $n\%$ of 50.
$\frac{n}{100} = \frac{35}{50}.$ $\quad 50n = 3500. \quad n = 70.$

2. 85% of n is 34.
$\frac{85}{100} = \frac{34}{n}$ $\quad 85n = 3400. \quad n = 40.$

Find the values for n:

162 **1.** 135% of 120 is n 19.2 **6.** 3.2% of 600 is n 400 **11.** 24 is $n\%$ of 6

32 **2.** n is 80% of 40 45 **7.** $n\%$ of 40 is 18 15 **12.** 75 is $n\%$ of 500

0.198 **3.** 0.3% of 66 is n 75 **8.** 3.6 is $n\%$ of 4.8 125 **13.** 85 is 68% of n

3200 **4.** 16 is 0.5% of n 25 **9.** 0.4 is $n\%$ of 1.6 80 **14.** 40% of n is 32

273 **5.** 195% of 140 is n **10.** $n\%$ of 19 is 16 **15.** 625 is 2.5% of n

84.2 25,000

In dealing with conditional equations expressing the addend-addend-sum relationship, or the factor-factor-product relationship:
a. Identify the relationship expressed in the equation, and
b. Write the equivalent equation with the variable alone on one side.

D. *Solving conditional equations*

EXAMPLES

1. $56 - n = 18$ $\quad$ $s - a_1 = a_2$
$n = 56 - 18$ $\quad$ $a_1 = s - a_2$
$n = 38$

2. $28 \div n = 4$ $\quad$ $p \div f_1 = f_2$
$n = 28 \div 4$ $\quad$ $f_1 = p \div f_2$
$n = 7$

Solve for n:

1. $27 - n = 19$ 8
2. $12n = 48$ 4
3. $n \div 15 = 4$ 60
4. $n + 18 = 23$ 5
5. $43 = n - 17$ 60
6. $n \div 27 = 3$ 81
7. $19 + n = 45$ 26
8. $13n = 52$ 4
9. $18 - n = 11$ 7
10. $56 \div n = 7$ 8
11. $57 = n + 19$ 38
12. $n \div 5 = 13$ 65
13. $45 \div n = 3$ 15
14. $\frac{3}{4}n = 27$ 36
15. $27 + n = 83$ 56
16. $85 \div n = 17$ 5
17. $36 = n + 17$ 19
18. $37 - n = 19$ 18
19. $4\frac{1}{2}n = 81$ 18
20. $\frac{1}{8}n = 6$ 48
21. $n + 15 = 25$ 10

E. *Addition of signed numbers*

In adding numbers with like signs, add the absolute values of the numbers, and give the sum the common sign.

In adding numbers with unlike signs, subtract the smaller absolute value from the greater, and give the result the sign of the greater number.

EXAMPLES

1. $^+9 + (^+6) = {}^+16$
2. $^+13 + (^-27) = {}^-14$
3. $^-17 + (^-13) = {}^-30$
4. $^-19 + (^+25) = {}^+6$

Add:

1. $^+16 + (^-16)$ 0
2. $^-25 + (^+23)$ -2
3. $^+4\frac{1}{2} + (^-4\frac{1}{2})$ 0
4. $^+12 + (^-3)$ 9
5. $^-33 + (^-7)$ -40
6. $^-14 + (^-1)$ -15
7. $^+46 + (^-46)$ 0
8. $^+5.7 + (^-5.7)$ 0
9. $^+101 + (^+101)$ 202
10. $^-17 + (^-17)$ -34
11. $^-15\frac{1}{3} + (^-2\frac{1}{3})$ $-17\frac{2}{3}$
12. $^+7.6 + (^-1.6)$ 6
13. $^-143 + (^+143)$ 0
14. $^+49 + (^-26)$ 23
15. $^+34 + (^+4)$ 38
16. $^-42 + (^+8)$ -34
17. $^+36 + (^+16)$ 52
18. $^-17 + (^+145)$ 128
19. $^-12 + (^+1)$ -11
20. $^+39 + (^-39)$ 0
21. $^+21 + (^+20)$ 41

F. *Subtraction with signed numbers*

To subtract one signed number from another, add the additive inverse of the subtrahend to the minuend.

1. $^{+}45 - (^{-}11)$ 56	**6.** $^{-}18 - (^{-}14)$ $^{-}4$	**11.** $^{-}47 - (^{-}47)$ 0
2. $^{+}39 - (^{-}7)$ 46	**7.** $^{+}26 - (^{-}42)$ 68	**12.** $^{+}23 - (^{+}3)$ 20
3. $^{-}51 - (^{-}30)$ $^{-}21$	**8.** $^{-}8 - (^{-}16)$ 8	**13.** $^{-}28 - (^{+}5)$ $^{-}33$
4. $^{+}54 - (^{-}7)$ 61	**9.** $^{+}28 - (^{+}17)$ 11	**14.** $^{+}16 - (^{+}16)$ 0
5. $^{-}33 - (^{+}33)$ $^{-}66$	**10.** $^{-}12 - (^{+}76)$ $^{-}88$	**15.** $^{-}48 - (^{-}49)$ 1

PRACTICE TEST

A. *Solve:*

1. $\frac{n}{15} = \frac{2}{3}$ 10	**3.** $\frac{8}{n} = \frac{16}{20}$ 10	**5.** $\frac{15}{18} = \frac{n}{36}$ 30
2. $\frac{16}{24} = \frac{12}{n}$ 18	**4.** $\frac{5}{16} = \frac{n}{100}$ 31.25	**6.** $\frac{16}{44} = \frac{n}{11}$ 4

B. Find n:

1. 125% of 76 is n 95	**3.** n% of 45 is 36 80	**5.** 15% of n is 45 300
2. 0.8% of n is 40 5000	**4.** 4 is n% of 160 2.5	**6.** n is 18% of 50 9

C. *Solve:*

1. $18n = 90$ 5	**3.** $96 \div n = 16$ 6	**5.** $75 - n = 13$ 62
2. $45 = 18 + n$ 27	**4.** $54 \div n = 9$ 6	**6.** $n \div 18 = 20$ 360

D. Write each as per cent to the nearest tenth of 1%:

1. $\frac{5}{8}$ 62.5% **2.** $\frac{2}{3}$ 66.7% **3.** $\frac{3}{16}$ 18.8% **4.** $\frac{5}{23}$ 21.7% **5.** $\frac{3}{11}$ 27.3%

E. *Add:*

1. $^{+}19 + (^{-}15)$ 4	**3.** $^{-}25 + (^{-}17)$ $^{-}42$	**5.** $^{+}13 + (^{+}42)$ 55
2. $^{-}13 + (^{+}55)$ 42.	**4.** $^{+}27 + (^{+}35)$ 62	**6.** $^{-}35 + (^{-}21)$ $^{-}56$

F. *Subtract:*

1. $^{+}27 - (^{+}32)$ $^{-}5$	**3.** $^{-}52 - (^{-}19)$ $^{-}33$	**5.** $^{+}29 - (^{-}24)$ 53
2. $^{-}46 - (^{-}45)$ $^{-}1$	**4.** $^{+}41 - (^{+}34)$ 7	**6.** $^{-}76 - (^{+}55)$ $^{-}131$

A. *Addition with signed numbers* (Review page 489)

1. ${}^{+}16 + ({}^{+}9)$ *25*
2. ${}^{-}17 + ({}^{-}25)$ *${}^{-}42$*
3. ${}^{+}39 + ({}^{-}15)$ *24*
4. ${}^{-}36 + ({}^{+}39)$ *3*
5. ${}^{+}18 + ({}^{+}17)$ *35*
6. ${}^{-}45 + ({}^{+}28)$ *${}^{-}17$*
7. ${}^{+}13 + ({}^{-}20)$ *${}^{-}7$*
8. ${}^{-}16 + ({}^{-}19)$ *${}^{-}35$*
9. ${}^{-}20 + ({}^{+}40)$ *20*
10. ${}^{-}12 + ({}^{+}13)$ *1*
11. ${}^{+}26 + ({}^{+}16)$ *42*
12. ${}^{+}43 + ({}^{-}29)$ *14*

B. *Subtraction* (Review page 490)

1. ${}^{+}16 - ({}^{+}9)$ *7*
2. ${}^{+}31 - ({}^{-}17)$ *48*
3. ${}^{-}17 - ({}^{-}31)$ *14*
4. ${}^{+}43 - ({}^{-}26)$ *69*
5. ${}^{+}24 - ({}^{+}30)$ *${}^{-}6$*
6. ${}^{-}26 - ({}^{-}14)$ *${}^{-}12$*
7. ${}^{-}51 - ({}^{+}16)$ *${}^{-}67$*
8. ${}^{+}51 - ({}^{-}49)$ *100*
9. ${}^{+}45 - ({}^{+}18)$ *27*
10. ${}^{+}62 - ({}^{-}3)$ *65*
11. ${}^{-}51 - ({}^{-}39)$ *${}^{-}12$*
12. ${}^{+}45 - ({}^{-}15)$ *60*

C. *Multiplication* (Review pages 225–227)

1. ${}^{-}8 \cdot ({}^{-}9)$ *72*
2. ${}^{-}12 \cdot ({}^{+}11)$ *${}^{-}132$*
3. ${}^{+}16 \cdot ({}^{+}9)$ *144*
4. ${}^{+}12 \cdot ({}^{-}12)$ *${}^{-}144$*
5. ${}^{-}41 \cdot ({}^{-}7)$ *287*
6. ${}^{-}3 \cdot ({}^{+}9)$ *${}^{-}27$*
7. ${}^{+}18 \cdot ({}^{+}11)$ *198*
8. ${}^{+}22 \cdot ({}^{-}7)$ *${}^{-}154$*
9. ${}^{-}16 \cdot ({}^{-}5)$ *80*
10. ${}^{+}19 \cdot ({}^{+}8)$ *152*
11. ${}^{+}24 \cdot ({}^{-}5)$ *${}^{-}120$*
12. ${}^{-}16 \cdot ({}^{+}6)$ *${}^{-}96$*

D. *Division* (Review pages 225–227)

1. ${}^{-}72 \div ({}^{-}9)$ *8*
2. ${}^{+}198 \div ({}^{+}11)$ *18*
3. ${}^{+}144 \div ({}^{-}18)$ *8*
4. ${}^{-}396 \div ({}^{+}12)$ *${}^{-}33$*
5. ${}^{-}132 \div ({}^{-}11)$ *12*
6. ${}^{-}154 \div ({}^{+}22)$ *${}^{-}7$*
7. ${}^{+}156 \div ({}^{+}12)$ *13*
8. ${}^{-}396 \div ({}^{+}22)$ *${}^{-}18$*
9. ${}^{-}240 \div ({}^{+}15)$ *${}^{-}16$*
10. ${}^{-}144 \div ({}^{-}9)$ *16*
11. ${}^{-}160 \div ({}^{+}32)$ *${}^{-}5$*
12. ${}^{+}304 \div ({}^{+}16)$ *19*

E. *Solving equations*

EXAMPLE

Solve for x

$$\frac{7x}{4} - 11 = \frac{x}{2} + 4$$

$$\frac{7x}{4} - \frac{x}{2} - 11 + 11 = \frac{x}{2} - \frac{x}{2} + 4 + 11$$

$$\frac{5x}{4} = 15 \qquad \frac{5x}{4} \cdot \frac{4}{5} = 15 \cdot \frac{4}{5}$$

To get x alone on the left and the constant alone on the right, we add to both sides the additive inverses of ${}^{-}11$ and $\frac{{}^{+}x}{2}$:

The coefficient of x is $\frac{5}{4}$. So we multiply both sides by $\frac{4}{5}$. $x = 12$

Solve for the variable x.

1. $3x - 7 = 2x + 1$ 8
2. $\frac{3x}{4} + 5 = \frac{x}{8} + 9$ 6.4
3. $\frac{5x}{6} - 4 = \frac{x}{3}$ 8
4. $2x = \frac{x}{3} + 30$ 18
5. $\frac{4x}{3} + 5 = x + 10$ 15
6. $\frac{3x}{8} + 8 = \frac{x}{4} + 10$ 16
7. $\frac{13x}{4} + 7 = 2x + 15$ 6.4
8. $\frac{2x}{3} - 5 = \frac{x}{9}$ 9
9. $2x = \frac{x}{2} + 15$ 10
10. $5x - 3 = 3x + 1$ 2

PRACTICE TEST

A. Perform the indicated operation.

1. $^{-}18 + (^{-}27)$ -45
2. $^{+}25 + (^{-}25)$ 0
3. $^{-}17 - (^{-}26)$ 9
4. $^{-}24 - (^{+}24)$ -48
5. $^{+}26 - (^{+}20)$ 6
6. $^{-}17 + (^{-}20)$ -37
7. $^{+}17 + (^{+}25)$ 42
8. $^{-}34 - (^{-}25)$ -9
9. $^{+}50 - (^{+}60)$ -10
10. $^{+}29 - (^{-}29)$ 58
11. $^{+}26 + (^{+}18)$ 44
12. $^{-}25 + (^{-}14)$ -39
13. $^{+}30 - (^{-}26)$ 56
14. $^{-}18 - (^{+}30)$ -48
15. $^{+}37 - (^{+}25)$ 12

B. Solve.

-171 **1.** $^{-}19 \cdot (^{+}9) = x$
90 **2.** $^{-}18 \cdot (^{-}5) = a$
153 **3.** $^{+}17 \cdot (^{+}9) = b$
49 **4.** $^{+}343 \div (^{+}7) = d$
12 **5.** $^{-}180 \div (^{-}15) = m$
-217 **6.** $^{+}31 \cdot (^{-}7) = r$
144 **7.** $^{+}12 \cdot (^{+}12) = s$
152 **8.** $^{-}19 \cdot (^{-}8) = t$
13 **9.** $^{-}117 \div (^{-}9) = y$
18 **10.** $^{+}270 \div (^{+}15) = m$
84 **11.** $^{-}14 \cdot (^{-}6) = n$
-132 **12.** $^{-}15 \cdot (^{+}9) = p$
-256 **13.** $^{+}16 \cdot (^{-}16) = x$
19 **14.** $^{-}152 \div (^{-}8) = b$
28 **15.** $^{-}364 \div (^{+}13) = c$

16. $2x - 11 = \frac{x}{2} + 7$ 12
17. $3x - 5 = x + 7$ 6
18. $\frac{5x}{4} - 3 = \frac{x}{2} + 3$ 8
19. $3x + 4 = 2x + 11$ 7
20. $\frac{3x}{5} + 7 = x + 5$ 5
21. $\frac{3x}{2} - 11 = \frac{x}{7} + 8$ 14
22. $\frac{5x}{4} - 5 = \frac{x}{2} - 2$ 4
23. $3x + 1 = 2x + 4$ 3
24. $2x + 3 = x + 9$ 6
25. $\frac{5x}{4} - 20 = \frac{7x}{8} - 14$ 16

A. *Adding Integers* (Review page 489)

1. $^{+}9 + (^{+}7)$ 16
2. $^{+}19 + (^{-}15)$ 4
3. $^{-}16 + (^{+}5)$ $^{-}11$
4. $^{-}21 + (^{-}23)$ $^{-}44$
5. $^{+}46 + (^{-}25)$ 21
6. $^{+}3 + (^{+}31)$ 34
7. $^{+}11 + (^{-}45)$ $^{-}34$
8. $^{-}28 + (^{-}39)$ $^{-}67$
9. $^{-}91 + (^{+}76)$ $^{-}15$

Subtracting Integers

Subtraction of signed numbers is most easily performed as an equivalent addition.

Add to the minuend the additive inverse of the subtrahend. Then apply the rules for adding signed numbers.

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1. $^{+}15 - (^{+}13)$ 2
2. $^{+}18 - (^{-}13)$ 31
3. $^{-}16 - (^{+}21)$ $^{-}37$
4. $^{-}28 - (^{-}16)$ $^{-}12$
5. $^{-}43 - (^{+}58)$ 101
6. $^{+}32 - (^{+}12)$ 20
7. $^{+}45 - (^{-}14)$ 59
8. $^{+}17 - (^{-}29)$ 46
9. $^{-}35 - (^{-}11)$ $^{-}24$
10. $^{-}17 - (^{-}36)$ 19
11. $^{-}19 - (14)$ $^{-}33$
12. $^{-}12 - (3)$ $^{-}15$

B. *Conditional equations*

1. $x + 3 = 12$ 9
2. $3x + 8 = 29$ 7
3. $4n + 3 = 19$ 4
4. $29 = 5n + 4$ 5
5. $48 = 8y + 8$ 5
6. $32 - 2x = 22$ 5
7. $47 - 3y = 14$ 11
8. $19 - 9y = 1$ 2
9. $76 = 84 - 8y$ 1
10. $43 = 29 - 7y$ $^{-}2$
11. $2n + 36 = 78$ 21
12. $81 = 9y + 18$ 7
13. $53 - 10n = 3$ 5
14. $12 = 72 - 12n$ 5
15. $5z + 21 = 36$ 3

C. *Proportions*

Find the value of the variable in the following:

1. $\frac{n}{3} = \frac{24}{36}$ 2
2. $\frac{n}{5} = \frac{42}{35}$ 6
3. $\frac{50}{x} = \frac{12}{3}$ 12.5
4. $\frac{3}{21} = \frac{y}{42}$ 6
5. $\frac{56}{x} = \frac{24}{3}$ 7
6. $\frac{91}{13} = \frac{14}{n}$ 2
7. $\frac{144}{z} = \frac{30}{5}$ 24
8. $\frac{n}{16} = \frac{48}{16}$ 48
9. $\frac{1}{2} = \frac{17}{x}$ 34
10. $\frac{7}{203} = \frac{a}{29}$ 1
11. $\frac{4}{15} = \frac{12}{n}$ 45
12. $\frac{3}{16} = \frac{9}{n}$ 48
13. $\frac{15}{80} = \frac{x}{20}$ $3\frac{3}{4}$
14. $\frac{4}{y} = \frac{32}{100}$ 12.5
15. $\frac{75}{25} = \frac{n}{12}$ 36
16. $\frac{21}{y} = \frac{63}{21}$ 7

D. *Solving Percentage Statements* (Review page 488)

Solve for n:

1. 85% of n is \$255 \$ 300
2. 27% of n is \$108 \$ 400
3. 95% of n is \$475 \$ 500
4. 3% of n is \$24 \$ 800
5. 1.6% of n is \$1.44 \$ 90
6. \$21.60 is 3.6% of n \$600
7. 0.83% of n is \$4.98 \$600
8. \$14.40 is 16% of n \$ 90
9. 8.9% of n is \$13.35 \$ 150
10. 0.6% of n is 2.88 480
11. 16 is 2.5% of n 640
12. 8.8% of n is \$79.20 \$ 900
13. 87% of n is \$788 \$ 905.75
14. 43% of n is 301 700

E. *The Interest Formula* (Review page 156)

The interest formula is: $i = prt$ where i is the interest, p is principal in dollars, r is rate, expressed in hundreths, t is time in years or fractions of a year.

1. \$250, 60 days, 6% \$ 2.50
2. \$240, 90 days, $3\frac{1}{2}$% \$ 2.10
3. \$216, 2 years, 4% \$ 17.28
4. \$1600, 75 days, $4\frac{1}{2}$% \$ 15.00
5. \$200, 72 days, 3% \$ 1.20
6. \$360, 90 days, $4\frac{1}{2}$% \$ 4.05
7. \$420, 60 days, $3\frac{1}{2}$% \$ 2.45
8. \$750, 3 months, 3% \$ 5.63

PRACTICE TEST

A. Add or subtract as indicated:

1. $16 + (^+4)$ 20
2. $32 + (^-7)$ 25
3. $42 - (^+11)$ 31
4. $36 + (^-14)$ 22
5. $27 - (^-26)$ 53
6. $14 - (^+3)$ 11
7. $^-46 + (^-14)$ $^-60$
8. $46 - (^+14)$ 32
9. $28 - (^-18)$ 46

B. Find the value of the variable in each of these conditional statements to make the statement true:

17 **1.** $3y = 51$

13 **2.** $4x + 4 = 56$

14 **3.** $5y - 7 = 48 + 15$

3 **4.** $3z + 2z = 29 - 14$

9 **5.** $2x + 3 = 12 + x$

$^-20$ **6.** $45 - 3n = 105$

16 **7.** $3y - 5 = 43$

11 **8.** $14y = 154$

$16\frac{1}{3}$ **9.** $6z - z = 49 + 2z$

12 **10.** $\frac{2n}{3} = 8$

6 **11.** $2x + 5x + 3x = 48 - 12 + 24$

$3\frac{3}{7}$ **12.** $5n + n - 6 = 14 - n + 4$

C. Find the value of the variable in the following proportions:

1. $\frac{2}{3} = \frac{n}{18}$ 12	**4.** $\frac{y}{7} = \frac{72}{56}$ 9	**7.** $\frac{2}{24} = \frac{3.5}{k}$ 42
2. $\frac{20}{24} = \frac{5}{n}$ 6	**5.** $\frac{40}{x} = \frac{25}{10}$ 16	**8.** $\frac{48}{15} = \frac{16}{z}$ 5
3. $\frac{18}{x} = \frac{54}{9}$ 3	**6.** $\frac{68}{52} = \frac{n}{13}$ 17	**9.** $\frac{n}{9} = \frac{84}{108}$ 7

D. Find the value of n:

1. 40% of 75 is n 30	**4.** 90% of 56 is n 50.4	**7.** 2.5 is n% of 750 0.3
2. 37 is n% of 185 20	**5.** n is 25% of 65 16.25	**8.** 25% of n is 65 260
3. 76 is n% of 95 80	**6.** 37.5% of n is 72 19.2	**9.** 2% of 6400 is n 128

E. Find the interest:

1. \$100 at 6% for 72 days \$1.20	**4.** \$500 at $5\frac{1}{2}$% for 90 days \$6.88
2. \$200 at 7% for 8 months \$9.33	**5.** \$150 at 5% for 135 days \$2.81
3. \$120 at $6\frac{1}{2}$% for 290 days \$6.28	**6.** \$200 at 8% for 175 days \$7.78

PRACTICE EXERCISES

A. *Solving Proportions* (Review page 487)

Find the value of x in each of these proportions:

1. $\frac{x}{3} = \frac{4}{12}$ 1	**4.** $\frac{3}{7} = \frac{21}{x}$ 49	**7.** $\frac{8}{10} = \frac{16}{x}$ 20	**10.** $\frac{14}{70} = \frac{2}{x}$ 10
2. $\frac{x}{6} = \frac{12}{3}$ 24	**5.** $\frac{15}{4} = \frac{20}{x}$ $5\frac{1}{3}$	**8.** $\frac{16}{20} = \frac{40}{x}$ 50	**11.** $\frac{19}{x} = \frac{95}{120}$ 24
3. $\frac{x}{7} = \frac{15}{21}$ 5	**6.** $\frac{x}{4} = \frac{10}{8}$ 5	**9.** $\frac{3}{x} = \frac{24}{32}$ 4	**12.** $\frac{15}{25} = \frac{45}{x}$ 75

B. *Multiplication of Decimals* (Review page 480)

1. 64.1 .291 18.6531	**4.** .362 9.45 3.4209	**7.** 6.01 .11 0.6611	**10.** 35.3 .004 0.1412	**13.** 2.06 .015 0.3090
2. 28.7 3.19 91.553	**5.** 20.9 .078 1.6302	**8.** 2.98 11.3 33.674	**11.** 68.2 .175 11.935	**14.** 426.9 32.05 13,682.145
3. .705 .54 0.3807	**6.** 6.25 11.8 73.75	**9.** 7.98 13.1 104.538	**12.** 36.82 5.072 186.75104	**15.** 9.681 118.4 1146.2304

C. *Division of Decimals* (Review page 481)

Find the quotients to the nearest hundredth:

1. 4.280 ÷ .2 21.4
2. 29.6 ÷ .04 740
3. 36.036 ÷ .08 450.45
4. 62.4 ÷ .008 7800
5. .468 ÷ .13 3.6
6. 4.242 ÷ .0021 2020
7. 36.42 ÷ 6.07 6.0
8. 562.4 ÷ 1.6 351.8
9. 2.856 ÷ .24 11.9

Find the quotients to the nearest thousandth:

10. 19.36 ÷ 1.6 12.1
11. 148.2 ÷ .02 7410
12. 487.2 ÷ 2.4 20.30
13. 481.5 ÷ .15 3210
14. 11.088 ÷ .42 26.4
15. 4.0992 ÷ .056 73.2

D. Find the value for n:

1. 80% of 85 is n 68
2. 74 is n% of 148 50
3. n% of 84 is 21 25
4. 28% of n is 224 800
5. 45 is n% of 180 25
6. 76 is n% of 95 80
7. 90% of 160 is n 144
8. 37.5% of n is 54 144
9. 2.5 is n% of 500 5

PRACTICE TEST

A. Find the value for x:

3 **1.** $\frac{45}{27} = \frac{5}{x}$

$55\frac{5}{9}$ **2.** $\frac{5}{9} = \frac{x}{100}$

42 **3.** $\frac{18}{x} = \frac{3}{7}$

20 **4.** $\frac{x}{100} = \frac{14}{70}$

5 **5.** $\frac{15}{27} = \frac{x}{9}$

300 **6.** $\frac{73}{100} = \frac{219}{x}$

12 **7.** $\frac{x}{16} = \frac{3}{4}$

400 **8.** $\frac{83}{100} = \frac{332}{x}$

B. Find the products:

1. 0.14 × 1.3 0.182
2. 5.19 × 4.3 22.317
3. 12.09 × 0.12 1.4508
4. 16.8 × 8.7 146.16
5. 15.6 × 0.011 0.1716
6. 3.13 × 1.027 3.21451

C. Divide:

1. 27.2 ÷ .04 680
2. 5.46 ÷ 30 0.182
3. 43.83 ÷ .09 487.0
4. 106.4 ÷ .014 7600
5. 465 ÷ 1.5 310
6. 627.5 ÷ 1.25 502.0

D. Find the value for n:

40 **1.** n% of 40 is 16

16.5 **2.** n is 30% of 55

50 **3.** 15 is 30% of n

62.5 **4.** 25 is n% of 40

64 **5.** 25% of n is 16

140 **6.** 84 is n% of 60

45 **7.** n is 125% of 36

20 **8.** 150% of n is 30

55.6 **9.** 45 is n% of 81

PRACTICE EXERCISES

A. *Expressing a ratio as per cent* (Review page 488)

Express in per cent, to the nearest tenth of 1%:

1. $\frac{3}{8}$ 37.5% **3.** $\frac{5}{6}$ 83.3% **5.** $\frac{18}{23}$ 78.3% **7.** $\frac{1}{2}$ 50.0% **9.** $\frac{9}{10}$ 90.0%

2. $\frac{19}{50}$ 38.0% **4.** $\frac{5}{16}$ 31.3% **6.** $\frac{5}{8}$ 62.5% **8.** $\frac{17}{40}$ 42.5% **10.** $\frac{3}{20}$ 15.0%

B. *Writing a decimal as a per cent*

1. .75 75% **3.** .945 94.5% **5.** .109 10.9% **7.** 3.39 339% **9.** 32.05 3025%

2. .68 68% **4.** .021 2.1% **6.** .0002 .02% **8.** 1.65 165% **10.** 8.5 850%

C. Solve for n.

1. n% of 160 is 40 25 **5.** n is 40% of 160 64 **9.** 35% of n is 14 40

2. 25% of 230 is n 57.5 **6.** n% of 160 is 20 12.5 **10.** 36 is n% of 180 20

3. 16 is 25% of n 64 **7.** 7.5% of 180 is n 13.5 **11.** n% of 360 is 18 5

4. 60 is n% of 300 20 **8.** 170 is n% of 850 20 **12.** n is 30% of 85 25.5

PRACTICE TEST

A. Express as decimals:

1. 76% 0.76 **2.** 183% 1.83 **3.** 5.5% .055 **4.** 175.5% 1.755 **5.** 0.7% .007

B. Express as per cent:

1. .92 92% **2.** .07 7% **3.** .073 7.3% **4.** 1.95 195% **5.** .0375 3.75%

C. Express as per cent, to the nearest tenth of 1%:

1. $\frac{5}{8}$ 62.5% **2.** $\frac{3}{7}$ 42.9% **3.** $\frac{5}{9}$ 55.6% **4.** $\frac{7}{4}$ 175.0% **5.** $1\frac{2}{9}$ 122.2%

D. Copy and complete each statement:

1. 27% of 60 is n 16.2 **3.** 217.5% of 60 is n 130.5 **5.** 3.5% of 16 is n 0.56

2. 125% of 64 is n 80 **4.** n is 5.28% of 72 3.8016 **6.** 0.03% of 1600 is n 0.48

E. Copy and complete to the nearest tenth of 1%:

1. 36 is n% of 48 75.0 **3.** 12 is n% of 1200 1.0 **5.** 85 is n% of 68 125.0

2. 18 is n% of 96 18.8 **4.** 2250 is n% of 1750 128.6 **6.** 120 is n% of 48 250.0

F. Copy and complete:

1. 45 is n% more than 30 50

2. 16 is n% less than 20 20

3. 1600 is n% more than 1280 25

4. 25% less than 80 is n 60

A. *Expressing a ratio in different forms*

EXAMPLES

1. Express the ratio of 15 to 24 in three ways.

$\frac{15}{24} = \frac{5}{8}$, in lowest terms, $5 \div 8 = .625$, and $.625 = 62.5\%$

2. Express $\frac{11}{17}$ as per cent, to the nearest tenth of 1%.

$\frac{11}{17} = 0.647$, to the nearest thousandth. $0.647 = 64.7\%$

Express as per cent:

1. 0.35 35%	**4.** 0.27 27%	**7.** .049 4.9%	**10.** .025 2.5%
2. .09 9%	**5.** .03 3%	**8.** 2.27 227%	**11.** 0.176 17.6%
3. 0.17 17%	**6.** 1.15 115%	**9.** 0.314 31.4%	**12.** 3.14 314%

Express as per cent, to the nearest tenth of 1%:

1. $\frac{3}{8}$ 37.5%	**4.** $\frac{4}{5}$ 80.0%	**7.** $\frac{7}{4}$ 175.0%	**10.** $\frac{5}{9}$ 55.6%
2. $\frac{7}{40}$ 17.5%	**5.** $\frac{8}{3}$ 266.7%	**8.** $\frac{23}{25}$ 92.0%	**11.** $\frac{2}{3}$ 66.7%
3. $\frac{3}{13}$ 23.1%	**6.** $\frac{2}{11}$ 18.2%	**9.** $\frac{5}{16}$ 31.3%	**12.** $\frac{4}{7}$ 57.1%

See front.

Express the ratio of the first number to the second in three ways:

1. 8 to 5	**3.** 7 to 4	**5.** 5 to 7	**7.** 3 to 16
2. 17 to 20	**4.** 9 to 8	**6.** 19 to 16	**8.** 5 to 8

B. *Finding a per cent of a number* (Review page 488)

22.1 **1.** 26% of 85 is n	5.1 **4.** 6% of 85 is n	7524 **7.** 627% of 1200 is n
3.5 **2.** 5% of 70 is n	15 **5.** 125% of 12 is n	81 **8.** 135% of 60 is n
4.32 **3.** n is 8% of 54	40.5 **6.** 45% of 90 is n	14.26 **9.** 92% of 15.5 is n

C. *Finding what per cent one number is of another*

25 **1.** 8 is n% of 32	10 **4.** n% of 50 is 5	62.5 **7.** n% of 80 is 50
75 **2.** 15 is n% of 20	50 **5.** 32 is n% of 64	12.5 **8.** n% of 96 is 12
20 **3.** n% of 60 is 12	87.5 **6.** 56 is n% of 64	73.9 **9.** 17 is n% of 23

D. *Finding a number when a per cent of it is given*

1. 25% of n is 14 56	**6.** 45 is 90% of n 50	**11.** 350% of n is 14 4
2. 40% of n is 82 205	**7.** 80 is 125% of n 64	**12.** 37.5% of n is 12 32
3. 15% of n is 6 400	**8.** 7 is 3.5% of n 200	**13.** 6.5% of n is 26 400
4. 6% of n is 18 300	**9.** 75 is 150% of n 50	**14.** 56 is 87.5% of n 64
5. 15 is 30% of n 50	**10.** 2.5% of n is 15 600	**15.** 95% of n is 76 80

E. *Per cent of increase and decrease*

1. 35% more than 40 is n 54	**4.** 120 decreased by 70% is n 36
2. 30% less than 80 is n 56	**5.** 50% less than 44 is n 22
3. 50 increased by 40% is n 70	**6.** 60% more than 50 is n 80

PRACTICE TEST

A. Express as decimals:

1. 78% 0.78	**3.** 17.5% 0.175	**5.** 0.25% 0.025	**7.** 169.2% 1.692
2. 19% 0.19	**4.** 2.3% 0.023	**6.** 0.4% 0.004	**8.** 425% 4.25

B. Express as per cent:

1. .35 35%	**3.** .05 5%	**5.** 0.125 12.5%	**7.** 1.5 150%
2. .95 95%	**4.** .375 37.5%	**6.** 0.1625 16.25%	**8.** 2.25 225%

C. Express as per cent, to the nearest tenth of 1%:

1. $\frac{4}{25}$ 16.0%	**3.** $\frac{5}{16}$ 31.3%	**5.** $\frac{3}{200}$ 1.5%	**7.** $\frac{7}{4}$ 175.0%
2. $\frac{3}{5}$ 60.0%	**4.** $\frac{17}{40}$ 42.5%	**6.** $\frac{1}{50}$ 2.0%	**8.** $\frac{5}{2}$ 250.0%

D. Copy and solve:

1. 60% of 85 is n 51	**4.** 28% of n is 14 30	**7.** 90% of 162 is n 145.8
2. 37 is n% of 185 20	**5.** 57 is n% of 95 60	**8.** 37.5% of n is 57 152
3. n% of 84 is 21 25	**6.** 45 is n% of 50 90	**9.** 2.5 is n% of 500 0.5

E. Find n:

1. 15% more than 80 is n 92	**4.** n% more than 30 is 42 40
2. 150 decreased by n% is 120 20	**5.** 60 is 20% less than n 75
3. n increased by 30% is 65 50	**6.** 48 decreased by n% is 36 25

PRACTICE EXERCISES

A. *Add:*

1. $(^{+}2) + (^{-}5) + (^{-}4)$ $^{-}7$ **2.** $(^{-}3) + (^{-}7) + (^{-}13)$ $^{-}23$ **3.** $(^{-}7) + (^{+}8) + (^{+}13)$ 14

4.	**5.**	**6.**	**7.**	**8.**	**9.**	**10.**
$^{+}3$	$^{-}7$	$^{-}3\frac{1}{2}$	$^{-}3.61$	$^{-}4.11$	$^{-}2\frac{5}{8}$	$^{-}3\frac{1}{2}$
$^{+}7$	$^{-}8$	$^{+}2\frac{3}{4}$	$^{-}2.58$	$^{+}2.68$	$^{+}8\frac{2}{3}$	$^{+}1$
$^{-}5$	$^{-}9$	$^{-}5\frac{2}{3}$	$^{+}4.11$	$^{+}7.03$	$^{+}4\frac{1}{2}$	$^{+}2\frac{1}{2}$
$^{-}13$	$^{-}14$	$^{-}6\frac{5}{12}$	$^{-}1.38$	$^{-}.21$	$10\frac{13}{24}$	0
$^{-}8$	$^{-}38$		$^{-}3.46$	5.39		

B. *Subtract:*

1. $(^{+}7) - (^{+}3)$ 4 **2.** $(^{-}3) - (^{-}9)$ 6 **3.** $(^{+}18) - (^{+}13)$ 5 **4.** $(^{-}3) - (^{-}14)$ 11

C. *Multiply:*

1. $(^{+}5) \times (^{-}5)$ $^{-}25$

2. $(^{-}17) \times (^{-}7)$ 119

3. $(^{-}3) \times (^{-}5) \times (^{+}4)$ 60

4. $(^{-}\frac{2}{3}) \times (^{-}\frac{3}{4}) \times (^{+}\frac{1}{2})$ $\frac{1}{4}$

5. $(^{-}2.61) \times (^{-}3.55)$ 9.2655

6. $(^{-}3\frac{2}{3}) \times (^{+}4\frac{2}{3})$ $^{-}17\frac{1}{9}$

D. *Divide:*

4 **1.** $(^{-}64) \div (^{-}16)$

$^{-}14.7$ **2.** $(^{-}49.98) \div (^{+}3.4)$

18 **3.** $(^{-}45) \div (^{-}2.5)$

4. $(^{+}1250) \div (^{-}25)$ $^{-}50$

20 **5.** $(^{-}2.610) \div (^{-}.1305)$

$^{-}2$ **6.** $(^{-}6\frac{2}{3}) \div (^{+}3\frac{1}{3})$

PRACTICE TEST

A. *Add:*

1. $(^{+}5) + (^{-}7) + (^{+}9)$ 7

2. $(^{-}8) + (^{-}6) + (^{+}5) + (^{+}7)$ $^{-}2$

3. $(^{+}17) + (^{-}8) + (^{-}15) + (^{+}13)$ 7

B. *Subtract:*

1. $(^{+}17) - (^{+}13)$ 4

2. $(^{+}25) - (^{+}33)$ $^{-}8$

3. $(^{+}31) - (^{-}32)$ 63

4. $(^{-}11) - (^{-}15)$ 4

5. $(^{-}23) - (^{-}8)$ $^{-}15$

6. $(^{-}17) - (^{+}19)$ $^{-}36$

C. *Multiply:*

1. $(^{+}3) \times (^{+}7)$ 21

2. $(^{+}7) \times (^{-}9)$ $^{-}63$

3. $(^{-}15) \times (^{+}13)$ $^{-}195$

4. $(^{-}8) \times (^{-}13)$ 104

5. $(^{-}14) \times (^{+}7)$ $^{-}98$

6. $(^{+}18) \times (^{-}5)$ $^{-}90$

D. *Divide:*

1. $(^{+}84) \div (^{+}7)$ 12

2. $(^{+}96) \div (^{-}8)$ $^{-}12$

3. $(^{-}343) \div (^{-}49)$ 7

4. $(^{-}56) \div (^{-}7)$ 8

5. $(^{-}112) \div (^{+}14)$ $^{-}8$

6. $(^{+}288) \div (^{-}18)$ $^{-}16$

Tables

LINEAR MEASURE

12 inches (in.)	= 1 foot (ft.)
3 feet	= 1 yard (yd.)
$5\frac{1}{2}$ yards, or $16\frac{1}{2}$ feet	= 1 rod (rd.)
320 rods, 1760 yards, or 5280 feet	= 1 mile (mi.)

SQUARE MEASURE

144 square inches (sq. in.)	= 1 square foot (sq. ft.)
9 square feet	= 1 square yard (sq. yd.)
$30\frac{1}{4}$ square yards	= 1 square rod (sq. rd.)
160 square rods, or 43,560 square feet	= 1 acre (A.)
640 acres	= 1 square mile (sq. mi.)

CUBIC MEASURE

1728 cubic inches (cu. in.)	= 1 cubic foot (cu. ft.)
27 cubic feet	= 1 cubic yard (cu. yd.)

LIQUID MEASURE

3 teaspoons	= 1 tablespoon
16 tablespoons	= 1 cup
2 cups	= 1 pint
1 pint	= 16 ounces (liquid)
2 pints	= 1 quart
4 quarts	= 1 gallon
7.5 gallons	$\approx$ 1 cubic foot

DRY MEASURE

2 pints (pt.)	= 1 quart (qt.)
8 quarts	= 1 peck (pk.)
4 pecks	= 1 bushel (bu.)

AVOIRDUPOIS WEIGHT

16 ounces (oz.)	= 1 pound (lb.)
100 pounds	= 1 hundredweight (cwt.)
20 hundredweight, or 2000 pounds	= 1 ton (T.)

ANGLES AND ARCS

60 seconds (″)	= 1 minute (′)
60 minutes	= 1 degree (°)
90 degrees	= 1 right angle
360 degrees of arc	= 1 circumference
360 degrees of angle	= 1 complete rotation

TIME

60 seconds (sec.)	= 1 minute (min.)
60 minutes	= 1 hour (hr.)
24 hours	= 1 day (da.)
7 days	= 1 week (wk.)
365 days	= 1 common year (yr.)
366 days	= 1 leap year
12 months (mo.)	= 1 year
360 days	= 1 commercial year

METRIC LINEAR UNITS

1 millimeter (mm.)	= 0.001 meter (m.)
1 centimeter (cm.)	= 0.01 meter
1 decimeter (dm.)	= 0.1 meter
1 dekameter (dkm.)	= 10 meters
1 hectometer (hm.)	= 100 meters
1 kilometer (km.)	= 1000 meters

METRIC MEASURES OF CAPACITY

10 milliliters	= 1 centiliter (cl.)
10 centiliters	= 1 deciliter (dl.)
10 deciliters	= 1 liter (l.)
10 liters	= 1 dekaliter (dkl.)
10 dekaliters	= 1 hectoliter (hl.)
10 hectoliters	= 1 kiloliter (kl.)

METRIC WEIGHT

10 milligrams (mg.) = 1 centigram (cg.)
10 centigrams = 1 decigram (dg.)
10 decigrams = 1 gram (g.)
10 grams = 1 dekagram (dkg.)
10 dekagrams = 1 hectogram (hg.)
10 hectograms = 1 kilogram (kg.)

COMMON EQUIVALENTS (APPROXIMATE)

1 bu. $\approx$ 2150 cu. in. or $1\frac{1}{4}$ cu. ft.
1 gal. $\approx$ 231 cu. in.
1 cu. ft. $\approx$ $7\frac{1}{2}$ gal.
1 cu. ft. water $\approx$ 62.5 lb.
1 gal. water $\approx$ $8\frac{1}{3}$ lb.

METRIC EQUIVALENTS

Linear

1 inch = 2.54 centimeters
1 yard = 0.9144 meter
1 mile $\approx$ 1.609 kilometers
1 centimeter $\approx$ 0.39 inch
1 meter $\approx$ 39.37 inches, or 1.1 yard
1 kilometer $\approx$ 0.62 mile

Liquid

1 liter $\approx$ 1.056 quarts
1 quart $\approx$ 0.95 liters
1 gallon $\approx$ 3.8 liters

Dry

1 liter $\approx$ 0.91 quarts
1 quart $\approx$ 1.1 liters
1 hectoliter $\approx$ 2.8 bushels

Weight

1 gram $\approx$ 0.035 oz.
1 kg. $\approx$ 2.2 lb.
1 MT $\approx$ 2204.6 lb.
1 oz. $\approx$ 28.4 g.
1 lb. $\approx$ 0.453 kg.

TRIGONOMETRIC RATIOS

Angle Measure	*Sin*	*Cos*	*Tan*	*Angle Measure*	*Sin*	*Cos*	*Tan*
0°	0.0000	1.000	0.0000	46	.719	.695	1.04
1	.0175	1.000	.0175	47	.731	.682	1.07
2	.0349	.999	.0349	48	.743	.669	1.11
3	.0523	.999	.0524	49	.755	.656	1.15
4	.0698	.998	.0699	**50°**	**.766**	**.643**	**1.19**
5	.0872	.996	.0875	51	.777	.629	1.24
6	.105	.995	.105	52	.788	.616	1.28
7	.122	.993	.123	53	.799	.602	1.33
8	.139	.990	.141	54	.809	.588	1.38
9	.156	.988	.158	55	.819	.574	1.43
10°	**.174**	**.985**	**.176**	56	.829	.559	1.48
11	.191	.982	.194	57	.839	.545	1.54
12	.208	.978	.213	58	.848	.530	1.60
13	.225	.974	.231	59	.857	.515	1.66
14	.242	.970	.249	**60°**	**.866**	**.500**	**1.73**
15	.259	.966	.268	61	.875	.485	1.80
16	.276	.961	.287	62	.883	.469	1.88
17	.292	.956	.306	63	.891	.454	1.96
18	.309	.951	.325	64	.899	.438	2.05
19	.326	.946	.344	65	.906	.423	2.15
20°	**.342**	**.940**	**.364**	66	.914	.407	2.25
21	.358	.934	.384	67	.921	.391	2.36
22	.375	.927	.404	68	.927	.375	2.48
23	.391	.921	.425	69	.934	.358	2.61
24	.407	.914	.445	**70°**	**.940**	**.342**	**2.75**
25	.423	.906	.466	71	.946	.326	2.90
26	.438	.899	.488	72	.951	.309	3.08
27	.454	.891	.510	73	.956	.292	3.27
28	.469	.883	.532	74	.961	.276	3.49
29	.485	.875	.554	75	.966	.259	3.73
30°	**.500**	**.866**	**.577**	76	.970	.242	4.01
31	.515	.857	.601	77	.974	.225	4.33
32	.530	.848	.625	78	.978	.208	4.71
33	.545	.839	.649	79	.982	.191	5.15
34	.559	.829	.675	**80°**	**.985**	**.174**	**5.67**
35	.574	.819	.700	81	.988	.156	6.31
36	.588	.809	.727	82	.990	.139	7.12
37	.602	.799	.754	83	.993	.122	8.14
38	.616	.788	.781	84	.995	.105	9.51
39	.629	.777	.810	85	.996	.0872	11.4
40°	**.643**	**.766**	**.839**	86	.998	.0698	14.3
41	.656	.755	.869	87	.999	.0523	19.1
42	.669	.743	.900	88	.999	.0349	28.6
43	.682	.731	.933	89	1.000	.0175	57.3
44	.695	.719	.966	90°	1.000	.0000	. . . .
45	.707	.707	1.000				

SQUARES AND SQUARE ROOTS

n	n^2	$\sqrt{n}$	n	n^2	$\sqrt{n}$
1	1	1.000	51	2,601	7.141
2	4	1.414	52	2,704	7.211
3	9	1.732	53	2,809	7.280
4	16	2.000	54	2,916	7.348
5	25	2.236	55	3,025	7.416
6	36	2.449	56	3,136	7.483
7	49	2.646	57	3,249	7.550
8	64	2.828	58	3,364	7.616
9	81	3.000	59	3,481	7.681
10	100	3.162	60	3,600	7.746
11	121	3.317	61	3,721	7.810
12	144	3.464	62	3,844	7.874
13	169	3.606	63	3,969	7.937
14	196	3.742	64	4,096	8.000
15	225	3.873	65	4,225	8.062
16	256	4.000	66	4,356	8.124
17	289	4.123	67	4,489	8.185
18	324	4.243	68	4,624	8.246
19	361	4.359	69	4,761	8.307
20	400	4.472	70	4,900	8.367
21	441	4.583	71	5,041	8.426
22	484	4.690	72	5,184	8.485
23	529	4.796	73	5,329	8.544
24	576	4.899	74	5,476	8.602
25	625	5.000	75	5,625	8.660
26	676	5.099	76	5,776	8.718
27	729	5.196	77	5,929	8.775
28	784	5.292	78	6,084	8.832
29	841	5.385	79	6,241	8.888
30	900	5.477	80	6,400	8.944
31	961	5.568	81	6,561	9.000
32	1,024	5.657	82	6,724	9.055
33	1,089	5.745	83	6,889	9.110
34	1,156	5.831	84	7,056	9.165
35	1,225	5.916	85	7,225	9.220
36	1,296	6.000	86	7,396	9.274
37	1,369	6.083	87	7,569	9.327
38	1,444	6.164	88	7,744	9.381
39	1,521	6.245	89	7,921	9.434
40	1,600	6.325	90	8,100	9.487
41	1,681	6.403	91	8,281	9.539
42	1,764	6.481	92	8,464	9.592
43	1,849	6.557	93	8,649	9.644
44	1,936	6.633	94	8,836	9.695
45	2,025	6.708	95	9,025	9.747
46	2,116	6.782	96	9,216	9.798
47	2,209	6.856	97	9,409	9.849
48	2,304	6.928	98	9,604	9.899
49	2,401	7.000	99	9,801	9.950
50	2,500	7.071	100	10,000	10.000

Glossary of Mathematical Terms

The explanations in this glossary are intended as useful clarifications of the terms listed. They are not to be interpreted as precise definitions.

Absolute value The value of a number independent of its sign

Acute angle An angle whose measure is less than 90°

Acute triangle A triangle all of whose angles are acute

Addends Numbers to be added

Additive inverse If the sum of two numbers is zero, each is the additive inverse of the other

Altitude of a triangle The segment from a vertex perpendicular to the line containing the opposite side

Angle The figure formed by two rays with a common endpoint

Angle of depression The angle between the horizontal and a ray extending along the line of sight to an object below the horizontal

Angle of elevation The angle between the horizontal and a ray extending along the line of sight to an object above the horizon

Approximate number A number that expresses nearly, but not exactly, a measurement or count

Area The number of square units of surface in a region

Associative property The way in which the addends (or factors) are grouped does not affect the sum (or product); For example:
$(a + b) + c = a + (b + c)$;
$(a \times b) \times c = a \times (b \times c)$

Axis of symmetry The line, in line symmetry, that divides a figure so that one half is a reflection of the other

Bar graph A graph in which relative values are represented by the lengths of the bars

Base (geometry) The line or surface on which a plane or solid figure rests

Bisect To divide into two equal parts

Capacity The measure of contents of a container: cubic units

Cardinal numbers Numbers used to tell how many elements are in a set

Chord A segment whose endpoints are on a circle

Circle A closed curve, each of whose points is equally distant from its center

Circle graph A graph using sectors of a circle to show how the whole of anything is divided up

Circumference The distance around a circle $C = \pi d$

Circumscribe To construct a figure around another

Closure The property that the result of an operation is a number that is an element of the same set as the numbers operated on

Coefficient A number by which another number (usually a variable following it) is to be multiplied. For example 3 is the coefficient of the x in the term $3x^4$

Collinear Lying in the same straight line

Commutative property The order in which the addends (or factors) are added (or multiplied) does not affect the sum (or product); For example:
$a + b = b + a, a \times b = b \times a$

Conditional equation A mathematical statement that is neither true nor false; an open sentence; For example: $|x| + |5| = |7|$

Congruent Having the same size or shape

Constant A term in a mathematical sentence whose value does not change (as opposed to a variable)

Coordinate The point on a number line (or plane) associated with a given number (or number pair)

Corresponding angles The equal angles in two similar or congruent triangles, or other figures

Corresponding sides The sides opposite the equal angles in two similar or congruent triangles, or other figures

Cosine The ratio of the measure of the leg adjacent to an acute angle to the measure of the hypotenuse, in a right triangle

Cosine ratio The ratio of the measure of the leg adjacent to an acute angle to the measure of the hypotenuse in a right triangle

Decimal A fraction whose denominator is some power of 10, expressed with a decimal point to the left of the tenths place

Decimal system of numeration The writing of numerals using powers of 10 as place value

Degree A unit of measure for angles

Denominator The number indicated below the bar in a fraction; the number of parts into which the whole is divided

Diagonal A segment, other than a side, connecting two vertices

Diameter A chord which contains the center of a circle

Difference The result of subtraction

Digit Any of the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9

Dimension The length of a segment in a given direction, as length, width, or depth

Distributive property A property of our number system; For example: $a(b + a) = ab + ac$

Dividend The number to be divided

Edge The intersection of two faces of a solid

Element A member of a set

Empty set The set which contains no elements. Indicated by $\{\ \}$ or ϕ

Equation A statement that two expressions name the same number or the same set

Equivalent fractions Fractions naming the same number

Evaluate To find the value of an expression

Exponent A number that tells how many times another number (base) is to be taken as a factor

Extremes The first and fourth terms of a proportion

Factor One of two or more numbers whose product is to be found. In 3×8 both 3 and 8 are factors

Factorization The process of naming a number as a product; i.e., the factorizations of 30 are $2 \times 3 \times 5$, 1×30, 2×15, 3×10, and 5×6

Finite set A set in which the number of elements can be described by a whole number

Formula A general rule stated as a mathematical sentence

Fractional numbers of arithmetic The positive rational numbers and zero

Graph A pictorial representation of numerical data

Hexagon A polygon of six sides

Horizontal Parallel to the horizon or to the ground

Hypotenuse The side of a right triangle that is opposite the right angle

Identity element A number that, as a result of an operation with n produces n as a result

Indirect measurement of distances Measurement of distances between inaccessible points

Infinite Extending without limit

Infinite set A set with an infinite number of members

Inscribe To construct a figure within another figure

Integers The set of numbers, $\{\ldots, {}^{-}3, {}^{-}2, {}^{-}1, 0, {}^{+}1, {}^{+}2, \ldots\}$

Intersect (of lines) To have a point in common

Intersection of sets The set containing all elements which are common to two or more given sets. Indicated by $\cap$

Inverse operations Operations that undo one another, as addition and subtraction, multiplication and division

Irrational numbers Numbers that cannot be expressed as $\frac{a}{b}$, where a and b are integers $b \neq 0$

Isosceles triangle A triangle in which two sides have the same measure

Lateral area The area of the surface of a solid, exclusive of the bases

Like fractions Fractions having the same denominator

Line graph A graph using line segments to show values (usually over a period of time)

Lowest terms A fraction is in lowest terms when numerator and denominator do not contain a common factor other than 1

Mathematical sentence A statement of equality or inequality which may be a true statement, a false statement, or a conditional statement

Means The second and third terms of a proportion

Multiplicative inverse If the product of two numbers is 1, each is the multiplicative inverse, or reciprocal, of the other

Natural numbers N The set of numbers used for counting, for example, $\{1, 2, 3, \ldots\}$

Negative integers The set of numbers, $\{\ldots, {}^-3, {}^-2, {}^-1\}$

Negative numbers Numbers having values less than zero

Number line A line used in graphing points associated with numbers

Numeral A name for a number

Numerator The number indicated above the bar in a fraction indicating how many of the equal parts are being considered

Oblique Deviating from the vertical or horizontal

Obtuse angle An angle whose measure is greater than 90° but less than 180°

Obtuse triangle A triangle which contains an obtuse angle

Octagon An eight-sided polygon

Open sentence A mathematical sentence which is neither true nor false; a conditional statement; i.e., $x + 2 = 6$

Parallel lines Lines in the same plane with no point in common

Pentagon A five-sided polygon

Perimeter The distance around a polygon

Perpendicular lines Lines which meet or intersect to form a right angle

Pi The number which is the ratio of the circumference to the length of a diameter of a circle

Plane A set of points making up a flat surface extending indefinitely in two dimensions

Plane symmetry The property of a plane figure that it can be divided by a line so that each half is a reflection of the other

Point The unit element in geometry

Polygon A closed plane figure, each of whose sides is a segment

Positive integers The set of natural numbers

Positive numbers Numbers whose value is greater than zero

Prime factorization Factorization utilizing only prime numbers

Prime number A number with no factors other than itself and 1

Prism A solid figure, each of whose faces is a parallelogram and whose bases are congruent and parallel

Product The result obtained from multiplying factors

Proportion A statement of equality between two ratios. If $\frac{e_1}{m_1} = \frac{m_2}{e_2}$ then $m_1m_2 = e_1e_2$

Protractor An instrument for measuring an angle in degrees

Pythagorean property If a and b are measures of the lengths of the legs of a right triangle, and c is the measure of the hypotenuse, then $c^2 = a^2 + b^2$

Quadrilateral A four-sided polygon

Quotient The number of times the divisor is contained in the dividend, or the second factor found when the product and one factor are given

Radius A segment, one of whose endpoints is the center of the circle and whose other endpoint is on the circle

Ratio A comparison, by division, between two numbers which may be expressed as a fraction, a decimal, or a per cent
Rational numbers The set of numbers that can be expressed as the ratio of two integers, $\frac{a}{b}$, $b \neq 0$
Ray A subset of a line formed by any point of the line and all the points on one side of the chosen point
Real numbers The sets of rational and irrational numbers
Reciprocal The multiplicative inverse of a number
Rectangle A quadrilateral whose angles are right angles
Regular polygon A polygon the measures of whose angles are the same and the measures of whose sides are the same
Relatively prime Two numbers are relatively prime if there is no prime number that is a factor of both
Remainder That which is left (less than the divisor) after division is performed, or the difference when the subtrahend is subtracted from the minuend
Right angle An angle whose measure is 90°
Right triangle A triangle with one right angle
Scalene triangle A triangle in which the measures of no two sides are the same
Segment A subset of a line formed by any two points of the line and all the points between
Semicircle Half of a circle
Set Any well-defined collection
Signed numbers Numbers whose values relative to zero are indicated by + and −
Similar figures Figures that have the same shape but not necessarily the same size
Sine ratio The ratio of the measure of the side opposite an acute angle to the hypotenuse of a right triangle
Square A regular quadrilateral
Subset A set, all of whose members also belong to the given set
Subtrahend The number to be subtracted from the minuend
Sum The result from addition
Surface area The area of the surface of a three-dimensional figure, including that of the bases
Tangent ratio The ratio of the measure of the side opposite an acute angle to the measure of the side adjacent in a right triangle
Terms The numerator and denominator of a fraction; the means and extremes in a proportion; the symbols separated by + or − signs or verbs in a mathematical sentence
Trapezoid A quadrilateral with one pair of parallel sides
Triangle A three-sided polygon
Unit of measurement A stated amount of quantity adopted for purposes of comparison
Variable A symbol used to hold the place of a numeral
Vertex The common endpoint of two rays forming an angle
Vertical In an upright position; perpendicular to the horizontal
Whole numbers W The set of numbers consisting of the natural numbers and zero, $\{0, 1, 2, \ldots\}$

Index